Minimal projective orbits of semi-simple Lie groups

Henrik Winther

UiT – The Arctic University of Norway

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Our problem

We are interested in actions of Lie groups on real projective space.

- Arising from (finite dimensional) irreducible real representations.
- What is the minimal dimension of a projective orbit?
- Uniqueness? Classification?
- What about topology? Closed projective orbits?
- Computational aspects?

Constructing geometries with large symmetries

Let M be (say) a Cartan geometry modelled on homogeneous space F/G.

- Suppose we want M to have large automorphism group $\mathrm{Aut}(M)$
- Biggest possible by dimension, but non-flat. (Submaximal problem)

The differential invariants are sections of some vector bundle $\Gamma(E \to M)$ associated to G-module $\mathbb{E}$

- At some point $x \in M$, the image H of the stabilizer subgroup $\mathrm{Stab}(x)$ under the isotropy representation acts on $E_x \simeq \mathbb{E}$

Finding large H is the same as finding small orbits.

What is the alternative?

One typical way to find minimal orbits is the following:

- Consider a maximal subalgebra $\mathfrak{h} \subset \mathfrak{g}$
- Compute the space of invariants $\mathbb{V}^{\mathfrak{h}}$ for the restricted representation $\rho|_{\mathfrak{h}}$
- If this is non-zero, we are done: The minimal orbit is G/H (more or less) and we have a representative.
- else, consider maximal subalgebras of $\mathfrak{h}$ and start over from the top.

Remark

This procedure will find vectors with maximal dimensional stabilizers, yielding minimal orbits. But walking the tree of subalgebras of $\mathfrak{g}$ is enormously labour intensive and ad-hoc.

The complex case

The solution to the problem in the case of complex representations is well-known

Theorem (A. Borel, 1956)

Let G be a complex semi-simple Lie group and $\rho : G \to GL(\mathbb{V})$ *a complex irreducible representation. Then the minimal projective orbit is the orbit of a highest weight vector, and this is the unique closed orbit in* $P_{\mathbb{C}}\mathbb{V}$

In light of this, our goal is to make the best possible generalization of the Borel theorem.

- We want to consider real groups, real representations and real projectivizations
- No access to highest weight vectors
- Failure of uniqueness

Let $\mathfrak{g}$ be a real, semi-simple Lie algebra.

Definition

A $\mathbb{Z}$–grading on $\mathfrak{g}$ is a vector space decomposition

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$$

such that

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}.$$

We say it's nontrivial if for some $i > 0$ we have $\mathfrak{g}_i \neq \{0\}$. There exists a unique element $Z \in \mathfrak{g}$ satisfying

$$[Z, g_i] = i g_i, \quad \mathfrak{ad}_Z|_{\mathfrak{g}_i} = i \,\mathrm{Id}\,.$$

This Z is called the grading element of the $\mathbb{Z}$–grading.

Some properties

- If $\mathfrak{g}$ admits a nontrivial $\mathbb{Z}$–grading, then $\mathfrak{g}$ is non-compact.
- Any non-compact $\mathfrak{g}$ admits a non-trivial $\mathbb{Z}$–grading.
- Any $\mathbb{Z}$–grading is symmetric around zero.

Definition

A subalgebra $\mathfrak{p}$ is called parabolic if it can be written as the non-negatively graded subalgebra of a nontrivial $\mathbb{Z}$–grading:

$$\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}_i$$

- The subalgebra $\mathfrak{p}_+ = \bigoplus_{i>0} \mathfrak{g}_i$ is the nilradical of $\mathfrak{p}$.
- The subalgebra $\mathfrak{g}_0 = \mathfrak{p}_0$ is the reductive Levi factor of $\mathfrak{p}$.
- $\mathfrak{g}_0$ contains a (maximally noncompact) Cartan subalgebra of $\mathfrak{g}$.

Minimal parabolic

Definition

A minimal parabolic subalgebra $\mathfrak{b}$ is the normalizer of a non-abelian maximal nilpotent subalgebra of $\mathfrak{g}$.

