Srni 26th Winter School Geometry and Physics

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Special geometries and superstring theory

II: Physical motivation & first applications

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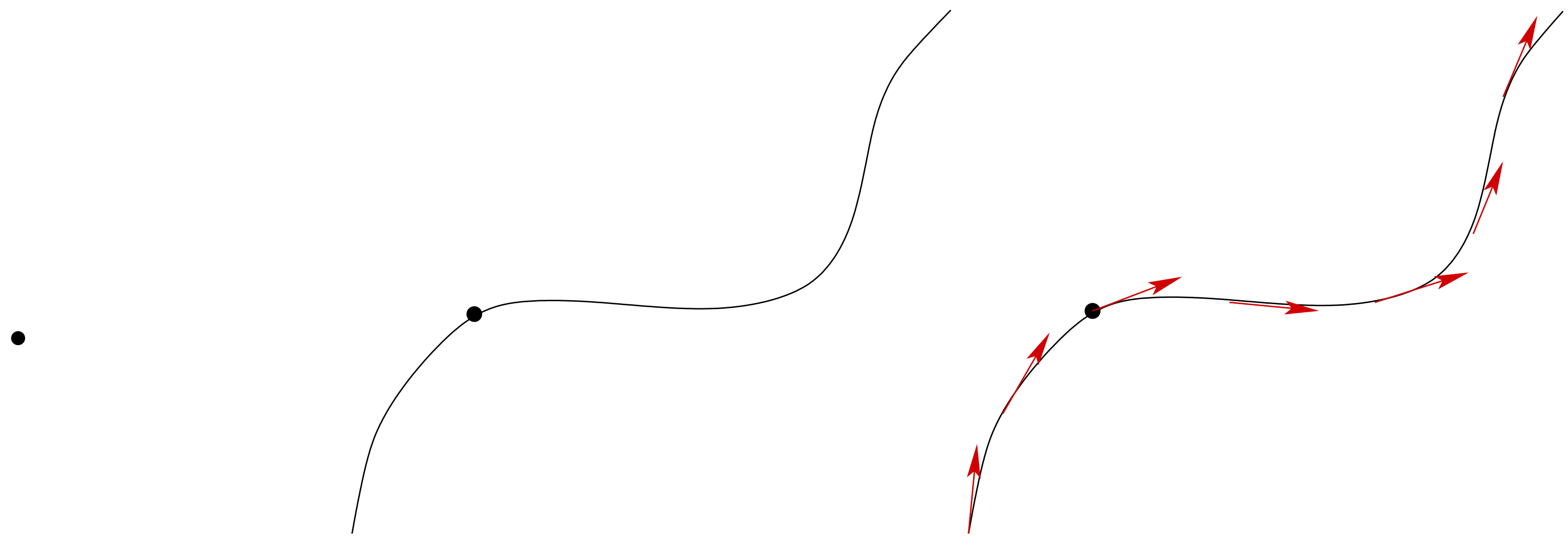
VolkswagenStiftung

