1)

Marburg, Sophus Lie seminar, 7.1.11

Universal principles for Kazdan-Warner type identities.

— with Rod Gover [arXiv].

Background:

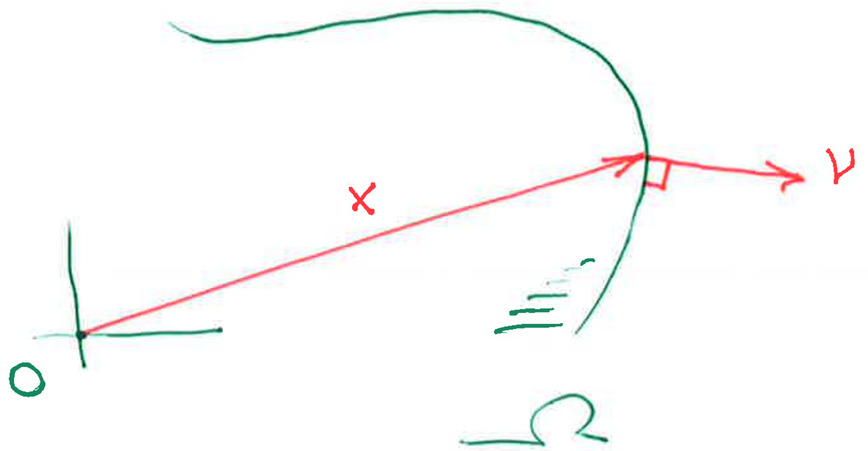
- Lie: conserved quantities in geometry & PDE
- Noether: symmetries $\rightsquigarrow$ conserved quant.
- Pohozaev [1965]: (non-) existence for

$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } \Omega \subseteq \mathbb{R}^n \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

using the identity ("Pohozaev")

$$\lambda n \int_\Omega F(u) + \frac{2-n}{2} \lambda \int_\Omega f(u) u = \frac{1}{2} \int_{\partial\Omega} x \cdot \nu \left(\frac{\partial u}{\partial \nu} \right)^2$$

2) where $f(0) = 0$, $F(u) = \int_0^u f(t)\,dt$



$$X = x_i \frac{\partial}{\partial x_i} \quad \text{Euler field}$$

- Kazdan - Warner [1975] : conformal conservation law

$$\boxed{\int_M X\, S_c \, dv_g = 0}$$

$$\begin{cases} (M, g) & \text{closed Riemannian manifold} \\ X & \text{conformal vector field} \\ S_c & \text{scalar curvature} \end{cases}$$

, i.e.

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$Ric(X,Z) = tr\,(Y \to R(X,Y)Z)$$

$$S_c = tr\, Ric$$

3)

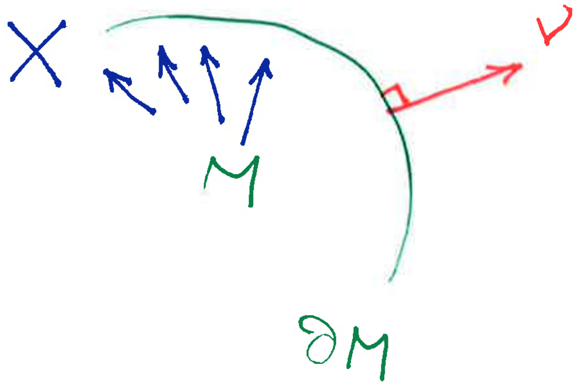
$$\mathcal{L}_X g = \frac{2}{n}(\operatorname{div} X)\, g \,, \quad n = \dim M$$

- Schoen [1988]:

$$\boxed{\int_M X\, Sc \, dv_g = \frac{2n}{n-2} \int_{\partial M} \left(Ric - \frac{1}{n} Sc\, g\right)(X, \nu)\, d\sigma_g}$$

for X conformal

and $n = \dim M \geq 3$



Aim: (1) Understand the relation

"Pohozaev" $\longleftrightarrow$ "Schoen"

(2) Find similar identities for other Riemannian invariants.