- The minimal parabolic is a parabolic, and unique up to conjugacy.
- If $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{g}$ with $\mathfrak{p}$ parabolic, then $\mathfrak{q}$ is parabolic.
- Any parabolic $\mathfrak{p}$ contains a minimal parabolic $\mathfrak{b}$, $\mathfrak{b} \subset \mathfrak{p}$.
- We could define parabolic as any subalgebra $\mathfrak{p}$ containing $\mathfrak{b}$.

Remark

In the complex (or split-real) setting, $\mathfrak{b}$ is spanned by a Cartan subalgebra and positive root vectors. For example, for $\mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{b}$ is the subalgebra of upper triangular matrices.

Modules and $\mathbb{Z}$–gradings

Let $\mathfrak{g}$ be real, semi-simple, with non-trivial $\mathbb{Z}$–grading. Consider the simple $\mathfrak{g}$–module $\mathbb{V}$ with rep. ρ.

- $\rho(Z)$ is diagonalizable.
- We have $\mathbb{V} = \bigoplus_\theta \mathbb{V}_\theta$ as a vector space, the eigenspace decomposition with respect to $\rho(Z)$.
- For later use we denote the greatest eigenvalue of $\rho(Z)$ by θ_{max}.

Proposition

Let $g_i \in \mathfrak{g}_i$. Then $\rho(g_i) : \mathbb{V}_\theta \mapsto \mathbb{V}_{\theta+i}$.

Proof.

Commutator relation between Z and g_i gives

$$\rho([Z, g_i]) = \rho(Z)\rho(g_i) - \rho(g_i)\rho(Z).$$

Apply to $v \in \mathbb{V}_\theta$, to get $\rho(Z)\big(\rho(g_i)(v)\big) = (\theta + i)\rho(g_i)(v)$. □

Proposition ($\mathbb{V}_{\theta_{\max}}$)

The subspace $\mathbb{V}_{\theta_{\max}}$ is a simple G_0-module and $\mathbb{V}_{\theta_{\max}} = \mathbb{V}^{\mathfrak{p}_+}$.

Proof.

$\mathbb{V}$ is simple, so given two nonzero vectors v, w, we have

$$v = \Big(\sum_{i=1}^{k} \prod_{j=1}^{l_i} \rho(g_{\alpha_{ij}}^{(i,j)})\Big) w$$

The products can be assumed ordered non-strictly increasing in gradation, because

$$g_i g_j = g_j g_i + [g_i, g_j]$$

Suppose $w \in \mathbb{V}^{\mathfrak{p}_+}$. Then all positively graded factors vanish. Next suppose $v \in \mathbb{V}_{\theta_{\max}}$. Then by Prop above, all factors are non-negatively graded. So $\mathfrak{g}_0$ and hence G_0 acts irreducibly on $\mathbb{V}_{\theta_{\max}} = \mathbb{V}^{\mathfrak{p}_+}$. □

Lemma (closed minimal orbits)

Let G be a real connected Lie group and $\mathbb{V}$ simple, such that center acts “nicely”. Then there exists a closed minimal projective orbit and all minimal projective orbits are closed.

Remark

This lemma seems obvious because it has a simple but false proof, which uses only assumption of minimality. Sketch: Let $O \in P\mathbb{V}$ be minimal. Its closure $\bar{O}$ can be decomposed into orbits and these are lower or equal dimension to O. Hence $O = \bar{O}$. This fails, counterexamples can be constructed. Introducing more assumptions (irreducibility) is essential, because it implies algebraicity.

Proof.

- Replace G by $\mathrm{Im}(\rho)$, now action is faithful.
- Then $\mathfrak{g} = \mathfrak{g}_{ss} \oplus Z(\mathfrak{g})$.
- $\mathbb{V}$ is a tensor product of algebraic representations of ideals.
- Take $G \cdot [v]$ to be a minimal projective orbit.
- $G^{\mathbb{C}} \cdot [v] \subset P_{\mathbb{C}}\mathbb{V}^{\mathbb{C}}$ has same complex dim as previous real dim.
- If there was another orbit in closure, this remains over $\mathbb{C}$.
- All actions are algebraic so closure consists of lower dim orbits.
- Now the complexification consists of one closed orbit, but it can have several real slices.
- Real algebraicity gives that they are finitely many and separated by inequalities, thus they cannot be in each other's real closures.