Classical general relativity and electromagnetism



point particle

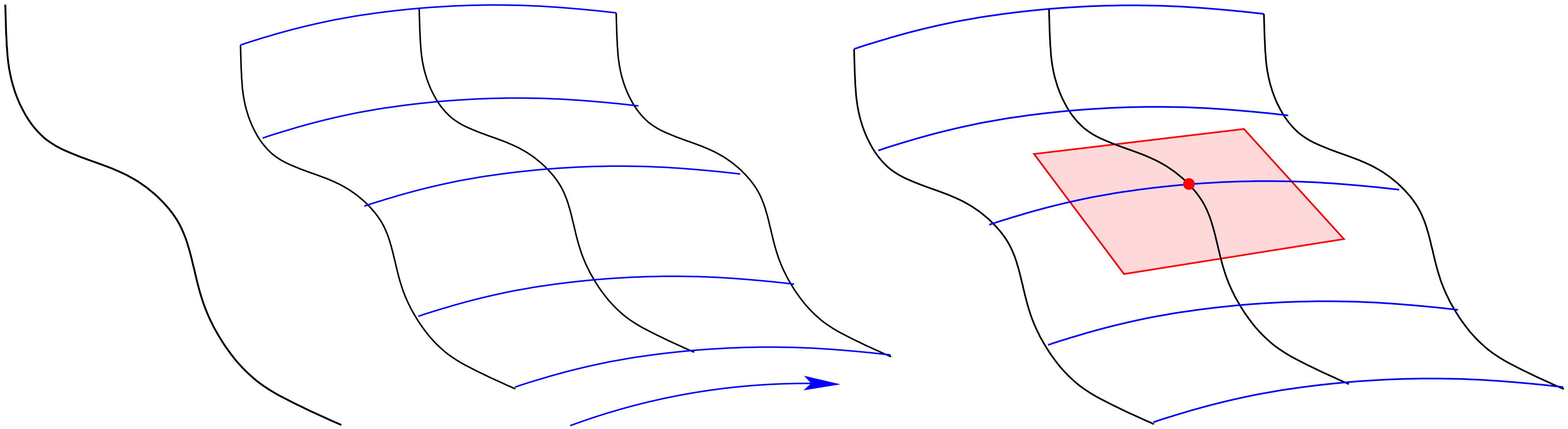
moves along
a curve

physical quantities: associate
with every tangent vector a number
= "potential" A

field strength
$F = dA$
$\Leftrightarrow$
how the potential A changes
in all directions
$\Leftrightarrow$
geometric concept
of curvature

curvature measures deviation from vacuum !

Modern unified models



string particle

moves along
a surface

physical quantities: associate
with every tangent plane a number
= “higher order potential” $\tilde{A}$

higher order field
strength $F = d\tilde{A}$ $\Leftrightarrow$ how the potential $\tilde{A}$ changes
in all directions $\Leftrightarrow$ geometric concept
of torsion

torsion measures deviation from vacuum (“integrable case”) !

Relativistic electromagnetism
1900-1940

describes point particles with electromagnetic charge,
relates it with abelian gauge transformations
gauge group = rotations in a plane

Maxwell Lorentz Einstein Weyl Dirac

↓

Standard model of elementary particles
1950-1980

describes point particles with additional gauge properties, like charge, isospin, colour. . .
internal symmetries are described by some Lie group

Yang Mills Salam Weinberg

↓

Unified theories
(super-)strings, supergravity
> 1980

Quantized internal symmetries are spinor fields
on some space with special geometric structure,
replacing the Lie group of the standard model

Nieuwenhuizen Strominger Seiberg Witten ...

Mathematical scheme for unified theories

No more described as Yang-Mills theories (electrodynamics, standard model of elementary particles), but rather:

- Particles are "oscillatory states" on some high dimensional configuration space

$$Y^{10,11} = V^{3\text{-}5} \otimes M^{5\text{-}8}$$

V: configuration space visible to the outside, i. e. Minkowski space or some solution from General Relativity (adS is popular here).

M: configuration space of *internal symmetries* $=$ Riemannian manifold with special geometric structure, quantized internal symmetries are described by spinor fields.

Example: Supersymmetry transformation, transform bosons into fermions and vice versa by tensoring with a (special) spin $1/2$ field.

[$>$ 1980 Nieuwenhuizen, Strominger, Witten, Seiberg. . .]

Common sector of Type II string equations

- A. Strominger, 1986: (M^n, g) Riemannian Spin mfd with

  a 3-Form T, a spinor field Ψ, and a function Φ.

  (field strength) (supersymmetry) (dilaton)

- Bosonic eq.: $R^g_{ij} - \frac{1}{4} T_{imn} T_{jmn} + 2 \nabla^g_i \partial_j \Phi \; = \; 0, \quad \delta(e^{-2\Phi} T) \; = \; 0\,.$

- Fermionic eq.: $\left(\nabla^g_X \; + \; \frac{1}{4} X \lrcorner T \right) \cdot \Psi \; = \; 0, \quad T \cdot \Psi \; = \; 2\, d\Phi \cdot \Psi\,.$

  As in general relativity, it is impossible to fix the manifold and look for solutions on it. Rather, finding the manifold is part of the solution process!

