Prospects in Geometry and Global Analysis
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Canonical Submersions in Nearly Kähler Geometry

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Holonomy I

If $\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is a connection on M and γ a curve from p to q, then we have parallel transport $\mathcal{P}_\gamma\colon T_pM \to T_qM$.

Definition

The holonomy group is the group

$$\mathrm{Hol}_0(\nabla) = \{\mathcal{P}^\nabla_\gamma \text{ parallel transport} \mid [\gamma] = 0 \in \pi_1(M)\}.$$

If ∇ is metric then $\mathrm{Hol}_0(\nabla) \subset \mathrm{SO}(n)$.

Theorem (deRham 1952)

If the representation of $\mathrm{Hol}^g(M)$ on $T_pM = T_1 \oplus T_2$ splits, then (M, g) is locally a product

$$(M, g) = (M_1 \times M_2, g_{M_1 \times M_2}).$$

Holonomy II

Theorem (Berger 1955)

Let (M, g) be a Riemannian manifold that is not a symmetric space. If the holonomy representation is irreducible, then $\mathrm{Hol}_0^g(M)$ is among the following list.

Hol_0^g	*NAME*	dim	*Parallel Obj.*	*Einstein*
$\mathrm{SO}(n)$	*generic*	n	$-$	$-$
$\mathrm{U}(n)$	*Kähler*	$2n$	$\nabla^g J = 0$	$-$
$\mathrm{SU}(n)$	*Calabi-Yau*	$2n$	$\nabla^g J = 0$	$\mathrm{Ric}^g = 0$
$\mathrm{Sp}(n)\mathrm{Sp}(1)$	*quaternionic Kähler*	$4n$	$\nabla^g \Omega = 0$	$\mathrm{Ric}^g = \lambda g$
$\mathrm{Sp}(n)$	*hyperkähler*	$4n$	$\nabla^g J_i = 0$	$\mathrm{Ric}^g = 0$
G_2	*parallel* G_2	7	$\nabla^g \phi = 0$	$\mathrm{Ric}^g = 0$
$\mathrm{Spin}(7)$	*parallel* $\mathrm{Spin}(7)$	8	$\nabla^g \psi = 0$	$\mathrm{Ric}^g = 0$

Connections with skew torsion

Torsion:

$$T(X,Y,Z) := g(\nabla_X Y - \nabla_Y X - [X,Y], Z)$$

The space of torsion tensors decomposes into $\mathrm{SO}(n)$-invariant classes

$$T \in \Lambda^2 TM \otimes TM = TM \oplus \Lambda^3 TM \oplus \mathcal{T}'.$$

If $T \in \Lambda^3 TM$ we say ∇ has skew-torsion

$$\Rightarrow \ \nabla_X Y = \nabla^g_X Y + \frac{1}{2} T(X,Y,\cdot)$$

In particular, ∇ inherits geodesics from ∇^g!

∇ has parallel skew-torsion if, in addition, $\nabla T = 0$.

Implies, for instance, pair-symmetry of the curvature!

Canonical Submersion Theorem

Theorem (Cleyton, Moroianu, Semmelmann '21, Agricola, Dileo, S '21, S '22)

Let ∇ be a metric connection with parallel skew torsion T, and suppose the tangent space decomposes orthogonally as a representation of the reduced holonomy group $\mathrm{Hol}_0(\nabla)$ into $TM = \mathcal{V} \oplus \mathcal{H}$. If the component in $\Lambda^2\mathcal{V} \otimes \mathcal{H}$ of $T \in \Lambda^3 TM$ vanishes then

- *there exists a locally defined Riemannian submersion $\pi \colon M \to N$ with totally geodesic leaves along $\mathcal{V}$,*
- *the purely horizontal part $T^{\mathcal{H}} \in \Lambda^3\mathcal{H}$ of T is projectable and the connection $\nabla^{T^{\mathcal{H}}} = \nabla^{g_N} + \frac{1}{2}T^{\mathcal{H}}$ on N has parallel skew torsion and satisfies*

$$\nabla^{T^{\mathcal{H}}}_X Y = \pi_*(\nabla_{\overline{X}}\overline{Y}).$$

Corollary

If $TM = H_1 \oplus H_2$ under $\mathrm{Hol}_0(\nabla)$ with $T \in \Lambda^3 H_1 \oplus \Lambda^3 H_2$ then locally

$$(M, g, \nabla) = (M_1, g_1, \nabla_1) \times (M_2, g_2, \nabla_2).$$

Known Applications

- Sasakian manifolds $\longrightarrow$ Kähler:

  Simplest case, φ projects to J

- 3-(α, δ)-Sasaki manifolds $\longrightarrow$ quaternionic Kähler:

  φ_i do not project individually but their bundle does!
  (Agricola, Dileo, S-, '21)

  - relate curvature (Agricola, Dileo, S-, '23),
  - simplifies heterotic G_2 system (Galdeano, S-, '23)

- parallel 3-(α, δ)-Sasaki manifolds $\longrightarrow$ nearly Kähler:

  Torsion of the base is a feature not a bug!

