# Multiplicity free $\mathrm{U}(2)$-manifolds

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joint work with Oliver Goertsches and Bart Van Steirteghem

Prospects in Geometry and Global Analysis
August 23, 2023

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1. *$m(G \cdot p) = m(p)$.*
2. *$\mathcal{P} := m(M)$, the 'invariant momentum polytope', is a convex polytope (Kirwan).*

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If $G$ is abelian, then $M$ is multiplicity free if and only if it is toric. There is a generalization of Delzant's theorem to the non-abelian setting, conjectured by Delzant and proven by Knop, building on important work of Luna, Camus and Losev in smooth affine spherical varieties.

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*Up to equivariant symplectomorphism, any multiplicity free manifold $M$ is uniquely determined by its principal isotropy type and its invariant momentum polytope $\mathcal{P}$.*

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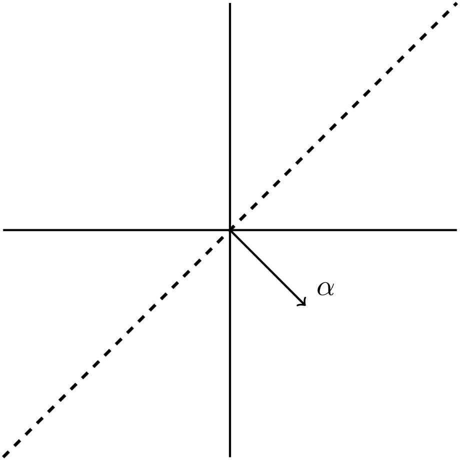
Now let $G = \mathrm{U}(2)$, $T \subset \mathrm{U}(2)$ the standard maximal torus, $\alpha = \varepsilon_1 - \varepsilon_2$, where $\varepsilon_1 = (1, 0)$ and $\varepsilon_2 = (0, 1)$ the simple root and $\mathfrak{t}_+$ the corresponding Weyl chamber.



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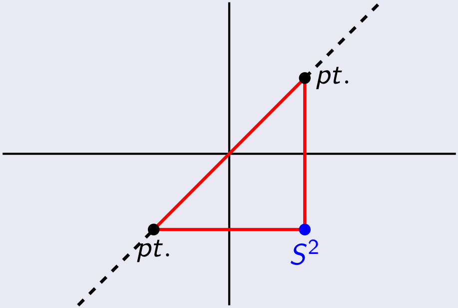
$$\{\varepsilon_1+\varepsilon_2, \varepsilon_1+k(\varepsilon_1+\varepsilon_2)\} \text{ and } \{-(\varepsilon_1+\varepsilon_2), -\varepsilon_2+k(\varepsilon_1+\varepsilon_2)\}$$
*for any* $k \in \mathbb{Z}$.



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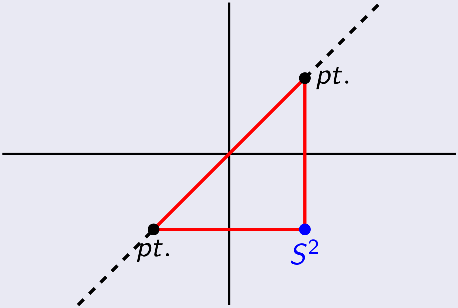


$\mathbb{P}^3$ *with standard Kähler form,* $\mathrm{U}(2)$ *acting linearly.*

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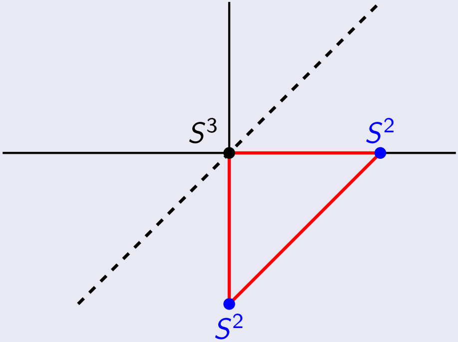