  $\longrightarrow$ Geometric meaning of the 3-form T ?

Idea: The first fermionic eq. means that the spinor field Ψ is parallel w.r.t. a new connection,

$$\nabla_X Y \ := \ \nabla^g_X Y \ + \ \frac{1}{2} T(X,Y,-)\,.$$

The 3-form T is then the torsion of the new metric connection ∇ and the eqs. are equivalent to:

- Bosonic eq.: $\mathrm{Ric}^{\nabla} + \frac{1}{2}\delta(T) + 2\nabla^g d\Phi \ = \ 0, \quad \delta(e^{-2\Phi}T) \ = \ 0\,.$
- Fermionic eq.: $\nabla\Psi \ = \ 0, \quad T\cdot\Psi \ = \ 2\,d\Phi\cdot\Psi\,.$

Remarks:

• Bosonic eq. generalizes Einstein's eq. of general relativity

• Calabi-Yau and Joyce mfds are exact solution with $T = 0$ and $\Phi =$ const $\longrightarrow$ Bergers' list + algebraic geometry

• For $T \neq 0$, the relation between curvature and parallel spinor is subtler, and there exists no holonomy theory for them

Intermezzo: Lifting metric connections into the spinor bundle

At first sight, the formulas on vectors and spinors look quite different!

Write $\nabla_X Y := \nabla^g_X Y + A_X Y$,

where A_X defines an endomorphism $TM \to TM$ for every X.

∇ metric $\Leftrightarrow g(A_X Y, Z) + g(Y, A_X Z) = 0$

$\Leftrightarrow A_X$ preserves $g \Leftrightarrow A_X \in \mathfrak{so}(n) \cong \Lambda^2(\mathbb{R}^n)$

So $A_X = \sum_{i<j} \alpha_{ij} e_i \wedge e_j$.

Since the lift into $\mathfrak{spin}(n)$ of $e_i \wedge e_j$ is $E_i \cdot E_j/2$, A_X defines an element in $\mathfrak{spin}(n)$ (= an endomorphism on the spinor bundle).

Observe: If A_X is written as a 2-form,

- its action on a vector Y as an element of $\mathfrak{so}(n)$ is just $A_X Y = Y \lrcorner A_X$, so

$$\nabla_X Y \;=\; \nabla^g_X Y + Y \lrcorner A_X,$$

- the action of A_X on a spinor ψ as an element of $\mathfrak{spin}(n)$ is just $A_X \psi = (1/2)\, A_X \cdot \psi$ (Clifford product of a k-form by a spinor), hence the lift of the connection ∇ to the spinor bundle SM is

$$\nabla_X \psi \;=\; \nabla^g_X \psi + \frac{1}{2} A_X \cdot \psi.$$

- Connection with vectorial torsion: $A_X = 2\, X \wedge V$, V a vector field

- Connection with skew symmetric torsion: $A_X = X \lrcorner T$, $T \in \Lambda^3(M)$.

Overview of general results

Non existence theorems

Thm. A full solution of Strominger's model with $\Phi = $ const satisfies necessarily $T = 0$ or $\Psi = 0$.

[M compact: IA, 2002; general case: IA, Friedrich, Nagy, Puhle, 2004]

$\Rightarrow$ physical meaning ?

• Investigation of the homogeneous case, in particular of the relation with *Kostant's cubic Dirac operator* and a generalized *Casimir operator*

• Investigation of the holonomy theory of metric connections with torsion, Weitzenböck formulas for their Dirac operators

Thm ('03). On a Calabi-Yau or Joyce mnfd, a metric connection with torsion T s.t. $dT = 0$ can have parallel spinors only for $T = 0$.

$\Rightarrow$ "rigidity" of CYJ's under deformation of the connection

• Non compact solvmanifolds for which the rigidity theorem does *not* hold

Existence results

Thm ('03). On every 7-dimensional 3-Sasaki mnfd, there exists a family of metric connections with torsion admitting parallel spinors.

