# Srni 26th Winter School Geometry and Physics

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## Special geometries and superstring theory

## I: Mathematical tools – geometry of metric connections

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# Outline

At border line between pure mathematics and theoretical physics

| formation of mathematical concepts | $\Leftrightarrow$ | building of physical models |
|---|---|---|
| differential geometry, analysis, group theory | | general relativity, unified field theories, string theory |

Lecture I: Mathematical tools – geometry of metric connections

Lecture II: Physical motivation & first applications

Lecture III: More on special geometries

Lecture IV: Geometric structures with parallel torsion and of vector type

## Symmetry I

- Classical mechanics: Symmetry considerations can simplify study of geometric problems (i.e., Noether's theorem)

- Felix Klein at his inaugural lecture at Erlangen University, 1872 ("Erlanger Programm"):

> "Es ist eine Mannigfaltigkeit und in derselben eine Transformationsgruppe gegeben; man soll die der Mannigfaltigkeit angehörigen Gebilde hinsichtlich solcher Eigenschaften untersuchen, die durch die Transformationen der Gruppe nicht geändert werden".

> "Let a manifold and in this a transformation group be given; the objects belonging to the manifold ought to be studied with respect to those properties which are not changed by the transformations of the group."

→ *Isometry group* of a Riemannian manifold $(M, g)$

## Symmetry II

- Around 1940-1950: Second intrinsic Lie group associated with a Riemannian manifold $(M, g)$ appeared, its *holonomy group*.

→ strongly related to curvature and parallel objects

A priori, the holonomy group is defined *for an arbitrary connection* $\nabla$ on $TM$. For reasons to become clear later, we concentrate mainly on

$$\text{Metric connections } \nabla : \; Xg(V,W) \; = \; g(\nabla_X V, W) + g(V, \nabla_X W).$$

The torsion (viewed as $(2,1)$- or $(3,0)$-tensor)

$$T(X,Y) \; := \; \nabla_X Y - \nabla_Y X - [X,Y], \quad T(X,Y,Z) \; := \; g(T(X,Y),Z)$$

can (for the moment. . . ) be arbitrary.

## Types of metric connections

$(M^n, g)$ oriented Riemannian mnfd, $\nabla$ any connection:

$$\nabla_X Y = \nabla^g_X Y + A(X,Y)\,.$$

Then: $\nabla$ is metric $\Leftrightarrow$ $g(A(X,Y),Z) + g(A(X,Z),Y) = 0$

$\Leftrightarrow$ $A \in \mathcal{A}^g := \mathbb{R}^n \otimes \Lambda^2(\mathbb{R}^n)$

This is also the space $\mathcal{T}$ of possible torsion tensors,

$$\mathcal{A}^g \cong \mathcal{T} \cong \mathbb{R}^n \otimes \Lambda^2(\mathbb{R}^n), \quad \dim = \frac{n^2(n-1)}{2}.$$

For metric connections: difference tensor $A$ $\Leftrightarrow$ torsion $T$

Decompose this space under $\mathrm{SO}(n)$ action (E. Cartan, 1925):

$$\mathbb{R}^n \otimes \Lambda^2(\mathbb{R}^n) = \mathbb{R}^n \oplus \Lambda^3(\mathbb{R}^n) \oplus \mathcal{T}\,.$$

- $A \in \Lambda^3(\mathbb{R}^n)$: "Connections with (totally) antisymmetric torsion":

$$\nabla_X Y \;:=\; \nabla^g_X Y + \frac{1}{2}\mathrm{T}(X,Y,-)\,.$$

**Lemma**. $\nabla$ is metric and geodesics preserving iff its torsion $T$ lies in $\Lambda^3(TM)$. In this case, $2\,A = T$, and the $\nabla$-Killing vector fields coincide with the Riemannian Killing vector fields.

⟶ Connections used in superstring theory (examples in Lecture II)

- $A \in \mathbb{R}^n$: "Connections with vectorial torsion", $V$ a vector field:

$$\nabla_X Y \;:=\; \nabla^g_X Y - g(X,Y)\cdot V + g(Y,V)\cdot X\,.$$

In particular, *any metric connection* on a surface is of this type!

## Mercator map

- conformal (angle preserving), hence maps loxodromes to straight lines
- Cartan (1923):

"On this manifold, the straight lines [of the flat connection] are the *loxodromes*, which intersect the meridians at a constant angle. The only straight lines realizing shortest paths are those which are normal to the torsion in every point: *these are the meridians*.