Note: (a) Many applications

(b) Previously found [with Branson '91]

$$\boxed{\int_M X\, U_i \, dv_g = 0 \quad (i = 0, 1, 2, \dots)}$$

4)

with X conformal on closed (M, g) and the heat asymptotics

$$\mathrm{tr}\, e^{-tA} \sim \sum_{i=0}^{\infty} t^{(2i-n)/2} \int_M U_i \, dv_g$$

$$A = -\Delta + \frac{n-2}{4(n-1)} Sc \quad \text{or} \quad (\text{Dirac})^2 \quad \text{or} \quad \ldots$$

$$U_i = \mathrm{pol}\,(R, \nabla R, \nabla^2 R, \ldots)$$

Fact: May derive "Pohozaev" from "Schoen" by $M = \Omega$, $X = x_i \frac{\partial}{\partial x_i}$,

$g = u^{4/(n-2)} g_0$, $0 < u$ smooth in Ω

$$Sc = -\frac{4(n-1)}{n-2} u^{-p} \Delta u, \quad p = \frac{n+2}{n-2}$$

$$X\, Sc\, dv_g = -\frac{4(n-1)}{n-2} \Big(x_i (-p) \frac{\partial u}{\partial x_i} \Delta u + u\, x_i \frac{\partial}{\partial x_i} \Delta u \Big)\, dv_{g_0}$$

$$Ric_{ij} = (2-n)\Big[\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\Big] + \Big[\frac{\partial^2 f}{\partial x_k^2} - (n-2)\frac{\partial f}{\partial x_k}\frac{\partial f}{\partial x_k}\Big]\delta_{ij}$$

5)

with $u^{p-1} = e^{2f}$

- integration by parts (Green) gives "Pohozaev" via

$$\int_\Omega |\nabla u|^2 = \lambda \int_\Omega u f(u) \quad \text{etc.}$$

Idea: build on Noether's principle with Diff(M) as symmetry.

def. Riemannian functional on metrics:

$$S : \mathcal{M} \to \mathbb{R}$$

$$S(\varphi^* g) = S(g) \quad (\varphi \in \mathrm{Diff}(M))$$

natural scalar = pol. jets of g, g^{-1} satisfying $\varphi^* L(g) = L(\varphi^* g)$

giving

$$\boxed{S(g) = \int_M L(g)\, dv_g}$$

def. S has gradient $B(g)$ at g provided $B(g) \in C^\infty(S^2 M)$ and

6)

$$\boxed{S'(g)(h) = \int_M (h, B(g))\, dv_g}$$

for all $h \in C^\infty(S^2 M)$. In particular for $h = \mathcal{L}_X g$

$$0 = \int_M (\mathcal{L}_X g, B(g))\, dv_g$$

$$= \int_M X^b \nabla^a B_{ab}\, dv_g \quad \text{so that}$$

$$\boxed{\nabla^a B_{ab} = 0}$$

ex. This applies to

$$S(g) = \int_M L(g)\, dv_g$$

with $B(g)$ a natural (tensor-valued) invariant.

Now: Conformal variations

$$g \to e^{2s\omega} g \quad (s \in \mathbb{R}, \omega \in C^\infty(M))$$

defines the conformal class $[g]$

7)

def. A natural scalar V is called conformally variational if for some S on $[g]$

$$\frac{d}{ds}\Big|_{s=0} S(e^{2s\omega}g) = S^{\cdot}(g)(\omega)$$

$$= 2\int_M \omega V \, dv_g \quad (\omega \in C^\infty(M))$$

def. V naturally conformally variational if for some Riemannian functional with gradient B on $[g]$

$$S^{\cdot}(g)(\omega) = 2\int_M (\omega g, B)\, dv_g$$

$$\text{i.e. } V = g^{ab} B_{ab}$$

Theorem. In this case, with

$$\mathcal{L}_X g = 2\omega g = \frac{2}{n}(\operatorname{div} X) g$$

$$0 = \int_M (\mathcal{L}_X g, B)\, dv_g = \frac{2}{n}\int_M (\operatorname{div} X) V \, dv_g$$

$$\boxed{\int_M X V \, dv_g = 0}$$

8)

Remark. There are many examples; the simplest case is $V = S_c$, scalar curvature, recovering Kazdan-Warner. Even if S does not have a gradient B we still get

$$0 = S'(g)(\mathcal{L}_X g) = S^{\cdot}(g)(\tfrac{1}{n}\operatorname{div} X)$$

$$= 2\int_M \omega V \, dv_g \qquad \text{so again}$$

$$\boxed{\int_M X V \, dv_g = 0}$$

ex. M a closed 2-manifold

$$S(g) = \det \Delta_g / A(g)$$

with Polyakov formula

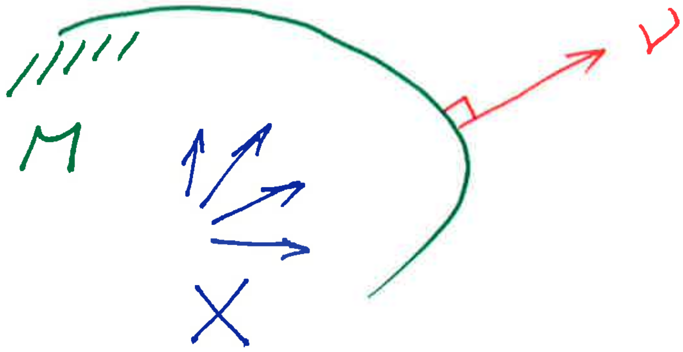
$$S^{\cdot}(g)(\omega) = c \cdot \int_M \omega Q \, dv_g$$