• Construction of partial solutions with particular properties, in particular, with parallel spinors

• Investigation of the case $\nabla T = 0$

• Solution of spinorial field eqs. with additional 4-flux-forms F,

$$\nabla_X \psi \;=\; \nabla^g_X \psi + \frac{1}{4}(X \lrcorner T) \cdot \psi + \frac{1}{144}(X \lrcorner F) - X \wedge F) \cdot \psi \;=\; 0.$$

Observe: These connections exist only in the spinor bundle, not in the tangent bundle!

The characteristic connection of a geometric structure

Fix $G \subset \mathrm{SO}(n)$, $\Lambda^2(\mathbb{R}^n) \cong \mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$, $\mathcal{F}(M^n)$: frame bundle of (M^n, g).

Dfn. A geometric G-structure on M^n is a G-PFB $\mathcal{R}$ which is subbundle of $\mathcal{F}(M^n)$: $\mathcal{R} \subset \mathcal{F}(M^n)$.

Choose a G-adapted local ONF $e_1, \ldots, e_n$ in $\mathcal{R}$ and define *connection 1-forms of ∇^g:*

$$\omega_{ij}(X) := g(\nabla^g_X e_i, e_j), \quad g(e_i, e_j) = \delta_{ij} \;\Rightarrow\; \omega_{ij} + \omega_{ji} = 0.$$

Define a skew symmetric matrix Ω with values in $\Lambda^1(\mathbb{R}^n) \cong \mathbb{R}^n$ by $\Omega(X) := \big(\omega_{ij}(X)\big) \in \mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$ und set

$$\Gamma := \mathrm{pr}_{\mathfrak{m}}(\Omega).$$

- Γ is a 1-Form on M^n with values in $\mathfrak{m}$, $\Gamma_x \in \mathbb{R}^n \otimes \mathfrak{m}$ $(x \in M^n)$

[“intrinsic torsion”, Swann/Salamon]

Fact: $\Gamma = 0 \Leftrightarrow \nabla^g$ is G-invariant $\Leftrightarrow \mathrm{Hol}(\nabla^g) \subset G$

Via Γ, geometric G-structures $\mathcal{R} \subset \mathcal{F}(M^n)$ correspond to irreducible components of the G-representation $\mathbb{R}^n \otimes \mathfrak{m}$.

- For the rest of this talk, consider only connections with antisymmetric torsion.

Thm ('02). A geometric G-structure $\mathcal{R} \subset \mathcal{F}(M^n)$ admits a G-invariant metric connection with antisymmetric torsion iff Γ lies in the image of Θ,

$$\Theta : \Lambda^3(M^n) \to T^*(M^n) \otimes \mathfrak{m}, \quad \Theta(T) \ := \ \sum_{i=1}^{n} e_i \otimes \mathrm{pr}_{\mathfrak{m}}(e_i \lrcorner T)\,.$$

If such a connection exists, it is called the *characteristic connection* ∇ and it is unique; its torsion is essentially Γ and $\mathrm{Hol}(\nabla) \subset G$.

If existent, we can thus replace the (unadapted) LC connection by some new unique G-invariant connection!

Examples.

- The *canonical connection* of a naturally reductive space (see below);
- The *Bismut connection* of an almost hermitian mnfd;
- The *Gray connection* of a nearly Kähler mnfd. . .

Example: G_2 structures in dimension 7

Fix $G_2 \subset \mathrm{SO}(7)$, $\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathfrak{m}^7 \cong \mathfrak{g}_2 \oplus \mathbb{R}^7$.

Intrinsic torsion Γ lies in $\mathbb{R}^7 \otimes \mathfrak{m}^7 \cong \mathbb{R}^1 \oplus \mathfrak{g}_2 \oplus \mathrm{S}_0(\mathbb{R}^7) \oplus \mathbb{R}^7 =: \bigoplus_{i=1}^4 W_i$

$\Rightarrow$ four classes of geometric G_2 structures [Fernandez-Gray, '82]

- Decomposition of 3-forms: $\Lambda^3(\mathbb{R}^7) = \mathbb{R}^1 \oplus \mathrm{S}_0(\mathbb{R}^7) \oplus \mathbb{R}^7$.