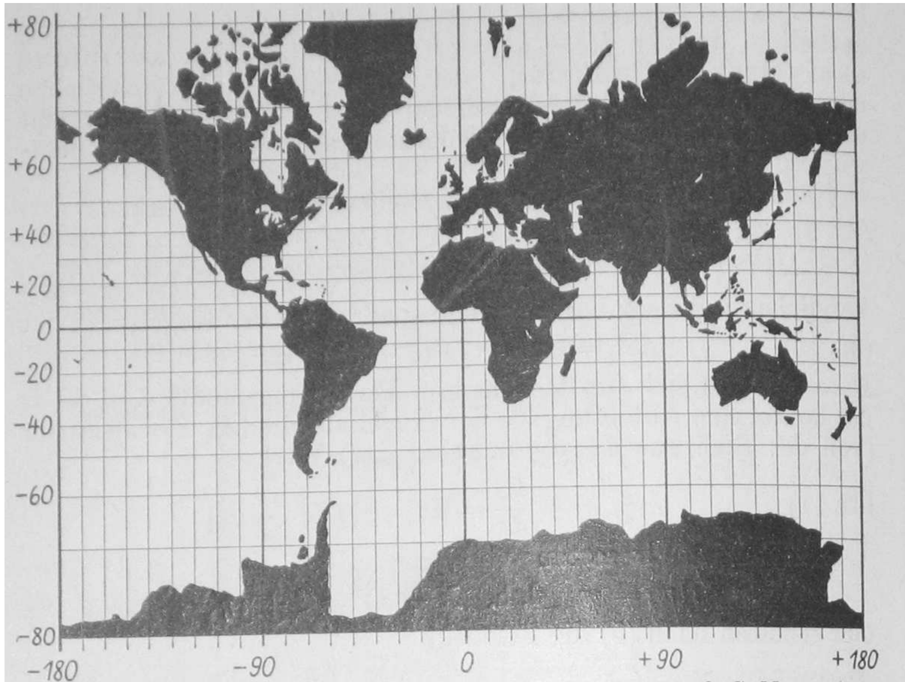



Bild 8. Winkeltreue Abbildung der Kugel auf die Ebene nach G. Mercator

$S^2 - \{N, S\} \to I \times \mathbb{R}$ (1569)

- Explanation & generalisation to arbitrary manifolds?
- Existence of a Clairaut style invariant?

**Thm (A-Thier, '03).** $(M, g)$ Riemannian manifold, $\sigma \in C^\infty(M)$ and $\tilde{g} = e^{2\sigma} g$ the conformally changed metric. Let

$\tilde{\nabla}^g$: metric connection with vectorial torsion $V = -\operatorname{grad} \sigma$ on $(M, g)$,

$$\tilde{\nabla}^g_X Y \;=\; \nabla^g_X Y - g(X, Y)V + g(Y, V)X$$

$\nabla^{\tilde{g}}$: Levi-Civita connection of $(M, \tilde{g})$. Then

(1) Every $\tilde{\nabla}^g$-geodesic $\gamma(t)$ is (up to reparametrisation) a $\nabla^{\tilde{g}}$-geodesic;

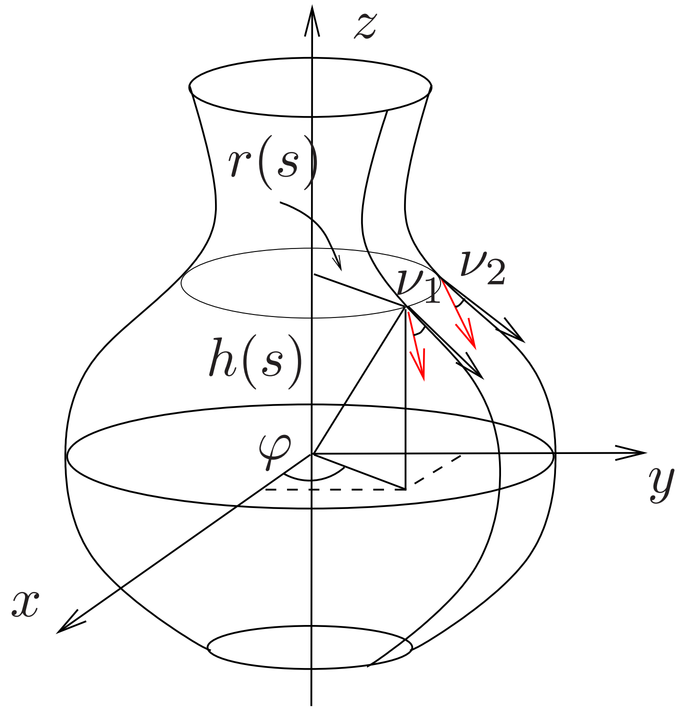
(2) If $X$ is a Killing vector field of $\tilde{g}$, then $e^\sigma g(\gamma', X)$ is a constant of motion for every $\tilde{\nabla}^g$-geodesic $\gamma(t)$.