□

Theorem

Let G and $\mathbb{V}$ be as in settings. Then any closed projective orbit contains an element $[w]$ for $w \in \mathbb{V}_{\theta_{max}}$.

Proof.

Any linear orbit contains an element $v = \sum_\theta v_\theta$ with nonzero component $v_{\theta_{\text{max}}}$ (or module is nonsimple). Consider the sequence $w_m = \exp(tZ)|_{t=m} v, m \in \mathbb{N}$. We have $w_m = \sum_\theta e^{m\theta} v_\theta$. Thus $[w_m]$ converges to $[v_{\theta_{\text{max}}}]$ in $P\mathbb{V}$. Orbit is closed, so let $w = v_{\theta_{\text{max}}}$. □

Corollary

Let G be split-real. Then there is a unique closed projective orbit, the orbit of a highest weight vector.

Proof.

We have a Borel subalgebra with noncompact Cartan as G_0. $\mathbb{V}_{\theta_{\max}}$ is one-dimensional by Prop, so $P\mathbb{V}_{\theta_{\max}}$ is a point. A closed projective orbit exists by Lemma, and by Theorem 1 it intersects $P\mathbb{V}_{\theta_{\max}}$. □

Proposition

Let G be non-compact and semi-simple and $\mathbb{V}$ a simple, finite dimensional and non-trivial G-module. There exists a non-trivial $\mathbb{Z}$-grading on $\mathfrak{g}$ such that for any closed projective orbit, there exists a representative $[w]$ with $\mathrm{Ann}(w) \subset \mathfrak{p}$.

Proof.

First, consider the minimal parabolic $\mathfrak{b}$. By the first theorem, any minimal orbit admits a representative in $\mathbb{V}_{\theta_{\max}} = \mathbb{V}^{\mathfrak{b}_+}$.

- Claim: if $x_- \in \mathfrak{g}_-$ annihilates some $v \in \mathbb{V}_{\theta_{\max}}$, then it annihilates all of $\mathbb{V}_{\theta_{\max}}$

Proof of claim: Let σ be a Cartan involution preserving the Cartan subalgebra in $\mathfrak{b}_0$. Then $x_-, x_+ = \sigma x_-$, and $x_0 = [x_-, x_+]$ forms an $\mathfrak{sl}_2$ triple with $x_0 \in \mathfrak{g}_0$ and $x_+ \in \mathfrak{p}_+$, and $\mathbb{V}_{\theta_{\max}}$ consists of highest weight vectors of weight 0 (x_0 acts on $\mathbb{V}_{\theta_{\max}}$ as a scalar).

- Now, take any $v \in \mathbb{V}_{\theta_{\max}}$ and let $\mathfrak{p} = \langle \mathrm{Ann}(v), \mathfrak{b} \rangle$.

This is parabolic because $\mathfrak{b} \subset \mathfrak{p}$, and each minimal orbit contains a representative in $w \in \mathbb{V}^{\mathfrak{b}_+} \subset \mathbb{V}^{\mathfrak{p}_+}$, which by the claim satisfies $\mathrm{Ann}(w) \subset \mathfrak{p}$. □

Theorem

Let G be a real, semisimple, non-compact Lie group and $\mathbb{V}$ simple. Then there exists a compact subgroup K such that the minimal projective G orbits are in bijective correspondence with the minimal projective K-orbits of a simple K-submodule $\mathbb{W} \subset \mathbb{V}$.

Proof.

We define the following operation on the set of pairs $(G, \mathbb{V})$.

- Apply previous Prop. to the pair, get $\mathbb{Z}$–grading.
- $(G, \mathbb{V}) \mapsto (G', \mathbb{V}')$, where:
- $\mathbb{V}' = \mathbb{V}_{\theta_{\max}} \subset \mathbb{V}$
- G' is generated by simple ideals of $\mathfrak{g}_0$ acting effectively on $\mathbb{V}'$.