G_2 is the isotropy group of a generic element of $\omega \in \Lambda^3(\mathbb{R}^7)$:

$$G_2 \;=\; \{A \in \mathrm{SO}(7) \mid A \cdot \omega = \omega\}.$$

Thm. A 7-dimensional Riemannian mfd (M^7, g, ω) with a fixed G_2 structure $\omega \in \Lambda^3(M^7)$ has a G_2-invariant characteristic connection ∇

$\Leftrightarrow$ the $\mathfrak{g}_2$ component of Γ vanishes

$\Leftrightarrow$ There exists a VF β with $\delta\omega = -\beta \lrcorner \omega$

The torsion of ∇ is then $T = -*d\omega - \frac{1}{6}(d\omega, *\omega)\omega + *(\beta \wedge \omega)$, and ∇ admits (at least) one parallel spinor.

Examples: Explicit constructions of G_2 structures:

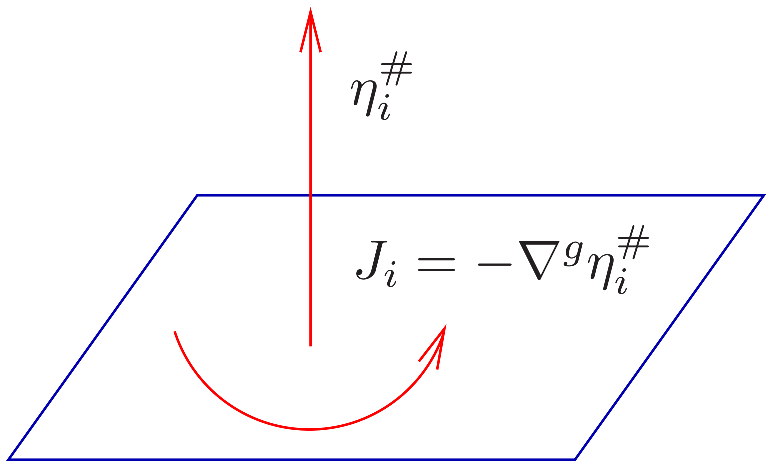
[Friedrich-Kath, Fernandez-Gray, Fernandez-Ugarte, Aloff-Wallach, Boyer-Galicki. . .]

M^7: 3-Sasaki mnfd, corresponds to $\mathrm{SU}(2) \subset G_2 \subset \mathrm{SO}(7)$.

- Has 3 compatible contact structures $\eta_i \in T^*M^7$ and 3 Killing spinors ψ_i $\Rightarrow$ Ansatz:

$$T = \sum_{i,j=1}^{3} \alpha_{ij}\eta_i \wedge \eta_j + \gamma \eta_1 \wedge \eta_2 \wedge \eta_3,$$

$$\psi = \sum_{i=1}^{3} \mu_i \psi_i.$$



Thm ('03). Every 7-dimensional 3-Sasaki mnfd admits a $\mathbb{P}^2$-family of metric connections with antisymmetric torsion and parallel spinors. Its holonomy is G_2.

$\Rightarrow$ First constructive global existence thm for supersymmetries!

Example: $\mathrm{U}(n)$ structures in dimension $2n$

Thm. An almost hermitian manifold (M^{2n}, g, J) admits a $\mathrm{U}(n)$-invariant characteristic connection if and only if the Nijenhuis tensor

$$N(X, Y, Z) \ := \ g\big(N(X, Y),\ Z\big)$$

is skew-symmetric. Its torsion is then

$$T(X, Y, Z) \ = -d\Omega(JX, JY, JZ) + N(X, Y, Z).$$

In particular for $n = 3$: [Gray-Hervella]

- $\mathfrak{so}(6) = \mathfrak{u}(3) \oplus \mathfrak{m}^6$, $\Gamma \in \mathbb{R}^6 \otimes \mathfrak{m}^6\big|_{\mathrm{U}(3)} \cong W_1^2 \oplus W_2^{16} \oplus W_3^{12} \oplus W_4^6$
- N is skew-symmetric $\Leftrightarrow$ Γ has no W_2-part
- $\Gamma \in W_1$: nearly Kähler manifolds $(S^6, S^3 \times S^3, F(1,2), \mathbb{CP}^3)$
- $\Gamma \in W_3 \oplus W_4$: hermitian manifolds $(N = 0)$

Example: Naturally reductive spaces

• Homogeneous *non symmetric* spaces provide a rich source for manifolds with characteristic connection.

