The kernel of the Gysin homomorphism for Chow groups of zero cycles

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Let S be a smooth projective connected surface over an algebraically closed field k and Σ the linear system of a very ample divisor D on S. Let $d := \dim(\Sigma)$ be the dimension of Σ and

$$\varphi_\Sigma : S \hookrightarrow \mathbb{P}^d$$

the closed embedding of S into $\mathbb{P}^d$, induced by Σ.

For any closed point $t \in \Sigma \cong \mathbb{P}^{d^*}$, let C_t be the corresponding hyperplane section on S, and let

$$r_t : C_t \hookrightarrow S$$

be the closed embedding of the curve C_t into S.

Let Δ be the discriminant locus of Σ, that is,

$$\Delta := \{t \in \Sigma : C_t \text{ is singular}\}.$$

Then

$$U := \Sigma \setminus \Delta = \{t \in \Sigma : C_t \text{ is smooth}\}.$$

Let

$$r_t^* : H^1(C_t, \mathbb{Z}) \to H^3(S, \mathbb{Z})$$

be the Gysin homomorphism on cohomology groups induced by r_t, whose kernel $H^1(C_t, \mathbb{Z})_{\mathrm{van}}$ is called the *vanishing cohomology of* C_t (see [VoiII], 3.2.3).

Let $J_t = J(C_t)$ be the Jacobian of the curve C_t and let B_t be the abelian subvariety of the abelian variety J_t corresponding to the Hodge substructure $H^1(C_t, \mathbb{Z})_{\mathrm{van}}$ of $H^1(C_t, \mathbb{Z})$.

Let $\mathrm{CH}_0(S)_{\deg=0}$ be the Chow group of zero cycles of degree 0 on S, and for any closed point $t \in \Sigma$, let $\mathrm{CH}_0(C_t)_{\deg=0}$ be the Chow group of zero cycles of degree 0 on C_t.

For any closed point $t \in \Sigma$, let

$$r_t^* : \mathrm{CH}_0(C_t)_{\deg=0} \to \mathrm{CH}_0(S)_{\deg=0}$$

be the Gysin pushforward homomorphism on the Chow groups of degree 0 zero cycles of C_t and S, respectively, induced by r_t, whose kernel

$$G_t = \ker(r_t^*)$$

is called the *Gysin kernel* associated with the hyperplane section C_t.

Intermezzo on Cycles and Chow groups

Let X denote a smooth projective variety over an algebraically closed field k.

Definition

- An algebraic cycle of dimension r or simply an r-cycle is a finite formal linear combination

$$Z = \sum n_i Z_i,$$

where $n_i \in \mathbb{Z}$ and Z_i is a subvariety of X of dimension r.

- The group of r-cycles is denoted by $Z_r(X)$.
- Thinking in terms of codimension and if X is of dimension n we write

$$Z_r(X) = Z^{n-r}(X).$$

$Z^{n-r}(X)$ is the group of cycles of codimension $(n-r)$ on X.

Examples

Let X be a smooth projective variety of dimension n.

1. The zero cycles on X are finite formal linear combinations

$$Z = \sum n_i P_i,$$

where $n_i \in \mathbb{Z}$ and P_i is a point on X. The group of zero cycles is denoted by $Z_0(X)$.

2. The cycles of codimension 1 on X are finite formal linear combinations

$$Z = \sum n_i Z_i,$$

where $n_i \in Z$ and Z_i is a subvariety of codimension 1 of X. The cycles of codimension 1 are also called divisors. The group of cycles of codimension 1 is denoted by $Z^1(X)$ or by $\mathrm{Div}(X)$.

Rational and algebraic equivalence

Definition

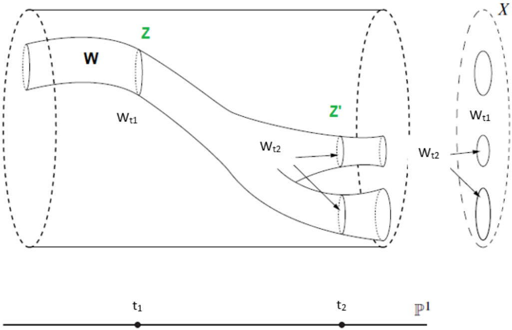
- Two r-cycles Z_1 and Z_2 on X are rationally equivalent if there exits a family of r-cycles parametrized by $\mathbb{P}^1$ interpolating between them, i.e. if there is $W \in Z_{r+1}(\mathbb{P}^1 \times X)$ not contained in any fiber $\{t\} \times X = \mathrm{pr}_1^{-1}(t), t \in \mathbb{P}^1$, such that defining by

$$W(t) := [W \cap (\{t\} \times X)]$$

the r-cycle obtained by intersecting W with the fiber $\{t\} \times X$ over t, we have

$$Z_1 = W(t_1) \text{ and } Z_2 = W(t_2) \text{ for some } t_1, t_2 \in \mathbb{P}^1.$$

- If in the above definition we replace $\mathbb{P}^1$ by any smooth curve then we say that Z_1 and Z_2 are algebraically equivalent.