Next consider the sequence of pairs given by

$$G^{m+1} = (G^m)', \quad \mathbb{V}^{m+1} = (\mathbb{V}^m)', \quad G^0 = G, \quad \mathbb{V}^0 = \mathbb{V}$$

This sequence ends at some step k, as soon as G^k is a compact group. Let $K = G^k$ and $\mathbb{W} = \mathbb{V}^k$. A combination of the same Prop and the last Theorem gives that the minimal orbits are in bijective correspondence at each step. □

Satake diagrams

Let $\mathfrak{h} \subset \mathfrak{g}$ be a maximally noncompact Cartan subalgebra. Then $\mathfrak{h} \to \mathfrak{h}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$. Take root space decomposition of $\mathfrak{g}^{\mathbb{C}} = \bigoplus_{\alpha \in \Phi} \mathfrak{g}^{\mathbb{C}}_\alpha$ via $\mathfrak{h}^{\mathbb{C}}$.

- α are complex valued covectors on $\mathfrak{h}$.
- Call a root compact if it is pure imaginary. Call them Φ^C
- There is an involution ζ on roots: $\zeta(\alpha) = \beta$ if $\beta = \bar{\alpha} \mod \Phi^C$.

Recipe

Start with D the Dynkin diagram of Φ.

1. Colour compact roots black.
2. Draw an arrow (not an edge) between α and β if $\zeta(\alpha) = \beta$

$\mathfrak{sl}(4,\mathbb{R})$: $\circ - \circ - \circ$ $\mathfrak{sl}(2,\mathbb{H})$: $\bullet - \circ - \bullet$

$\mathfrak{su}(4)$: $\bullet - \bullet - \bullet$ $\mathfrak{su}(3,1)$: $\circ \overset{\longleftrightarrow}{- \bullet -} \circ$ $\mathfrak{su}(2,2)$: $\circ \overset{\longleftrightarrow}{- \circ -} \circ$

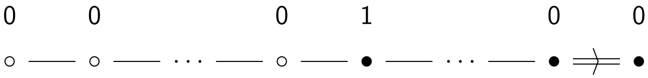
There is a combinatorial algorithm to compute $(K, \mathbb{W})$:

1. Let S be the Satake diagram of $\mathfrak{g}$.
2. Decorate S with highest weight coefficients over nodes.
3. Remove a node from S if either
   - It is white and has nonzero coefficient.
   - It is white and adjacent to fully black subdiagram with at least one nonzero coefficient.
4. If T is a connected component of S with all coefficients of T zero, then remove T from S.

Now S is the Satake diagram of K, decorated with the highest weight coefficients of $\mathbb{W}$.

Example

Consider $G = SO(p,q)$, $p < q$, and the module $\mathbb{V} = \Lambda^{p+1}(\mathbb{R}^{p+q})$. This admits a unique minimal projective orbit, because $K = SO(q-p)$, and $\mathbb{W}$ is the standard representation, which is sphere-transitive. A representative can be constructed by taking $\mathbb{R}^{p+q} = \mathbb{R}^{p,p} \oplus \mathbb{R}^{q-p}$, then a minimal orbit is generated by a null plane Π^p and $v \in \mathbb{R}^{q-p}$, $[\Lambda^p \Pi^p \wedge v]$.



Remark

More examples can be constructed by taking $\mathbb{W}$ to be a sphere-transitive representation of a compact group K, and extending it to a Satake diagram. These can be found from the original Berger list: $SO(n)$, $SU(n)$, $Sp(n)Sp(1)$, $Sp(n)$, $Spin(7)$ and $Spin(9)$. G_2 does not extend to a Satake diagram.

$$\overset{0}{\bullet} \text{——} \overset{0}{\bullet} \text{——} \overset{0}{\bullet} = \rangle = \overset{1}{\bullet} \quad \overset{\text{extend}}{\dashrightarrow} \quad \overset{\lambda_1}{\circ} \text{——} \overset{\lambda_2}{\circ} \text{——} \overset{0}{\bullet} \text{——} \overset{0}{\bullet} \text{——} \overset{0}{\bullet} = \rangle = \overset{1}{\bullet}$$

Here's a possible extension of the Spin representation of $Spin(9)$ to a representation of $Spin(11,2)$.