Consider $M = G/H$ with isotropy repr. $\mathrm{Ad} : H \to \mathrm{SO}(\mathfrak{m})$.

Lie algebra: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $\langle\, , \,\rangle$ a p. d. scalar product on $\mathfrak{m}$.

The PFB $G \to G/H$ induces a distinguished connection on G/H, the so-called *canonical connection* ∇^1.Its torsion is

$$T^1(X, Y, Z) \;=\; -\langle [X, Y]_{\mathfrak{m}}, Z\rangle\,.$$

Dfn. The metric $\langle\, , \,\rangle$ is called *naturally reductive* if T^1 defines a 3-form,

$$\langle [X, Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X, Z]_{\mathfrak{m}}\rangle \;=\; 0 \text{ for all } X, Y, Z \in \mathfrak{m}\,.$$

They generalize symmetric spaces: $\nabla^1 T^1 = 0, \nabla^1 \mathcal{R}^1 = 0$.

A oneparametric family of connections

Dfn. $\nabla^t_X Y \;:=\; \nabla^g_X Y - \frac{t}{2}[X,Y]_{\mathfrak{m}}$ for $X, Y \in \mathfrak{m}$.

Torsion: $T^t(X,Y) = -t[X,Y]_{\mathfrak{m}}$.

Special t values:
- $\underline{t=0}$: LC connection
- $\underline{t=1}$: canonical connection
- $\underline{t=1/3}$: "Kostant-Slebarski connection"

M spin manifold $\Rightarrow$ lift ∇^t into Spinor bundle, associated Dirac operator:

$$D\!\!\!/^t\psi \;=\; \sum_{i=1}^{n} Z_i(\psi) + \frac{1-t}{2} H\cdot\psi \qquad (Z_1,\ldots,Z_n:\ \text{ONB of } \mathfrak{m}),$$

H: the element in the Clifford algebra induced by torsion:

$$H \;:=\; \frac{3}{2}\sum_{i<j<k}\langle [Z_i,Z_j]_{\mathfrak{m}}, Z_k\rangle\, Z_i\cdot Z_j\cdot Z_k$$

The symmetric case

Want: Weitzenböck formula for $(\not{D}^t)^2$.

For M symmetric ($[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$), one would have:

Thm (Parthasarathy, 1972). $$(\not{D})^2 = \Omega_{\mathfrak{g}} + \frac{1}{8}\mathrm{Scal},$$

with $\Omega_{\mathfrak{g}}$: Casimir operator of $\mathfrak{g}$.

Consequences:

- Computation of spectrum of $\not{D}$
- Realisation of discrete series representations in the (twisted) kernel of $\not{D}$ for G non compact
- Character formulas (interpret character as an index)

In the homogeneous *non symmetric* case, this formula does no longer hold!

The general Kostant-Parthasarathy formula

Thm [Kostant, '99 / IA, '01]. For $n \geq 5$ and arbitrary t:

$$
\begin{aligned}
(D\!\!\!/^{t})^2\psi \;=\; & \Omega_G(\psi) + \tfrac{1}{4}(3t-1)\sum_{i,j,k}\langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k\rangle\, Z_i\cdot Z_j\cdot Z_k(\psi)\\
& -\tfrac{1}{2}\sum_{i<j<k<l}\Big\langle Z_i, \mathcal{J}_{\mathfrak{h}}(Z_j, Z_k, Z_l) + \tfrac{9(1-t)^2}{4}\mathcal{J}_{\mathfrak{m}}(Z_j, Z_k, Z_l)\Big\rangle\, Z_i\cdot Z_j\cdot Z_k\cdot Z_l\cdot\psi\\
& +\tfrac{1}{8}\Big(\sum_{i,j}||[Z_i, Z_j]||_{\mathfrak{h}} + \tfrac{3(1-t)^2}{4}\sum_{i,j}||[Z_i, Z_j]||_{\mathfrak{m}}\Big)\psi
\end{aligned}
$$