Rational equivalence between two cycles z and z' on X.

Properties

- The r-cycles rationally equivalent to 0 on X, denoted by

$$Z_r(X)_{\mathrm{rat}} = \{Z \in Z_r(X) : Z \sim_{\mathrm{rat}} 0\}$$

form a subgroup of $Z_r(X)$.

- The r-cycles algebraically equivalent to 0 on X, denoted by

$$Z_r(X)_{\mathrm{alg}} = \{Z \in Z_r(X) : Z \sim_{\mathrm{alg}} 0\}$$

form a subgroup of $Z_r(X)$.

Chow groups

Definition

The Chow group of r-cycles of X is the factor group

$$\mathrm{CH}_r(X) = Z_r(X)/Z_r(X)_{\mathrm{rat}}$$

of the group of r-cycles modulo the group of r-cycles rationally equivalent to 0, i.e. the group of rational equivalence classes of r-cycles.

Results on codimension 1 cycles or divisors

In this case one has good results.

- We have

$$\mathrm{CH}^1(X) = Z^1(X)/Z^1(X)_{\mathrm{rat}} \cong \mathrm{Pic}(X) \cong H^1_{\mathrm{Zar}}(X, \mathcal{O}_X^*),$$

  where $\mathrm{Pic}(X)$ is the group of isomorphism classes of invertible sheaves on X called the Picard group, and $H^1_{\mathrm{Zar}}(X, \mathcal{O}_X^*)$ is the group of isomorphism classes of line bundles on X. One can make this group into a scheme.

- We have

$$\mathrm{NS}(X) := Z^1(X)/Z^1(X)_{\mathrm{alg}},$$

  it is a finitely generated abelian group, called the Neron-Severi group of X.

The quotient group

$$A^1(X) := Z^1(X)_{\mathrm{alg}}/Z^1(X)_{\mathrm{rat}}$$

is the connected component of unity of the scheme $\mathrm{Pic}(X)$ denoted by $\mathrm{Pic}^0(X)$. In $\mathrm{char}(k) = 0$ it has the structure of an Abelian variety called the Picard variety. $A^1(X)$ is representable and we can think of it as a torus.

In contrast to the case of cycles of codimension 1 very little is known about the above groups of cycle classes. For example, if $r > 1$

- We do not know if
$$Z^r(X)/Z^r(X)_{\text{alg}}$$
is a finitely generated abelian group.
- In general
$$A^r(X) = Z^r(X)_{\text{alg}}/Z^r(X)_{\text{rat}}$$
is not representable. That is:

  In studying algebraic cycles we encounter objects which are geometric in content and simultaneously not representable!

Let $U = \Sigma \setminus \Delta$.

Theorem (Paucar, 2022)

(a) *For each $t \in U$ there is an abelian variety $A_t \subset B_t$ such that*

$$G_t = \ker(r_{t*}) = \bigcup_{\text{countable}} \text{translates of } A_t$$

(b) *For a very general $t \in U$ (i.e. for every t in a c-open subset U_0 of U) either*

1. *$A_t = B_t$, and then $G_t = \bigcup_{\text{countable}}$ translates of B_t, or*
2. *$A_t = 0$, and then G_t is countable.*

(c) *If $alb_S : \mathrm{CH}_0(S)_{\mathrm{hom}} \to \mathrm{Alb}(S)$ is not an isomorphism, for a very general t in U, then G_t is countable.*

- The subset U_0 is countable open $\rightsquigarrow$ allows to apply for all t in U_0 in a uniform way the irreducibility of the monodromy representation on the vanishing cohomology of a smooth section (see [DK73], [D74] for the étale cohomology, [La81] for the singular cohomology and [VoiII] in a Hodge theoretical context for complex algebraic varieties).
- this is done by viewing $U = \Sigma \setminus \Delta$ as an integral algebraic scheme over k and by passing to the general fiber, i.e. for each closed point t in U_0 there exists a scheme-theoretic isomorphism to the geometric generic point $\overline{\xi}$ over k', where k' is the minimal field of definition of S [Wei62]. This induces a scheme-theoretic isomorphism κ_t between the corresponding varieties C_t and $C_{\overline{\xi}}$ over k'.