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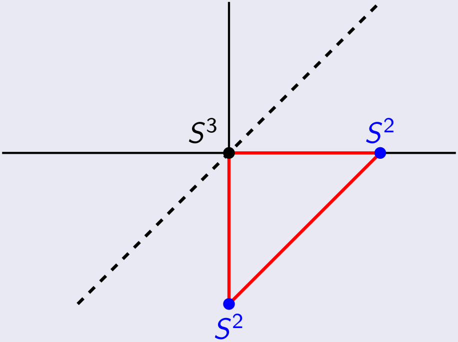
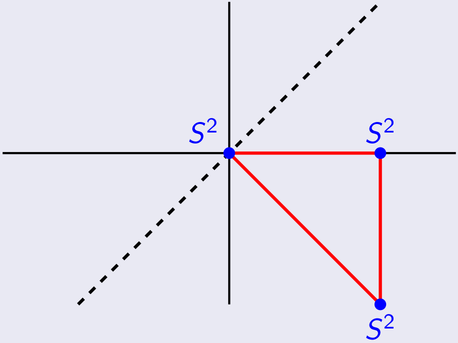



$$\mathrm{SO}(5)/[\mathrm{SO}(2)\times\mathrm{SO}(3)].$$

## Theorem

*Up to reflection at* $\mathbb{R}\alpha$, $\{\alpha, j\alpha + \varepsilon_1\}$ *for some* $j \in \mathbb{N}$.



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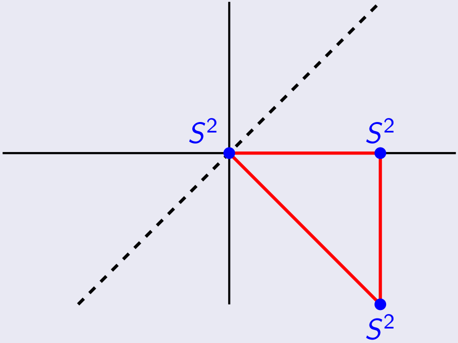
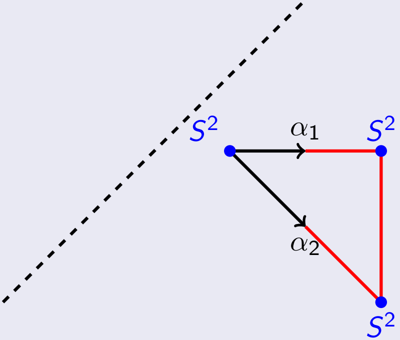
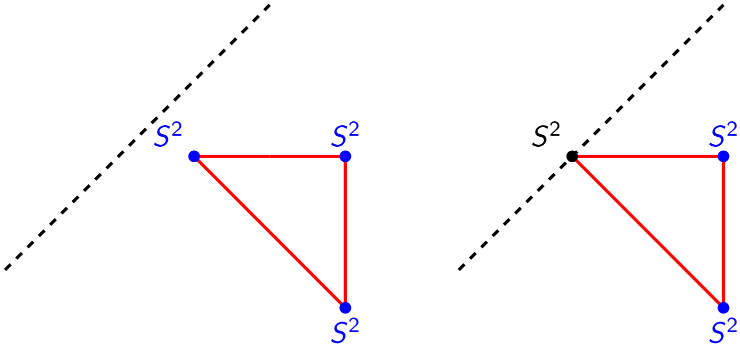



$\mathrm{U}(2) \times_T \mathbb{P}(\mathbb{C}^2 \oplus \mathbb{C}_{-j\alpha})$ *with certain projective Kähler form.*

## Theorem

*If $\mathcal{P}$ does not intersect the Weyl wall, then $M = \mathrm{U}(2) \times_T \mathbb{P}(\mathbb{C} \oplus \mathbb{C}_{\alpha_1} \oplus \mathbb{C}_{\alpha_2})$ with a projective Kähler form.*

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- $\mathbb{P}^1 \times \mathbb{P}^2$.
- $\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1(-1)} \oplus \mathcal{O}_{\mathbb{P}^1(0)} \oplus \mathcal{O}_{\mathbb{P}^1(0)}\right)$ *(a certain projectivized* $\mathbb{P}^2$*-bundle over* $\mathbb{P}^1$*).*