Notation:

- $\mathcal{J}_{\mathfrak{m}}(X, Y, Z) := [X, [Y, Z]_{\mathfrak{m}}]_{\mathfrak{m}} +$ cyclic
- $\mathcal{J}_{\mathfrak{h}}(X, Y, Z) := [X, [Y, Z]_{\mathfrak{h}}] +$ cyclic
- Q : the unique $\mathrm{Ad}\, G$-invariant continuation of $\langle\, ,\, \rangle$ to $\mathfrak{g}$. It satisfies:

$$\mathfrak{h} \perp \mathfrak{m}, \quad Q\big|_{\mathfrak{m}} = \langle\, ,\, \rangle\,, \quad Q\big|_{\mathfrak{h}} \text{ not degenerate}$$

The Kostant-Parthasarathy formula for $t = 1/3$

Thm [Kostant, '99 / IA, '01]. For $n \geq 5$ and $t = 1/3$:

$$(\not{D}^{1/3})^2 \psi \ = \ \Omega_G(\psi) + \frac{1}{8}(*)\,\psi\,,$$

where $(*)$ denotes the scalar

$$(*) = \sum_{i,j} |||[Z_i, Z_j]|||_{\mathfrak{h}} + \frac{1}{3}\sum_{i,j} |||[Z_i, Z_j]|||_{\mathfrak{m}}.$$

It can be rewritten as

$$(*) = Q(\varrho_G, \varrho_G) - Q_{\mathfrak{h}}(\varrho_H, \varrho_H)$$

and is thus *always* strictly positive.

First applications

Corollary ('01). If ψ satisfies $\nabla^t\psi = 0$ and $T^t \cdot \psi = 0$ on $M = G/H$, then $t = 0$ and ∇^t is the LC connection.

. . . purely mathematical applications:

Corollary ('01). On $M = G/H$, there exists a G-invariant differential operator of first order which has no symmetric counterpart:

$$\mathcal{D}(\psi) \;:=\; \sum_{i,j,k} \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k\rangle \, Z_i \cdot Z_j \cdot Z_k(\psi)\,.$$

Corollary ('01). If the Casimir operator is non negative, the first eigenvalue $\lambda^{1/3}$ satisfies $(\lambda^{1/3})^2 \geq (*)/8$. In particular, $D\!\!\!/^{1/3}$ has then no kernel.

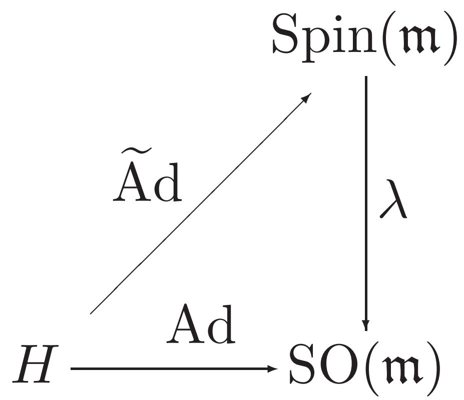
N.B. Character formulas generalize, too $\longrightarrow$ splitting of H-representations into families with similar properties

[$>$ 1999: Kostant, Sternberg, Ramond, Brink. . .]

• Realisation of infinite dimensional representations for G non compact inside kernels of twisted Dirac operators [$>$ 2003, Zierau-Mehdi . . .]

• Computation of the spectrum of $(\not{D}^{1/3})^2$

N.B. Consider lift of isotropy representation, $\widetilde{\mathrm{Ad}} : H \to \mathrm{Spin}(\mathfrak{m})$:



Assume that it contains the trivial representation. Any such spinor induces a section of the spinor bundle $S = G \times_{\kappa(\widetilde{\mathrm{Ad}})} \Delta_n$ if viewed as a constant map $G \to \Delta_n$.

These are exactly the *parallel spinors of the canonical connection*!

Another application: Construction of Lie algebras

Kostant's work was based on the following extension idea for Lie algebras. We formulate his work geometrically:

Let M^n be an *Ambrose-Singer manifold,* i. e.. a Riemannian manifold with a connection ∇ with antisymmetric torsion T s. t.

$$\nabla T = 0\,, \quad \nabla \mathcal{R} = 0\,.$$

Assumption: Universal cover of G_T is compact.