- specific nearly Kähler manifolds $\longrightarrow$ quaternionic Kähler:

  Modeled after Nagy classification of nK,
  but explicit construction of quaternionic structure!

3-(α, δ)-Sasaki Manifolds

$(M, \varphi_i, \xi_i, \eta_i, g)_{i=1,2,3}$ is an almost 3-contact metric manifolds if

- $\xi_i \in \mathfrak{X}(M)$ Reeb vector fields, $g(\xi_i, \xi_j) = \delta_{ij}$,
- $\eta_i \in \Omega^1(M)$ metric dual to ξ_i,
- $\varphi_i \in \mathrm{End}(TM)$ almost hermitian structures on $\ker \eta_i$,
- compatibility cond. $\varphi_i \varphi_j|_{\mathcal{H}} = \varphi_k|_{\mathcal{H}}$, $\varphi_i \xi_j = \xi_k$,

where (ijk) is an even permutation of (123) and $\mathcal{H} := \bigcap \ker \eta_i$.
Denote $\mathcal{V} = \langle \xi_1, \xi_2, \xi_3 \rangle$ the vertical space and $\Phi_i(X, Y) = g(X, \varphi_i Y)$ the fundamental 2-form.

Definition

A 3-(α, δ)-Sasaki manifold, $\alpha \neq 0$, δ real constants, is an almost 3-contact metric manifold satisfying

$$\mathrm{d}\eta_i = 2\alpha \Phi_i + 2(\alpha - \delta)\eta_j \wedge \eta_k,$$

for alle even permutations (ijk) of (123).

$\mathcal{H}$-Homothetic Deformation

For $a > 0$, $c \neq 0$ consider

$$g = g_{\mathcal{H}} + g_{\mathcal{V}} \quad \xrightarrow{\text{scaling}} \quad \tilde{g} = a g_{\mathcal{H}} + c^2 g_{\mathcal{V}},\ \tilde{\xi}_i = \frac{1}{c}\xi_i,\ \tilde{\varphi}_i = \varphi_i$$

Proposition (Agricola-Dileo, '20)

The deformed structure $(M, \tilde{\varphi}_i, \tilde{\xi}_i, \tilde{\eta}_i, \tilde{g})_{i=1,2,3}$ *for real parameters* $a > 0$ *and* c *is* 3-$(\tilde{\alpha}, \tilde{\delta})$*-Sasaki with*

$$\tilde{\alpha} = \frac{c}{a}\alpha\,, \qquad \tilde{\delta} = \frac{1}{c}\delta.$$

Up to $\mathcal{H}$*-homothetic deformation there are only the cases* $\alpha = 1$, $\delta \in \{\pm 1, 0\}$ *called positive, negative and degenerate.*

The canonical connection

Theorem (Agricola, Dileo, '20)

Let $(M, g, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ *be a* 3-(α, δ)*-Sasaki manifold. Then* M *admits a unique metric connection* ∇ *with skew torsion such that for a smooth function* β*,*

$$\nabla_X \varphi_i \;=\; \beta(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k) \quad \forall X \in \mathfrak{X}(M)$$

for every even permutation (ijk) *of* (123)*.*

This connection ∇ *has parallel skew-torsion, as* $\mathrm{Hol}_0 \subset \mathrm{Sp}(n)\mathrm{Sp}(1)$*, i.e. it preserves* $TM = \mathcal{V} \oplus \mathcal{H}$ *and the function* β *is a constant given by*

$$\beta = 2(\delta - 2\alpha).$$

∇ is called the canonical connection of M.

If $\delta = 2\alpha \Rightarrow \nabla\varphi_i \equiv 0$ for all $i = 1, 2, 3$, $\to$ parallel 3-(α, δ)-Sasaki

$\Rightarrow \mathrm{Hol}_0(\nabla) \subset \mathrm{Sp}(n)$

The canonical submersion

$TM = \mathcal{V} \oplus \mathcal{H}$ is a $\mathrm{Hol}_0(\nabla)$-invariant decomposition.
The torsion of ∇ is

$$T = 2\alpha \sum_{i=1}^{3} \eta_i \wedge \Phi_i^{\mathcal{H}} + 2(\delta - 4\alpha)\eta_{123} \in \Lambda^2\mathcal{H} \wedge \mathcal{V} \oplus \Lambda^3\mathcal{V}.$$

$\Rightarrow$ the projection onto $\Lambda^2\mathcal{V} \otimes \mathcal{H} \oplus \Lambda^3\mathcal{H}$ is trivial
$\Rightarrow$ locally defined Riemannian submersion $\pi\colon M \to N$ with

$$\nabla^{g_N}_X Y = \pi_*(\nabla_{\overline{X}}\overline{Y}).$$

Definition

The submersion $\pi\colon M \to N$ is called the canonical submersion of the 3-(α,δ)-Sasaki manifold M.