**N.B.** The curvatures of $\tilde{\nabla}^g$ and $\nabla^{\tilde{g}}$ coincide, but the curvatures of $\nabla^g$ and $\nabla^{\tilde{g}}$ are unrelated.

⟶ Beltrami's theorem does not hold anymore ["If a portion of a surface $S$ can be mapped LC-geodesically onto a portion of a surface $S^*$ of constant Gaussian curvature, the Gaussian curvature of $S$ must also be constant"]

# Connections with vectorial torsion on surfaces

- Curve: $\alpha = \big(r(s), h(s)\big)$
- Surface of revolution: $\big(r(s)\cos\varphi, r(s)\sin\varphi, h(s)\big)$
- Riemannian metric: $g = \mathrm{diag}\big(r^2(s), 1\big)$
- Orthonormal frame: $e_1 = \frac{1}{r}\partial_\varphi,\ e_2 = \partial_s$



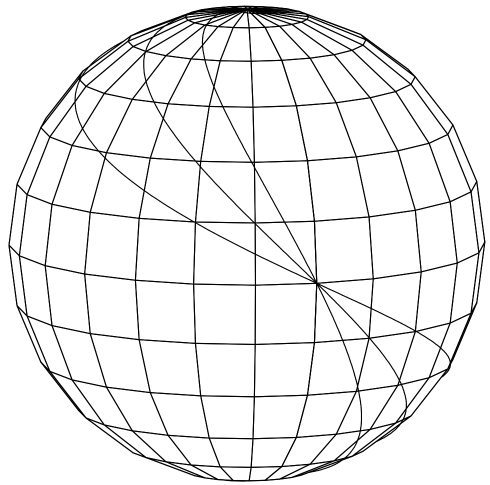

**Dfn:** Call two tangent vectors $v_1$ and $v_2$ of same length parallel if their angles $\nu_1$ and $\nu_2$ with the generating curves through their origins coincide.

- Hence $\nabla e_1 = \nabla e_2 = 0$
- Torsion: $T(e_1, e_2) = \frac{r'(s)}{r(s)} e_2$
- Corresponding vector field:

$V = \frac{r'(s)}{r(s)} e_1 = -\text{grad}\big(-\ln r(s)\big)$

- geodesics are LC geodesics of the conformally equivalent metric $\tilde{g} = e^{2\sigma} g = \text{diag}(1/r^2, 1)$

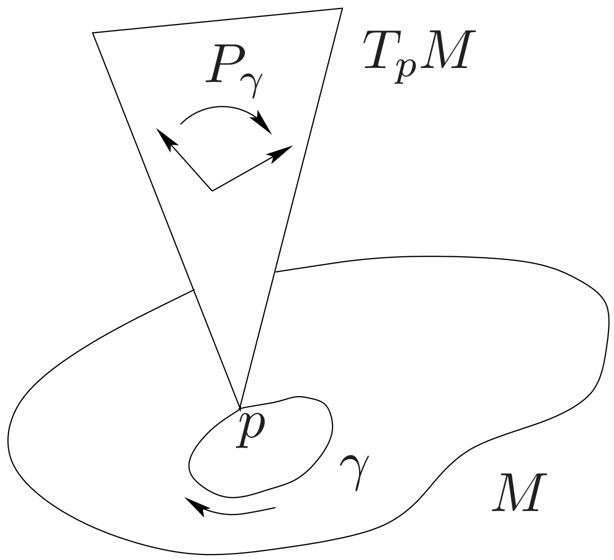
(coincides with euclidian metric under $x = \varphi$, $y = \int ds/r(s)$)