$\Rightarrow$ M^n is regular and locally isometric to a homogeneous space G/G_T. The Lie algebra of G is $\mathfrak{g} := \mathfrak{g}_T \oplus \mathbb{R}^n$ with the commutator

[Cleyton/Swann, 2002]

$$\big[A+X, B+Y\big] := \big([A,B] - \mathcal{R}(X,Y)\big) + \big(AY - BX - T(X,Y)\big).$$

Bianchi I $\Rightarrow$ $\mathcal{R}$ is *unique*:

Lemma. The curvature of ∇ is proportional to the orthogonal projection onto $\mathfrak{g}_T$,

$$\mathcal{R} : \Lambda^2(\mathbb{R}^n) = \mathfrak{so}(n) \longrightarrow \mathfrak{g}_T, \quad \mathcal{R}(X,Y) = 4\,\mathrm{pr}_{\mathfrak{g}_T}(X \wedge Y).$$

Choose an ONF of 2-forms ω_i for $\mathfrak{g}_T$.

Lemma. The commutator defines an extension of $\mathfrak{g}_T$ iff

$$T^2 + 4\sum \omega_i^2$$

is a scalar in the Clifford algebra of $\mathbb{R}^n$.

[a priori: parts of degree 4 + scalar]

– this identity can be understood as a Kostant-Parthasarathy type formula for the symbol of the operator $\not{D}^{1/3}$.

Construction of naturally reductive spaces

General construction:

Consider $M = G/H$ with restriction of the Killing form to $\mathfrak{m}$:

$$\beta(X,Y) := -\mathrm{tr}(X^tY),\ \langle X,Y\rangle = \beta(X,Y) \text{ for } X,Y \in \mathfrak{m}.$$

Suppose that $\mathfrak{m}$ is an orthogonal sum $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ such that

$$[\mathfrak{h},\mathfrak{m}_2] = 0,\ [\mathfrak{m}_2,\mathfrak{m}_2] \subset \mathfrak{m}_2\,.$$

Then the new metric, depending on a parameter $s > 0$

$$\langle X,Y\rangle_s = \begin{cases} 0 & \text{for } X \in \mathfrak{m}_1, Y \in \mathfrak{m}_2 \\ \langle X,Y\rangle & \text{for } X,Y \in \mathfrak{m}_1 \\ s\cdot\langle X,Y\rangle & \text{for } X,Y \in \mathfrak{m}_2 \end{cases}$$

is naturally reductive for $s \neq 1$ w. r. t. the realisation as

$$M = (G \times M_2)/(H \times M_2) =: \overline{G}/\overline{H}\,.$$

[Chavel, 1969; Ziller / D'Atri, 1979]

Jensen metrics

$M^5 = G/H$ with $G = SO(4)$, $H = \mathrm{SO}(2)$ and embed H in G as $\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \mathrm{SO}(2) \end{array}\right]$. Then $\mathfrak{so}(4) = \mathfrak{so}(2) + \mathfrak{m}$ with $(a \in \mathbb{R},\ X \in \mathcal{M}_{2,2}(\mathbb{R}))$

$$\mathfrak{m} = \left\{ \left[\begin{array}{cc|c} 0 & -a & \\ a & 0 & -X^t \\ \hline & & 0 \quad 0 \\ & X & 0 \quad 0 \end{array}\right] =: (a, X) \right\}.$$

Set $\mathfrak{m}_1 := \{(0, X)\}$ and $\mathfrak{m}_2 := \{(a, 0)\}$ $\Rightarrow$ new metric

$$\langle (a, X), (b, Y) \rangle_s = \frac{1}{2}\beta(X, Y) + \frac{s}{2} a \cdot b\,.$$

Properties: • Two ∇^0-parallel spinors for $s = 1$, none for other values of t and s;

• $\mathrm{Ric}^0 = (2 - s)\mathrm{diag}(0, 1, 1, 1, 1)$, Ricci-flat only for $s = 2$ und $t = 0$.