Theorem (Agricola, Dileo, S-, '21)

Let $(M, g, \eta_i, \xi_i, \varphi_i)$ *be a* 3-(α, δ)*-Sasaki manifold. The base space* (N, g_N) *of the canonical submersion* $\pi\colon M \to N$ *inherits a quaternionic Kähler structure spanned by*

$$\check{\varphi}_i = \pi_* \circ \varphi_i \circ s_*$$

for any local section $s\colon N \to M$*. The covariant derivative of* $\check{\varphi}_i$ *is given by*

$$\nabla^{g_N}_X \check{\varphi}_i = 2\delta(\check{\eta}_k(X)\check{\varphi}_j - \check{\eta}_j(X)\check{\varphi}_k),$$

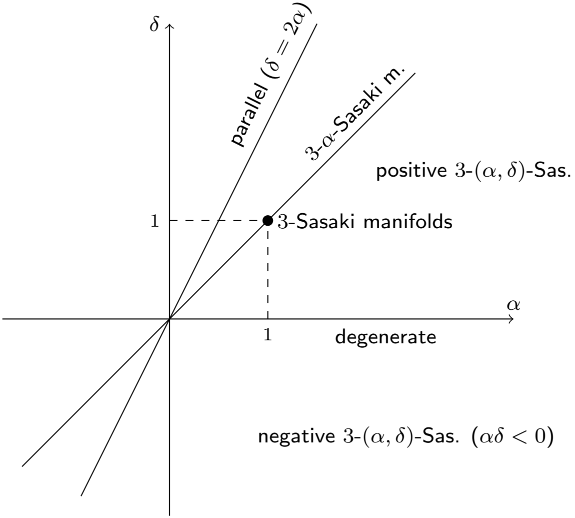
where $\check{\eta}_i(X) = \eta_i(s_* X)$*, for any even permutation* (ijk) *of* (123)*.*

The base space N *has scalar curvature* $\mathrm{scal}_{g_N} = 16n(n+2)\alpha\delta$*.*

Corollary

For degenerate 3-(α, δ)*-Sasaki manifolds the base is Hyperkähler.*

(α, δ)-plane of structures



Interlude: Constructions

Positive Case:

$(N, g, \mathcal{Q})$ positive quaternionic Kähler manifold, i.e. $\mathcal{Q} \subset \mathrm{End}(TM)$
$\rightsquigarrow M = \mathrm{Fr}(\mathcal{Q})$ frame bundle (Konishi-bundle).
Then M can be equipped with a positive 3-(α, δ)-Sasaki structure.

Examples: $W_{1,1}, \mathcal{S}^7, \mathcal{S}_p = \mathrm{SU}(3)//S^1_p$,...

Negative Case:

$(N, g, \mathcal{Q})$ negative qK $\rightsquigarrow M = \mathrm{Fr}(\mathcal{Q})$ Konishi bundle admits a negative 3-(α, δ)-Sasaki structure

Examples: $\mathrm{SU}(1,2)/S^1, \hat{\mathcal{T}}(1)$, ...

Degenerate Case:

$(N, g, \omega_1, \omega_2, \omega_3)$ hyper Kähler w $[\omega_i] \in \mathrm{H}^2(M, \mathbb{Z})$
M fibre product bundle of 3 Boothby-Wang S^1-bundles.
Then M can be equipped with degenerate 3-(α, δ)-Sasaki structure

Examples: $H^{n,\mathbb{H}}$, ...

Nearly Kähler

Definition

An almost Hermitian manifold (N, g_N, J) is nearly Kähler if for all $X \in TM$

$$(\nabla^{g_N}_X J)X = 0.$$

A nearly Kähler manifold admits a unique Hermitian connection ∇^{T_N}, called Gray connection, with parallel skew torsion

$$T^N(X, Y, Z) = g((\nabla^{g_N}_X J)JY, Z).$$

Theorem (Nagy '02)

A complete, strict nearly Kähler manifold that is not locally a product is

- *a homogeneous nearly Kähler manifold,*
- *a nearly Kähler manifold of dimension 6,*
- *the twistor space of a positive quaternionic Kähler manifold.*