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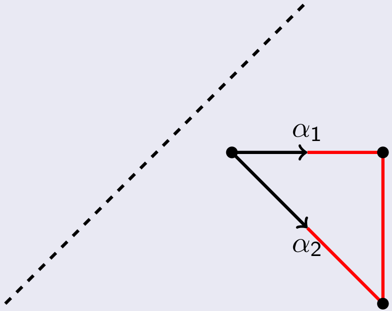


*$M \cong \mathbb{P}^1 \times \mathbb{P}^2$ if and only if $a_1 + a_2 - b_1 - b_2$ is a multiple of 3, where $\alpha_1 = a_1\varepsilon_1 + b_1\varepsilon_2$, $\alpha_2 = a_2\varepsilon_1 + b_2\varepsilon_2$.*

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**Theorem (G., V.S., W.)**

*If $\mathcal{P}$ does not intersect the Weyl wall at exactly one point, then $M$ admits a compatible, invariant complex structure. In fact, the $\mathrm{U}(2)$-action extends to a Hamiltonian $\mathrm{U}(2) \times S^1$-action.*

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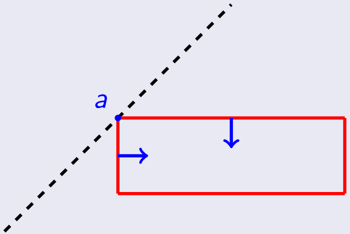


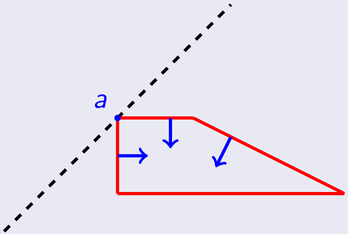

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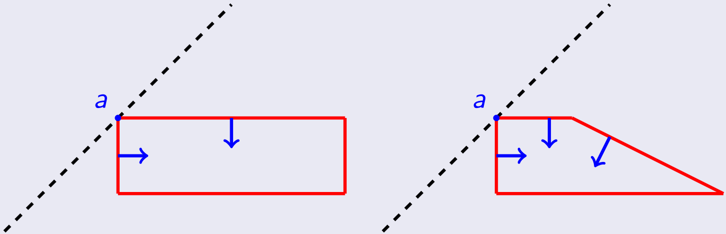


## Proof.

Using results of Martens and Thaddeus about partial compatibility results of local symplectic cutting and Kähler structures. □

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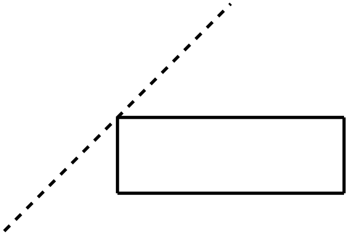

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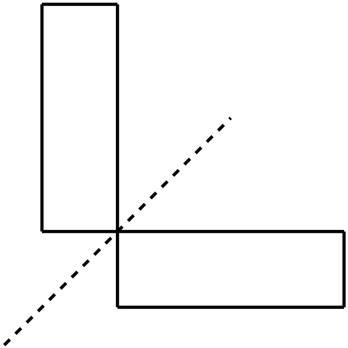

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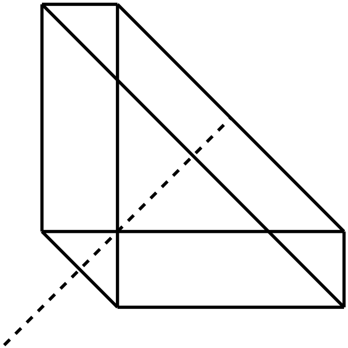

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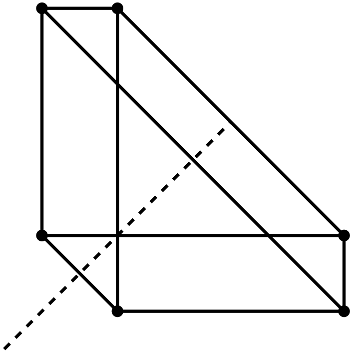