Q = Gauss curvature, $c \neq 0$

$$\boxed{\int_M X Q \, dv_g = 0 \quad (X \text{ conformal})}$$

Next: $\partial M \neq \emptyset$

$$\int_M \nabla^a (B_{ab} X^b)\, dv_g = \int_{\partial M} B_{ab} X^a \nu^b \, d\sigma_g$$



suppose $\nabla^a B_{ab} = 0$, X conformal

$$\text{GET}\begin{cases}\boxed{\int_M XV\, dv_g = -n \int_{\partial M} \mathring{B}_{ab} X^a \nu^b \, d\sigma_g} \\ V = g^{ab} B_{ab}, \quad \mathring{B}_{ab} = B_{ab} - \frac{1}{n} g_{ab} V \end{cases}$$

Theorem. This holds in particular when V is naturally conformally variational.

ex. B_{ab} the Einstein tensor corr. to the Einstein-Hilbert action

$$\boxed{S(g) = \int_M Sc \, dv_g}$$

10)

$$\left|\begin{array}{l} B_{ab} = P_{ab} - g_{ab} J \\ Ric_{ab} = (n-2) P_{ab} + J g_{ab} \\ Sc = 2(n-1) J \\ J = g^{ab} P_{ab} \\ V = (1-n) J \end{array}\right.$$

so in this case our Theorem says

$$\boxed{\int_M X \, Sc \, dv_g = \frac{2n}{n-2} \int_{\partial M} \mathring{Ric}_{ab} X^a \nu^b \, d\sigma_g}$$

i.e. exactly "Schoen".

Examples. (1) Branson's Q-curvatures

$$\int_M X \, Q_m \, dv_g = 0 \quad (\text{even } m \neq n+2, n+4, \ldots)$$

$m \neq n$

$$\left|\begin{array}{l} \overline{Q}_m = u^{(n+m)/(m-n)} (\delta S_m d + Q_m) u \\ u = e^{\omega(n-m)/2}, \quad \overline{g} = e^{2\omega} g \\ \omega \in C^\infty(M) \end{array}\right.$$

(2) Higher Einstein tensors

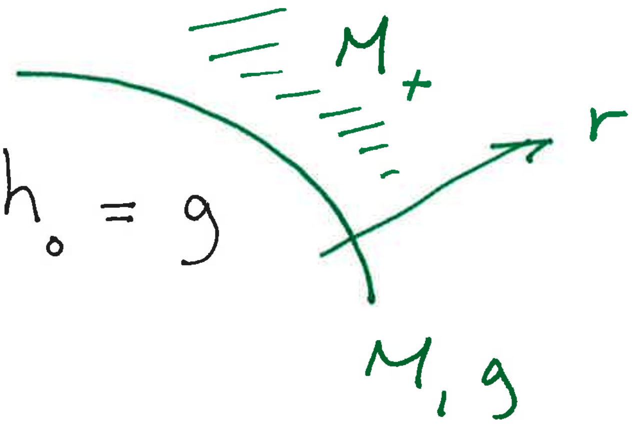
$$E^{(m)} = \text{the gradient of } (n-m)^{-1} \int_M Q_m \, dv_g \quad (m \neq n)$$

e.g. $m = 2 < n$: usual Einstein

$$\boxed{\int_M X Q_m \, dv_g = -n \int_{\partial M} \mathring{E}^{(m)}_{ab} X^a \nu^b \, d\sigma_g}$$

Note : (M, g) Einstein $\Rightarrow$ $\mathring{E}^{(m)}_{ab} = 0$

(3) Renormalized volume coefficients



$$g_+ = \frac{dr^2 + h_r}{r^2}, \quad h_0 = g$$

$$Ric^{g_+} = -n g_+$$

$$\left(\frac{\det g_\rho}{\det g_0} \right) = 1 + \sum_{k=1}^{\infty} v_k \rho^k$$

$$\rho = -\tfrac{1}{2} r^2, \quad g_\rho = h_r$$

ref. Fefferman-Graham [1985-2010]