- $X = \partial_\varphi$ is Killing vector field for $\tilde{g}$, invariant of motion:

$$\text{const} = e^\sigma g(\dot{\gamma}, X) = \frac{1}{r(s)} g(\dot{\gamma}, \partial_\varphi) = g(\dot{\gamma}, e_2)$$

## Holonomy of arbitrary connections




- $\gamma$ from $p$ to $q$, $\nabla$ any connection
- $P_\gamma : T_pM \to T_qM$ is the unique map s. t. $V(q) := P_\gamma V(p)$ is parallel along $\gamma$, $\nabla V(s)/ds = \nabla_{\dot\gamma} V = 0$.
- $C(p)$: closed loops through $p$
  $\mathrm{Hol}(p;\nabla) = \{P_\gamma \mid \gamma \in C(p)\}$
- $C_0(p)$: null-homotopic el'ts in $C(p)$
  $\mathrm{Hol}_0(p;\nabla) = \{P_\gamma \mid \gamma \in C_0(p)\}$

Independent of $p$, so drop $p$ in notation: $\mathrm{Hol}(M;\nabla)$, $\mathrm{Hol}_0(M;\nabla)$.

**A priori:**

(1) $\mathrm{Hol}(M;\nabla)$ is a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$,

(2) $\mathrm{Hol}_0(p)$ is the connected component of the identity of $\mathrm{Hol}(M;\nabla)$.

## Holonomy of metric connections

Assume: $M$ carries a Riemannian metric $g$, $\nabla$ *metric*

$\Rightarrow$ parallel transport is an isometry:

$$\frac{d}{ds}g\big(V(s),W(s)\big) \;=\; g\big(\frac{\nabla V(s)}{ds},W(s)\big)+\big(V(s),\frac{\nabla W(s)}{ds}\big) \;=\; 0.$$

and $\mathrm{Hol}(M;\nabla)\subset \mathrm{O}(n,\mathbb{R})$, $\mathrm{Hol}_0(M;\nabla)\subset \mathrm{SO}(n,\mathbb{R})$.

**Notation:** $\mathrm{Hol}_{(0)}(M;\nabla^g)=$ "Riemannian (restricted) holonomy group"

**N.B.** (1) $\mathrm{Hol}_{(0)}(M;\nabla)$ needs not to be closed!

(2) The holonomy representation needs not to be irreducible on irreducible manifolds!

⟶ Larger variety of holonomy groups, but classification difficult

## Curvature & Holonomy

Holonomy can be computed through curvature:

**Thm (Ambrose-Singer, 1953).** For any connection $\nabla$ on $(M,g)$, the Lie algebra $\mathfrak{hol}(p;\nabla)$ of $\mathrm{Hol}(p;\nabla)$ in $p \in M$ is exactly the subalgebra of $\mathfrak{so}(T_pM)$ generated by the elements

$$P_\gamma^{-1} \circ \mathcal{R}(P_\gamma V, P_\gamma W) \circ P_\gamma \quad V, W \in T_pM, \quad \gamma \in C(p).$$

But only of restricted use:

**Thm (Bianchi I).** (1) For a metric connection with vectorial torsion $V \in TM^n$: $\quad \overset{X,Y,Z}{\sigma} \mathcal{R}(X,Y)Z = \overset{X,Y,Z}{\sigma} dV(X,Y)Z.$

(2) For a metric connection with skew symmetric torsion $T \in \Lambda^3(M^n)$:

$$\overset{X,Y,Z}{\sigma} \mathcal{R}(X,Y,Z,V) = dT(X,Y,Z,V) - \sigma^T(X,Y,Z,V) + (\nabla_V T)(X,Y,Z),$$

$2\,\sigma^T := \sum_{i=1}^{n} (e_i \lrcorner T) \wedge (e_i \lrcorner T)$ for any orthonormal frame $e_1, \ldots, e_n$.