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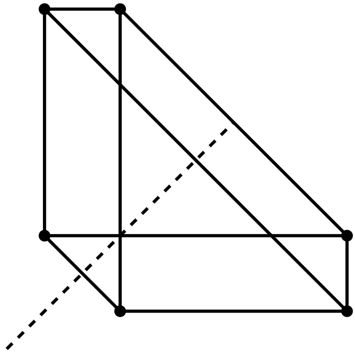
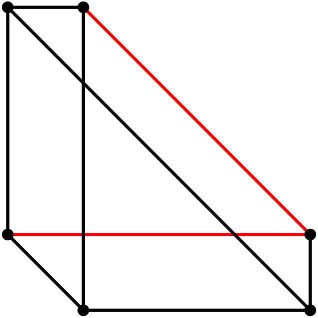


Circles are images of $T$-fixpoints, lines are images of $T$-invariant 2-spheres.

For any two lines $l_1$ and $l_2$ emerging from a circle, there is a convex polytope extending $l_1$ and $l_2$ and whose edges are lines ('$M$ satisfies the extension criterion').



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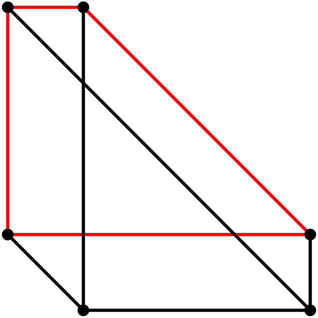

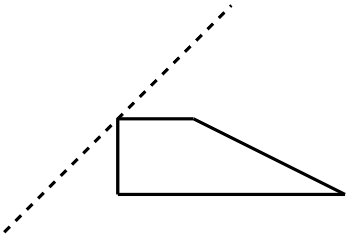

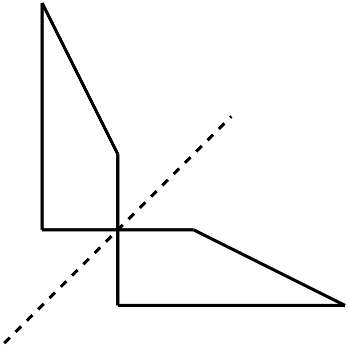

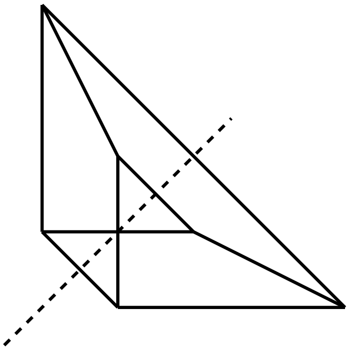

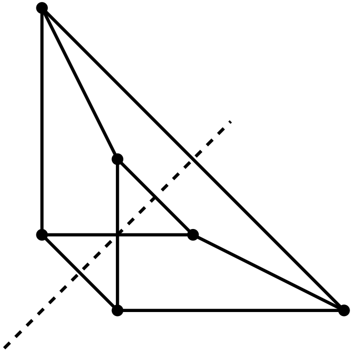

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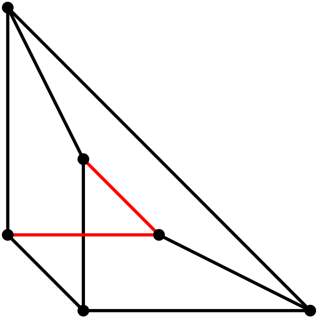

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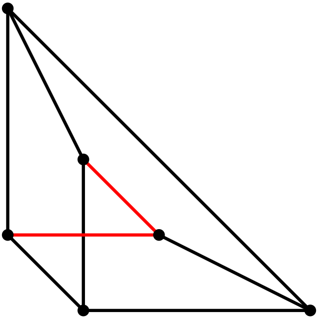

By the 'extension criterion' of Tolman, this can not happen if the symplectic form is a $T$-invariant Kähler form.

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