Domination of manifolds and formality

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Prospects in Geometry and Global Analysis
Rauischholzhausen 2023

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Example

- every manifold dominates S^n
- for $n = 2$ a surface dominates every surface of smaller genus
- finite coverings

Formality I

M manifold $\rightsquigarrow$ cdga of differential forms $(\Omega(M), d)$ $\rightsquigarrow$ $H^*(M)$

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The cdgas with a given cohomology partition into quasi isomorphism types

"Formality = the simplest quasi iso type with a given cohomology"

Formality II

Consequences of formality:

- the quasi isomorphism type is encoded in $H^*(M)$
- M 1-connected: the real homotopy type is encoded in $H^*(M)$
- in particular: $\pi_*(M)$ can up to torsion be recovered from $H^*(M)$

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- S^n, $\mathbb{C}P^n$
- G/U where G cpt Lie, $U \subset G$ max rank
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Not formal:

- All nilmanifolds except T^n
- a generic linear quotient $(S^3 \times S^3 \times S^3)/T^2$

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- However: Taylors theorem does not generalize to quadruple and higher Massey Products [Milivojevich, Stelzig, Z., 2022]

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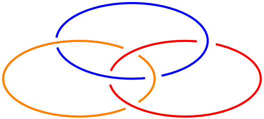
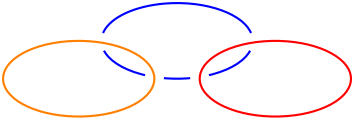


vs.



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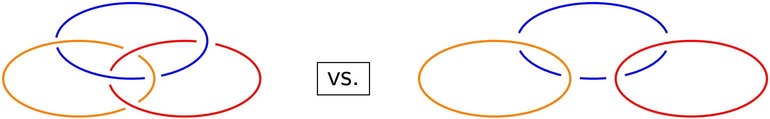

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- choose $x, y \in \Omega(M)$ with $dx = ab$, $dy = bc$

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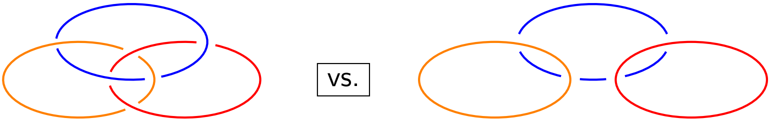

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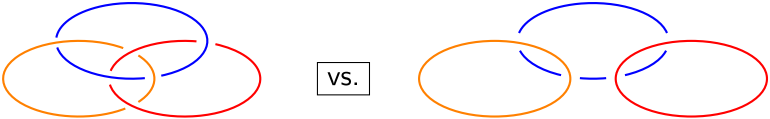

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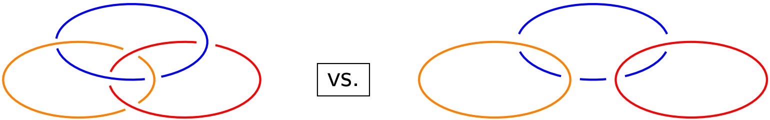

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$$M \text{ formal} \implies \text{all Massey products vanish}$$

$$M \text{ formal} \iff \text{all (operadic) Massey products vanish uniformly}$$

A proof of Taylors theorem

Theorem (Taylor)

Given a non-zero degree $f\colon M \to N$ *and* $\alpha, \beta, \gamma \in H^*(N)$ *s.t.* $0 \neq \langle \alpha, \beta, \gamma \rangle \in \frac{H^*(N)}{(\alpha,\gamma)}$ *is defined. Then* $\langle f^*(\alpha), f^*(\beta), f^*(\gamma) \rangle \neq 0$

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Proof.

One has

$$\overline{f^*}\colon \frac{H^*(N)}{(\alpha, \gamma)} \to \frac{H^*(M)}{(f^*(\alpha), f^*(\gamma))}$$

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Hence we obtain an induced left inverse $\frac{1}{\deg(f)}\overline{f_!}\colon \frac{H^*(M)}{(f^*(\alpha),f^*(\gamma))} \to \frac{H^*(N)}{(\alpha,\gamma)}$ □

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In fact ϕ_M is a quasi isomorphism by Poincaré duality since $H(D\Omega(M)) \cong DH^*(M)$

The key diagram

If $f\colon M \to N$ has degree $d \neq 0$ we obtain commutative diagram

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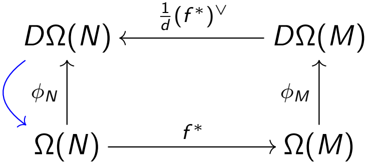

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Theorem (Milivojevic, Stelzig, Z.)

Let $A \to B$ be a morphism of A_∞-algebras that admits an A_∞ A-bimodule homotopy retract $B \to A$. If B is formal then so is A.

What is an A_∞-algebra/module?

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- the classical Massey products are very much related to an A_∞-structure on $H^*(M)$

Thank you!