**Theorem (Berger, Simons, $>1955$).** For a non symmetric Riemannian manifold $(M, g)$ and the Levi-Civita connection $\nabla^g$, the possible holonomy groups are $\mathrm{SO}(n)$ or

| | | | | | | |
|---|---|---|---|---|---|---|
| $4n$ | $2n$ | $2n$ | $4n$ | $7$ | $8$ | $16$ |
| $\mathrm{Sp}_n\mathrm{Sp}_1$ | $\mathrm{U}(n)$ | $\mathrm{SU}(n)$ | $\mathrm{Sp}_n$ | $G_2$ | $\mathrm{Spin}(7)$ | $(\mathrm{Spin}(9))$ |
| quatern. Kähler | Kähler | Calabi-Yau | hyper-Kähler | par. | par. | par. |
| $\nabla J \neq 0$ | $\nabla^g J = 0$ | $\nabla^g J = 0$ | $\nabla^g J = 0$ | $\nabla^g \omega^3 = 0$ | $--$ | $--$ |
| $\mathrm{Ric} = \lambda g$ | $--$ | $\mathrm{Ric} = 0$ | $\mathrm{Ric} = 0$ | $\mathrm{Ric} = 0$ | $\mathrm{Ric} = 0$ | $--$ |

Existence of Ricci flat compact manifolds:

- Calabi-Yau, hyper-Kähler: Yau, 1980's.
- $G_2$, $\mathrm{Spin}(7)$: D. Joyce since $\sim$ 1995, Kovalev (2003). Both rely on heavy analysis and algebraic geometry !

No such theorem for metric connections!

## General Holonomy Principle

**Thm (General Holonomy Principle).** $(M, g)$ a Riemannian manifold, $E$ a (real or complex) vector bundle over $M$ with (any!) connection $\nabla$. Then the following are equivalent:

(1) $E$ has a global section $\alpha$ which is invariant under parallel transport, i. e. $\alpha(q) = P_\gamma(\alpha(p))$ for any path $\gamma$ from $p$ to $q$;

(2) $E$ has a parallel global section $\alpha$, i. e. $\nabla\alpha = 0$;

(3) In some point $p \in M$, there exists an algebraic vector $\alpha_0 \in E_p$ which is invariant under the holonomy representation on the fiber.

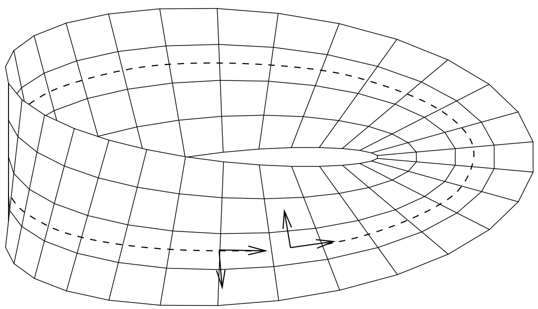
**Corollary.** The number of parallel global sections of $E$ coincides with the number of trivial representations occuring in the holonomy representation on the fibers.

**Example.** Orientability from a holonomy point of view:

**Lemma.** The determinant ist an $\mathrm{SO}(n)$-invariant element in $\Lambda^n(\mathbb{R}^n)$ that is not $\mathrm{O}(n)$-invariant.

**Corollary.** $(M^n, g)$ is orientable iff $\mathrm{Hol}(M;\nabla) \subset \mathrm{SO}(n)$ for *any metric connection* $\nabla$, and the volume form is then $\nabla$-parallel.

[Take $dM_p := \det = e_1 \wedge \ldots \wedge e_n$ in $p \in M$, then apply holonomy principle to $E = \Lambda^n(TM)$.]



An orthonormal frame that is parallel transported along the drawn curve reverses its orientation.

## Geometric stabilizers

**Philosophy:** Invariants of geometric representations are candidates for parallel objects. Find these!

- Invariants for $G \subset \mathrm{SO}(m)$ in tensor bundles (as just seen)
- Assume that $G \subset \mathrm{SO}(m)$ can be lifted to a subgroup $G \subset \mathrm{Spin}(m)$

$\Rightarrow$ $G$ acts on the spin representation $\Delta_m$ of $\mathrm{Spin}(m)$

Recall: • $m = 2k$ even: $\Delta_m = \Delta_m^+ \oplus \Delta_m^-$, both have dimension $2^{k-1}$

• $m = 2k+1$ odd: $\Delta_m$ is irreducible, of dimension $2^k$

Elements of $\Delta_m$: "algebraic spinors" (in opposition to spinors on $M$ that are sections of the spinor bundle)

Now decompose $\Delta_m$ under the action of $G$.

In particular: Are there invariant algebraic spinors?