<u>Theorem.</u> Let $k \in \mathbb{Z}$ with $2k < n$ if n is even and $B^{(k)}_{ab}(g)$ the gradient of $S(g) = (n-2k)^{-1} \int_M v_k(g)\, dv_g$; then for X conformal

$$\boxed{\int_M X v_k(g)\, dv_g = -n \int_{\partial M} \overset{\circ}{B}{}^{(k)}_{ab} X^a \nu^b \, d\sigma_g}$$

and for $n = 2k$ on (M^n, g) closed

$$\boxed{\int_M X v_k(g)\, dv_g = 0}$$

(4) Gauss - Bonnet invariants

$S^{(2k)}$ = contraction of R^k

= Pfaffian in dimension $2k$

$G^{(2k)}_{ab}$ = the gradient of $2 \int_M S^{(2k)}$

"Einstein - Lovelock" , $n \neq 2k$

$$\boxed{\int_M X S^{(2k)} dv_g = -\frac{n}{2(n-2k)} \int_{\partial M} \overset{\circ}{G}{}^{(2k)}_{ab} X^a \nu^b \, d\sigma_g}$$

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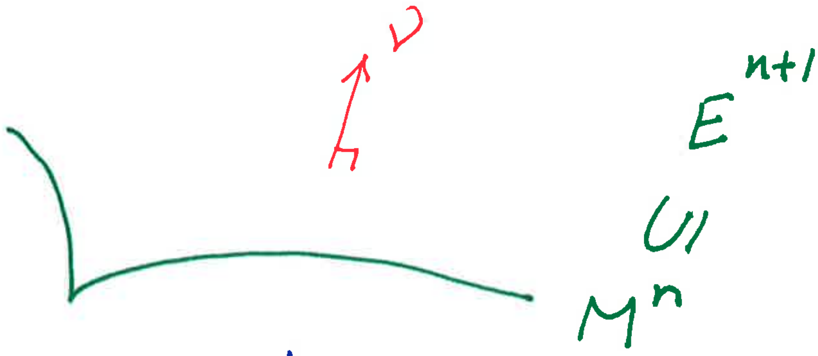
ref. Guo-Han-Li [arXiv] for $\partial M = \phi$

(5) Mean curvature of Euclidian hypersurfaces.



g_{ab} = first fundamental form

$\underline{II}_{ab}$ = second fundamental form, i.e.

$$\underline{II}(v, w) = (\nu, \nabla^E_v w)$$

where

$$\left| \begin{array}{l} \nabla^a \underline{II}_{ab} - n \nabla_b H = 0 \\ H = \frac{1}{n} g^{cd} \underline{II}_{cd} \quad \text{mean curvature} \end{array} \right.$$

and hence $B_{ab} = \underline{II}_{ab} - n g_{ab} H$ is locally preserved:

$$\boxed{\nabla^a B_{ab} = 0}$$

Thus we get a Pohozaev-Schoen type identity, and as a special case recover, since $V = g^{ab} B_{ab} = n(1-n)H$

14)

Theorem. Let $S^n \to E^{n+1}$ be a conformal immersion with mean curvature H. Then for any conformal vector field X on S^n we have

$$\boxed{\int_{S^n} XH \, dv_g = 0}$$

where we view H as a function on S^n, and dv_g is the pullback of the first fundamental form measure.

ref. Ammann et al. [2007]; they use

- restricting a parallel spinor
- a semilinear Dirac equation
- a Pohozaev-Schoen identity for such spinors

Specifically for (M, g, ξ) spin and

$$D\psi = H|\psi|^{p-2}\psi, \quad p > 1$$

where $H \in C^\infty(M)$ and ψ a smooth spinor field, we get for X conformal

$$\boxed{\int_M \left[\left(\frac{1}{n} - \frac{p-2}{p}\right) H \, \mathrm{div} X + \frac{2}{p} X H\right] |\psi|^p = \int_{\partial M} \langle \nu \cdot \mathcal{L}_X \psi, \psi \rangle - \frac{p-2}{p} H |\psi|^p g(X, \nu)}$$

– proof via $\langle D \mathcal{L}_X \psi, \psi \rangle$ & Green.

Open questions:

(1) Find an intrinsic proof of ☆.

(2) Make explicit some of the Pohozaev - Schoen identities found for (non-) existence results corresponding to semilinear elliptic equations.

(3) Similar results in other parabolic geometries, e.g.

$$\boxed{\int_M X W \, dvol = 0}$$

$M = CR$, $X = CR$, W = CR scalar curv.

Final remark

ALLES GUTE ZUM GEBURTSTAG

Harald Upmeier