Parallel to Nearly Kähler

If $\delta = 2\alpha \rightsquigarrow \nabla\xi_1 = 0$ (same for all ξ in the associated sphere)
$\rightsquigarrow \langle\xi_1\rangle \oplus \langle\xi_1\rangle^\perp$ is invariant under $\mathrm{Hol}_0(M)$

Look at torsion:

$$T = 2\alpha \sum_{i=1}^{3} \eta_i \wedge \Phi_i^{\mathcal{H}} - 4\alpha\eta_{123}$$

$$= \underbrace{2\alpha \sum_{i=2,3} \eta_i \wedge \Phi_i^{\mathcal{H}}}_{\Lambda^3\langle\xi_1\rangle^\perp} + \underbrace{2\alpha\eta_1 \wedge \Phi_1^{\mathcal{H}} - 4\alpha\eta_{123}}_{\langle\xi_1\rangle \wedge \Lambda^2\langle\xi_1\rangle^\perp}$$

$$\Lambda^3\langle\xi_1\rangle^\perp \quad \oplus \quad \langle\xi_1\rangle \wedge \Lambda^2\langle\xi_1\rangle^\perp$$

Theorem

Let $(M, g, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ be a parallel 3-(α,δ)-Sasaki manifold. Then there exists a loc. def. Riemannian submersion $\pi\colon (M,g) \to (N, g_N)$ with vertical space $\langle\xi_1\rangle$. Let $\tilde\varphi = \varphi_1|_{\mathcal{H}} - \varphi_1|_{\mathcal{V}}$ and set $J = \pi_* \circ \tilde\varphi \circ s_*$, for a section s of π. Then (N, g_N, J) defines a nearly Kähler space such that

$$\nabla^{T_N}_X Y = \pi_*(\nabla_{\overline{X}}\overline{Y}),$$

where ∇^{T_N} is the Bismut connection of (N, g_N, J).

NK to qK

In this case $TN = \mathcal{H} \oplus \pi_*\langle \xi_2, \xi_3 \rangle$ under $\mathrm{Hol}_0(\nabla^{T_N})$.
We want to use the canonical submersion once more:

Theorem

Let (N, g_N, J) be a nearly Kähler manifold with Gray connection ∇^{T_N}. Assume the tangent space splits $TN = \mathcal{V} \oplus \mathcal{H}$ into $\mathrm{Hol}_0(\nabla^{T_N})$ and J-invariant moduls and $T_N \in \Lambda^2\mathcal{H} \wedge \mathcal{V}$. If $2(\nabla^{g_N}_V J)^2 = -k^2\,\mathrm{id}$ for all unit length $V \in \mathcal{V}$ then there exists a local submersion $\pi\colon N \to \check{N}$ where $\check{N}$ admits a quaternionic Kähler structure locally defined by

$$I_1 = \pi_* \circ J \circ s_*,$$

$$I_2 = \sqrt{\frac{2}{k}}\,\pi_* \circ (JV \lrcorner T_N) \circ s_*, \qquad I_3 = \sqrt{\frac{2}{k}}\,\pi_* \circ (V \lrcorner T_N) \circ s_*,$$

where $s\colon \check{N} \to N$ is a section of π.

Proof

The canonical submersion theorem guarantees $\pi\colon N \to \check{N}$.

Check that I_1, I_2, I_3 locally define a quaternionic structure on $\check{N}$.

Now use $\nabla^{T_N} T = \nabla^{T_N} J = 0$ and $\nabla^{g_{\check{N}}}_X Y = \pi_*(\nabla^{T_N}_{\overline{X}} \overline{Y})$ to compute

$$\begin{aligned}(\nabla^{g_{\check{N}}}_X I_1)Y &= \nabla^{g_{\check{N}}}_X (I_1 Y) - I_1(\nabla^{g_{\check{N}}}_X Y) \\ &= \pi_*(\nabla^{T^N}_{\overline{X}} \overline{(\pi_*(J(s_* Y)))} - J(\nabla^{T^N}_{\overline{X}} \overline{Y}))\end{aligned}$$

But: $\overline{(\pi_*(J(s_* Y)))} = J(s_* Y)$ only along the image of s

$\rightsquigarrow$ Cannot take the covariant derivative in direction of $\overline{X}$

$\rightsquigarrow$ Set $\hat{X} = \overline{X} - s_* X \in \mathcal{V}$ and compute $\nabla_{\hat{X}} I_1$ individually

$$\begin{aligned}\ldots &= \pi_*((\nabla^{T^N}_{s_* X} J)(s_* Y) + (\hat{X} \lrcorner T^N)(J(s_* Y)) - J(\hat{X} \lrcorner T^N)(s_* Y)) \\ &= 2\pi_*((J\hat{X} \lrcorner T^N)(s_* Y))\end{aligned}$$

Analogous for I_2, I_3.

Thank You for your Attention!