## $\mathrm{U}(n)$ in dimension $2n$

- Hermitian metric $h(V,W) = g(V,W) - ig(JV,W)$

- $h$ is invariant under $A \in \mathrm{End}(\mathbb{R}^{2n})$ iff $A$ leaves invariant $g$ and the Kähler form $\Omega(V,W) := g(JV,W)$ $\Rightarrow$

$$\mathrm{U}(n) \;=\; \{A \in \mathrm{SO}(2n) \mid A^*\Omega = \Omega\}.$$

**Lemma.** Under the restricted action of $\mathrm{U}(n)$, $\Lambda^{2k}(\mathbb{R}^{2n}), k = 1, \ldots, n$ contains the trivial representation once, namely, $\Omega, \Omega^2, \ldots, \Omega^n$.

$\mathrm{U}(n)$ can be lifted to a subgroup of $\mathrm{Spin}(2n)$, but it has no invariant algebraic spinors:

$\Omega$ generates the one-dimensional center of $\mathfrak{u}(n)$ (identify $\Lambda^2(\mathbb{R}^{2n}) \cong \mathfrak{so}(2n)$).

Set $S_r = \{\psi \in \Delta_{2n} \,:\, \Omega\psi = i(n-2r)\psi\}$, $\quad \dim S_r = \binom{n}{r}$, $\quad 0 \le r \le n$.

$S_r \cong (0,r)$-forms with values in $S_0$ and

$$\Delta^+_{2n}\big|_{\mathrm{U}(n)} \cong S_n \oplus S_{n-2} \oplus \dots, \quad \Delta^-_{2n}\big|_{\mathrm{U}(n)} \cong S_{n-1} \oplus S_{n-3} \oplus \dots$$

$\Rightarrow$

- no trivial $\mathrm{U}(n)$-representation for $n$ odd

- For $n = 2k$ even, $\Omega$ has eigenvalue zero on $S_k$, but this space is an irreducible representation of dimension $\binom{2k}{k} \neq 1$

- $S_0$ and $S_n$ are one-dimensional, and they become trivial under $\mathrm{SU}(n)$

**Lemma.** $\Delta^\pm_{2n}$ contain no $\mathrm{U}(n)$-invariant spinors. If one restricts further to $\mathrm{SU}(n)$, there are exactly two invariant spinors.

## $G_2$ in dimension $7$

• Geometry of $3$-forms plays an exceptional role in Riemannian geometry, as it ocurs only in dimension seven:

| $n$ | $\dim \mathrm{GL}(n,\mathbb{R}) - \dim \Lambda^3\mathbb{R}^n$ | $\dim \mathrm{SO}(n)$ |
|---|---|---|
| 3 | $9 - 1 = 8$ | 3 |
| 4 | $16 - 4 = 12$ | 6 |
| 5 | $25 - 10 = 15$ | 10 |
| 6 | $36 - 20 = 16$ | 15 |
| 7 | $49 - 35 = 14$ | 21 |
| 8 | $64 - 56 = 8$ | 28 |

$\Rightarrow$ stabilizer $G^n_{\omega^3} := \{A \in \mathrm{GL}(n,\mathbb{R}) \mid \omega^3 = A^*\omega^3\}$ of a generic $3$-form $\omega^3$ cannot lie in $\mathrm{SO}(n)$ for $n \leq 6$ (for example: $G^3_{\omega^3} = \mathrm{SL}(3,\mathbb{R})$).

Reichel, 1907 (Ph D student of F. Engel in Greifswald):

- computed a system of invariants for a $3$-form in seven variables

- showed that there are exactly two $\mathrm{GL}(7,\mathbb{R})$-open orbits of $3$-forms

- showed that stabilizers of any representatives $\omega^3$, $\tilde{\omega}^3$ of these orbits are $14$-dimensional simple Lie groups of rank two, a compact and a non-compact one:

$$G^7_{\omega^3} \;\cong\; G_2 \;\subset\; \mathrm{SO}(7), \quad G^7_{\tilde{\omega}^3} \;\cong\; G_2^* \;\subset\; \mathrm{SO}(3,4)$$

- realized $\mathfrak{g}_2$ and $\mathfrak{g}_2^*$ as explicit subspaces of $\mathfrak{so}(7)$ and $\mathfrak{so}(3,4)$

As in the case of almost hermitian geometry, one has a favourite normal form for a $3$-form with isotropy group $G_2$:

$$\omega^3 \;:=\; e_{127} + e_{347} - e_{567} + e_{135} - e_{245} + e_{146} + e_{236}.$$

An element of the second orbit ($\rightarrow G_2^*$) may be obtained by reversing any of the signs in $\omega^3$.