This induces an isomorphism κ_t' between A_t and $A_{\overline{\xi}}$ compatible with the isomorphism on Chow groups induced by the isomorphism κ_t. Then by [BG20] $\kappa_t'(A_t) = A_{\overline{\xi}}$ and $\kappa_t'(B_t) = B_{\overline{\xi}}$ for every k-point in U_0.

- Goal: describe the Gysin kernel G_t for the points t in $U \setminus U_0$ where the local and global monodromy representations, i.e. the action of the fundamental groups $\pi_1(V, t)$, where $V = (\Sigma \setminus \Delta) \cap D$ with D a line containing t in the dual space $\mathbb{P}^{n^*}$ such that f_D is a Lefschetz pencil for S, and $\pi_1(U, t)$ on the vanishing cohomology $H^1(C_t, k')_{\mathrm{van}}$, are not fully understood.
- Approach: construct a stratification $\{\mathcal{U}_i \subseteq U\}_{i \in I}$ of U by countable open subsets for each of which the monodromy argument applies for all $t \in \mathcal{U}_i$ in a uniform way (i.e. for a partially ordered, at most countable set I we have $\mathcal{U}_i \subseteq \mathcal{U}_j$ if $i \leq j$ and two additional conditions on I for finiteness from below and for every map $\alpha : \mathrm{Spec}(k) \to U$ the set $\{i \in I : \alpha \text{ factors through } \mathcal{U}_i\}$ has a smallest element (cf. [GL19], Def. 5.2.1.1).

We then apply a convergence argument for the stratification $\{\mathcal{U}_i\}_{i \in I}$ (cf. [GL19], Def. 5.2.2.1) such that the monodromy argument applies in a uniform way for all $t \in U$, seen as the set-theoretic directed union $U = \bigcup_{\overrightarrow{i}} \mathcal{U}_i$.

Definition (Gaitsgory - Lurie, 2019)

Let $\mathcal{X}$ be an algebraic stack. A **stratification** of $\mathcal{X}$ consists of the following data:

(a) A partially ordered set A.

(b) A collection of open substacks $\{\mathcal{U}_\alpha \subseteq \mathcal{X}\}_{\alpha \in A}$ satisfying $\mathcal{U}_\alpha \subseteq \mathcal{U}_\beta$ when $\alpha \leq \beta$.

This data is required to satisfy the following conditions:

- For each index $\alpha \in A$, the set $\{\beta \in A : \beta \leq \alpha\}$ is finite.
- For every field m and every map $\eta : \mathrm{Spec}(m) \to \mathcal{X}$, the set

  $$\{\alpha \in A : \eta \text{ factors through } \mathcal{U}_\alpha\}$$

  has a least element.

Let $\mathcal{X}$ be an algebraic stack equipped with a stratification $\{\mathcal{U}_\alpha\}_{\alpha \in A}$.
Notation: For each $\alpha \in A$, we let $\mathcal{X}_\alpha$ denote the reduced closed substack of $\mathcal{X}$ given by the complement of $\cup_{\beta<\alpha}\mathcal{U}_\beta$. Each $\mathcal{X}_\alpha$ is a locally closed substack of $\mathcal{X}$, called the **strata** of $\mathcal{X}$.

Remark

A stratification $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ is determined by the partially ordered set A together with a collection of locally closed substacks $\{\mathcal{X}_\alpha\}_{\alpha \in A}$: each $\mathcal{U}_\alpha$ is an open substack of $\mathcal{X}$ and if m is a field, then a map $\eta : \mathrm{Spec}(m) \to \mathcal{X}$ factors through $\mathcal{U}_\alpha$ iff it factors through $\mathcal{X}_\beta$ for some $\beta \leq \alpha$. So one identifies the stratification of $\mathcal{X}$ with the locally closed substacks $\{\mathcal{X}_\alpha \subseteq \mathcal{X}\}_{\alpha \in A}$ (where the partial ordering of A is understood to be implicitely specified).

Remark

*If m is a field, then for any map $\eta : \mathrm{Spec}(m) \to \mathcal{X}$ there is a **unique** index $\alpha \in A$ such that η factors through $\mathcal{X}_\alpha$. In other words, $\mathcal{X}$ is the **set-theoretic** union of the locally closed substacks $\mathcal{X}_\alpha$.*

Definition

Let $m = \mathbb{C}$ or $m = \mathbb{F}_q$ and let $\mathcal{X}$ be an algebraic stack of finite type over $\mathrm{Spec}(m)$. A stratification $\{\mathcal{X}_\alpha\}_{\alpha\in A}$ of $\mathcal{X}$ is **convergent** if there exists a finite collection of algebraic stacks $\mathcal{T}_1, \dots, \mathcal{T}_n$ over $\mathrm{Spec}(m)$ with the following properties:

(1) For each $\alpha \in A$ there exists an integer $i \in \{1, 2, \dots, n\}$ and a diagram of algebraic stacks $\mathcal{T}_i \xrightarrow{f} \tilde{\mathcal{X}}_\alpha \xrightarrow{g} \mathcal{X}_\alpha$ where the map f is a fiber bundle (locally trivial with respect to the étale topology) whose fibers are affine spaces of some fixed dimension d_α and the map g is surjective, finite and radicial.

(2) The nonnegative integers d_α in (1) satisfy $\sum_{\alpha\in A} q^{-d_\alpha} < \infty$.

(3) For $1 \leq i \leq n$, the algebraic stack $\mathcal{T}_i$ can be written as a stack-theoretic quotient Y/G, where Y is an algebraic space of finite type over m and G is a linear algebraic group over m acting on Y.

Remark

In the definition, for $\mathrm{char}(m) > 0$ *the hypothesis* (2) *guarantees that the set* A *is at most countable. For* $\mathrm{char}(m) = 0$ *we will assume* A *to be at most countable.*

We have the following

Lemma (Gysin sequence)

Let X *and* Y *be smooth quasi-projective varieties over an algebraically closed field* k*, let* $g : Y \to X$ *be a finite radicial morphism, and let* $U \subseteq X$ *be the complement of the image of* g*. Then there is a canonical fiber sequence*

$$C^{*-2d}(Y; \mathbb{Z}_l(-d)) \to C^*(X, \mathbb{Z}_l) \to C^*(U; \mathbb{Z}_l),$$

where d *denotes the relative dimension* $\dim(X) - \dim(Y)$*.*

This implies a corresponding result for algebraic stacks:

Lemma

Let $\mathcal{X}$ and $\mathcal{Y}$ be smooth algebraic stacks of constant dimension over an algebraically closed field k, let $g : \mathcal{Y} \to \mathcal{X}$ be a finite radicial morphism, and let $\mathcal{U} \subseteq \mathcal{X}$ be the open substack of $\mathcal{X}$ complementary to the image of g. Then there is a canonical fiber sequence

$$C^{*-2d}(\mathcal{Y};\mathbb{Z}_l(-d)) \to C^*(\mathcal{X},\mathbb{Z}_l) \to C^*(\mathcal{U};\mathbb{Z}_l),$$

where d denotes the relative dimension $\dim(\mathcal{X}) - \dim(\mathcal{Y})$.

We have the following

Theorem (Paucar - S., 2023)

Let $U := \Sigma \setminus \Delta$. There exists a convergent stratification $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ of U by countable open subsets for each of which the irreducibility of the monodromy representation applies for all $t \in \mathcal{U}_\alpha$ in a uniform way such that the monodromy argument applies for all $t \in U$ in a uniform way, seen as the set-theoretic directed union $U = \bigcup_{\overrightarrow{i}} \mathcal{U}_i$.

Proof.

Let $d = \dim(U)$ and let $\{\mathcal{X}_\alpha\}_{\alpha \in A}$ be the given stratification of U. The set A is at most countable (see Remark). By adding additional elements to A and assigning to each of those additional elements the empty substack of U, we may assume that A is infinite.

By assumption the set $\{\beta \in A : \beta \leq \alpha\}$ is finite for each $\alpha \in A$, hence we can chose an enumeration

$$A = \{\alpha_0, \alpha_1, \alpha_2, \ldots\}$$

where each initial segment is a downward-closed subset of A. We can then write U as the union of an increasing sequence of open substacks

$$\mathcal{U}_0 \hookrightarrow \mathcal{U}_1 \hookrightarrow \mathcal{U}_2 \hookrightarrow \ldots$$

where $\mathcal{U}_n$ is characterized by the requirement that if m is a field, then a map $\eta : \mathrm{Spec}(m) \to \mathcal{X}$ factors through $\mathcal{U}_n$ iff it factors through one of the substacks $\mathcal{X}_{\alpha_0}, \mathcal{X}_{\alpha_1}, \ldots, \mathcal{X}_{\alpha_n}$.