**Lemma.** Under $G_2$: $\Lambda^3(\mathbb{R}^7) \cong \mathbb{R} \oplus \mathbb{R}^7 \oplus S_0(\mathbb{R}^7)$, where

$\mathbb{R}^7$: 7-dimensional standard representation of $G_2 \subset \mathrm{SO}(7)$

$S_0(\mathbb{R}^7)$: traceless symmetric endomorphisms of $\mathbb{R}^7$ (has dimension $27$).

• $G_2$ can be lifted to a subgroup of $\mathrm{Spin}(7)$. From a purely representation theoretic point of view, this case is trivial:

$\dim \Delta_7 = 8$ and the only irreducible representations of $G_2$ of dimension $\leq 8$ are the trivial and the 7-dimensional representation $\Rightarrow$

**Lemma.** Under $G_2$: $\Delta_7 \cong \mathbb{R} \oplus \mathbb{R}^7$.

In fact, the invariant 3-form $\omega^3$ and the invariant algebraic spinor $\psi$ are equivalent data:

$$\omega^3(X, Y, Z) \ = \ \langle X \cdot Y \cdot Y \cdot \psi, \psi \rangle.$$

But $\dim \Delta_7 = 8 < \dim \Lambda^3(\mathbb{R}^7) = 35$, so the spinorial picture is easier to treat!

Assume now that $G \subset G_2$ fixes *a second spinor* $\Rightarrow G \cong \mathrm{SU}(3)$

• this is one of the three maximal Lie subgroups of $G_2$, $\mathrm{SU}(3)$, $\mathrm{SO}(4)$ and $\mathrm{SO}(3)$

• $\mathrm{SU}(3)$ has irreducible real representations in dimension $1$, $6$ and $8$, so

**Lemma.** Under $\mathrm{SU}(3) \subset G_2$: $\Delta_7 \cong \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}^6$ and $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{R}^6$.

This implies:

• If $\nabla^g$ on $(M^7, g)$ has two parallel spinors, $M$ has to be (locally) reducible, $M^7 = M^6 \otimes M^1$ and the situation reduces to the 6-dimensional case.

• If $\nabla$ is some other metric connection on $(M^7, g)$ with two parallel spinors, $M^7$ will, in general, not be a product manifold. Its Riemannian holonomy will typically be $\mathrm{SO}(7)$, so $\nabla^g$ does not measure this effect!

$\Rightarrow$ geometric situations not known from Riemannian holonomy will typically appear.

In a similar way, one treats the cases

**$\mathrm{Spin}(7)$ in dimension $8$.** As just seen, $\mathrm{Spin}(7)$ has an 8-dimensional representation, hence it can be viewed as a subgroup of $\mathrm{SO}(8)$. $\Delta_8$ has again one $\mathrm{Spin}(7)$-invariant spinor.

**$\mathrm{Sp}(n)$ in dimension $4n$.** Identifying quaternions with pairs $(z_1, z_2) \in \mathbb{C}^2$ yields $\mathrm{Sp}(n) \subset \mathrm{SU}(2n)$, and $\mathrm{SU}(2n)$ is then realized inside $\mathrm{SO}(4n)$ as before. It has $n+1$ invariant spinors.

## The easiest case: $\nabla^g$-parallel spinors

**Thm (Wang, 1989).**

$(M^n, g)$: complete, simply connected, irreducible Riemannian manifold

$N$: dimension of the space of parallel spinors w. r. t. $\nabla^g$

If $(M^n, g)$ is non-flat and $N > 0$, then one of the following holds:

(1) $n = 2m$ $(m \geq 2)$, Riemannian holonomy repr.: $\mathrm{SU}(m)$ on $\mathbb{C}^m$, and $N = 2$ ("Calabi-Yau case"),

(2) $n = 4m$ $(m \geq 2)$, Riemannian holonomy repr.: $\mathrm{Sp}(m)$ on $\mathbb{C}^{2m}$, and $N = m + 1$ ("hyperkähler case"),

(3) $n = 7$, Riemannian holonomy repr.: 7-dimensional representation of $G_2$, and $N = 1$ ("parallel $G_2$ case"),

(4) $n = 8$, Riemannian holonomy repr.: spin representation of $\mathrm{Spin}(7)$, and $N = 1$ ("parallel $\mathrm{Spin}(7)$ case").