By hypothesis, there exists a finite collection $\{\mathcal{T}_i\}_{1\leq i\leq s}$ of smooth algebraic stacks over $\mathrm{Spec}(m)$, where each $\mathcal{T}_i$ has some fixed dimension d_i and for each $n\geq 0$ there exists an index $i(n)\in\{1,\ldots,s\}$ and a diagram

$$\mathcal{T}_{i(n)} \overset{f_n}{\to} \tilde{\mathcal{X}}_{\alpha_n} \overset{g_n}{\to} \mathcal{X}_{\alpha_n},$$

where g_n is a finite radicial surjection and f_n is an étale fiber bundle whose fibers are affine spaces of some fixed dimension $e(n)$. Set

$$\overline{U} = U \times_{\mathrm{Spec}(m)} \mathrm{Spec}(\overline{m})$$

$$\overline{\mathcal{U}}_n = \mathcal{U}_n \times_{\mathrm{Spec}(m)} \mathrm{Spec}(\overline{m})$$

$$\overline{\mathcal{T}}_i = \mathcal{T}_i \times_{\mathrm{Spec}(m)} \mathrm{Spec}(\overline{m})$$

The map f_n induces an isomorphism in l-adic cohomology. Applying the above Lemma to the finite radicial map

$$g_n : \overline{\mathcal{X}}_{\alpha_n} \times_{\mathrm{Spec}(m)} \mathrm{Spec}(\overline{m}) \to \overline{\mathcal{U}}_n$$

we obtain fiber sequences

$$C^{*-2e_n'}(\overline{\mathcal{T}}_{i(n)}; \mathbb{Z}_l(-e_n')) \to C^*(\overline{\mathcal{U}}_n, \mathbb{Z}_l) \to C^*(\overline{\mathcal{U}}_{n-1}; \mathbb{Z}_l),$$

where $e_n' = e_n + d - d_{i(n)}$ denotes the relative dimension of the map $\tilde{\mathcal{X}}_{\alpha_n} \to U$. We have a canonical equivalence

$$\Theta : C^*(\overline{U}; \mathbb{Z}_l) \simeq \lim_{\overleftarrow{n}} C^*(\overline{\mathcal{U}}_n, \mathbb{Z}_l).$$

By the increasing sequence of c-open substacks

$$\mathcal{U}_0 \hookrightarrow \mathcal{U}_1 \hookrightarrow \mathcal{U}_2 \hookrightarrow$$

and using Galois descent with respect to the corresponding simplicial complex $\mathcal{N}(U)$ we obtain $U = \bigcup_{\vec{n}} \mathcal{U}_n$. □

By a result due to Zariski the monodromy

$$\rho : \pi_1(U,0) \to \mathrm{Aut}(H^k(C_t,\mathbb{Z}))$$

can be computed by restricting to a Lefschetz pencil:

Theorem

Let $\mathcal{Y} \subset \mathbb{P}^d$ be a hypersurface, and let $U = \mathbb{P}^d \setminus \mathcal{Y}$ be its complement. Then for $0 \in U$ and for every projective line $C_t \subset \mathbb{P}^d$ passing through 0 which meets $\mathcal{Y}$ transversally in its smooth locus, the natural map

$$\pi_1(C_t \setminus C_t \cap \mathcal{Y}, 0) \to \pi_1(U,0)$$

is surjective.

The irreducibility of the discriminant hypersurface implies:

Proposition (VoiII, 3.23)

All the vanishing cycles (defined up to sign) are conjugate (up to sign) under the monodromy action ρ.

Zariski's theorem and the proposition imply the following

Corollary

Let $(C_t)_{t \in \mathbb{P}^1}$ be a Lefschetz pencil of hyperplane sections of S, $0_i, i = 1, \ldots, M$ the critical values, and $0 \in \mathbb{P}^1$ a regular value. Then all the vanishing cycles $\delta_i \in H^{n-1}(C_0, \mathbb{Z})$ of the pencil are conjugate under the monodromy action of

$$\rho : \pi_1(\mathbb{P}^1 \setminus \{0_1, \ldots, 0_M\}, 0) \to \mathrm{Aut}(H^{n-1}(X_0, \mathbb{Z})).$$

Remark

These vanishing cycles are not well-defined without specifying the choice of paths γ_i from 0 to 0_i. However, changing the path comes down to letting a loop act via the monodromy action.

Recall that if $C_0 \overset{r_0}{\hookrightarrow} S$ is a smooth hyperplane section, the vanishing cohomology of C_0 is defined by

$$H^{n-1}(C_0, \mathbb{Q})_{\mathrm{van}} = \ker(r_* : H^{n-1}(C_0, \mathbb{Q}) \to H^{n+1}(S, \mathbb{Q})).$$

Let $U \subset \mathbb{P}^{d^*}$ denote the open set parametrising the smooth hyperplane sections of S, then the monodromy action

$$\rho : \pi_1(U, 0) \to \mathrm{Aut}(H^{n-1}(C_0, \mathbb{Q}))$$

leaves $H^{n-1}(C_0, \mathbb{Q})_{\mathrm{van}}$ stable.

We have the following

Theorem (Voill, 3.27)

Let the notation and hypotheses be as in the above corollary. Then the monodromy action

$$\rho : \pi_1(U, 0) \to \mathrm{Aut}(H^{n-1}(C_0, \mathbb{Q})_{\mathrm{van}})$$

is irreducible.

Proof.

It suffices to prove the irreducibility of the monodromy action

$$\pi_1(\mathbb{P}^1 \setminus \{0_1, \ldots, 0_M\}, 0) \to \mathrm{Aut}(H^{n-1}(C_0, \mathbb{Q})_{\mathrm{van}}),$$

where $(C_t)_{t \in \mathbb{P}^1}$ is a Lefschetz pencil.

The vanishing cohomology is generated by the vanishing cycles δ_i of the pencil. The restriction of the intersection form $\langle\ ,\ \rangle$ is non-degenerate on $H^{n-1}(C_0,\mathbb{Q})_{\mathrm{van}}$. Let $F \subset H^{n-1}(C_0,\mathbb{Q})_{\mathrm{van}}$ be a non-trivial vector subspace stable under the monodromy action ρ. For $i \in \{1,\ldots,M\}$, let $\tilde{\gamma}_i$ be the loop in S based at 0 which is equal to γ_i until t_i, winds around the disk Δ_i once in the positive direction, then returns to 0 via γ_i^{-1}. By the Picard-Lefschetz formula we have

$$\rho(\tilde{\gamma}_i)(\alpha) = \alpha \pm \langle \alpha, \delta_i \rangle \delta_i,\ \forall \alpha \in H^{n-1}(C_0,\mathbb{Q}).$$

Let $0 \neq \alpha \in F$. There exists $i \in \{1, \ldots, M\}$ such that $\langle \alpha, \delta_i \rangle \neq 0$. As $\rho(\tilde{\gamma}_i)(\alpha) - \alpha \in F$, the Picard-Lefschetz formula implies that $\delta_i \in F$. By the corollary all vanishing cycles are conjugate under the monodromy action, so F, stable under ρ, must contain all the vanishing cycles. Thus $F = H^{n-1}(C_0, \mathbb{Q})_{\text{van}}$. □

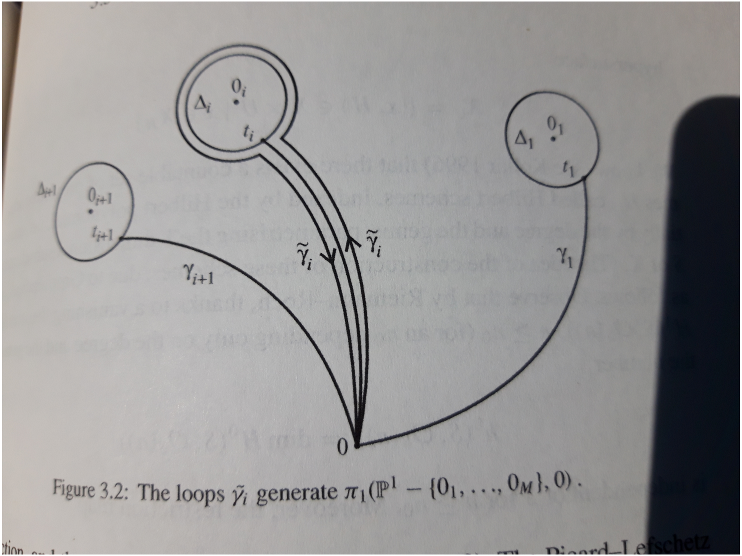


Figure 3.2: The loops $\tilde{\gamma}_i$ generate $\pi_1(\mathbb{P}^1 - \{0_1, \ldots, 0_M\}, 0)$.

Thank you for your attention!

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