Srni 26th Winter School Geometry and Physics

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Special geometries and superstring theory

III: Weitzenböck formulas for special geometries

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Summary

A. Strominger, 1986: (M^n, g) Riemannian Spin mfd (set dilaton=const) with a 3-form $T \in \Lambda^3(\mathbb{R}^n)$ (field strength) and a spinor field Ψ (supersymmetry) such that

- Bosonic eq.: $\mathrm{Ric}^{\nabla} = 0, \quad \delta T = 0$
- Fermionic eq.: $\nabla\Psi = 0, \quad T \cdot \Psi = 0$

with respect to the metric connection with antisymmetric torsion T

$$\nabla_X Y := \nabla^g_X Y + \frac{1}{2} T(X, Y, -), \quad \nabla_X \psi := \nabla^g_X \psi + \frac{1}{4}(X \lrcorner T) \cdot \psi.$$

This lecture: * Naturally reductive spaces and Kostant's cubic Dirac operator

* Generalisation: Weitzenböck formulas and a Casimir operator on non homogeneous spaces

Example: Naturally reductive spaces

• Homogeneous *non symmetric* spaces provide a rich source for manifolds with characteristic connection [= unique G-inv. metric ∇ with skew torsion]

Consider $M = G/H$ with isotropy repr. $\mathrm{Ad} : H \to \mathrm{SO}(\mathfrak{m})$.

Lie algebra: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $\langle \, , \rangle$ a pos. def. scalar product on $\mathfrak{m}$.

The PFB $G \to G/H$ induces a distinguished connection on G/H, the so-called *canonical connection* ∇^1. Its torsion is

$$T^1(X, Y, Z) \;=\; -\langle [X, Y]_{\mathfrak{m}}, Z\rangle \qquad \text{(}= 0 \text{ for } M \text{ symmetric)}$$

Dfn. The metric $\langle \, , \rangle$ is called *naturally reductive* if T^1 defines a 3-form,

$$\langle [X, Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X, Z]_{\mathfrak{m}}\rangle \;=\; 0 \text{ for all } X, Y, Z \in \mathfrak{m}\,.$$

They generalize symmetric spaces: $\nabla^1 T^1 = 0, \nabla^1 \mathcal{R}^1 = 0$.

A oneparametric family of connections

Dfn. $\nabla^t_X Y \;:=\; \nabla^g_X Y - \frac{t}{2}[X,Y]_{\mathfrak{m}}$ for $X, Y \in \mathfrak{m}$.

Torsion: $T^t(X,Y) = -t[X,Y]_{\mathfrak{m}}$.

Special t values:
- $\underline{t=0}$: LC connection
- $\underline{t=1}$: canonical connection
- $\underline{t=1/3}$: "Kostant-Slebarski connection"

M spin manifold $\Rightarrow$ lift ∇^t into spinor bundle, associated Dirac operator:

$$\not{D}^t\psi \;=\; \sum_{i=1}^{n} Z_i(\psi) + \frac{1-t}{2} H\cdot\psi \qquad (Z_1,\ldots,Z_n:\ \text{ONB of } \mathfrak{m}),$$

H: the element in the Clifford algebra induced by torsion:

$$H \;:=\; \frac{3}{2}\sum_{i<j<k} \langle [Z_i,Z_j]_{\mathfrak{m}}, Z_k\rangle\, Z_i\cdot Z_j\cdot Z_k$$

The symmetric case

Want: Weitzenböck formula for $(\not{D}^t)^2$.

For M symmetric ($[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$), one would have:

Thm (Parthasarathy, 1972). $(\not{D})^2 = \Omega_{\mathfrak{g}} + \frac{1}{8}\mathrm{Scal},$

with $\Omega_{\mathfrak{g}}$: Casimir operator of $\mathfrak{g}$.

Consequences:

- Computation of spectrum of $\not{D}$
- Realisation of discrete series representations in the (twisted) kernel of $\not{D}$ for G non compact
- Character formulas (interpret character as an index)

In the homogeneous *non symmetric* case, this formula does no longer hold!

The general Kostant-Parthasarathy formula

Thm [Kostant, '99 / IA, '01]. For $n \geq 5$ and arbitrary t:

$$
\begin{aligned}
(\not{D}^t)^2\psi \;=\; & \Omega_G(\psi) + \tfrac{1}{4}(3t-1)\sum_{i,j,k}\langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k\rangle\, Z_i \cdot Z_j \cdot Z_k(\psi) \\
& -\tfrac{1}{2}\sum_{i<j<k<l}\Big\langle Z_i, \mathcal{J}_{\mathfrak{h}}(Z_j, Z_k, Z_l) + \tfrac{9(1-t)^2}{4}\mathcal{J}_{\mathfrak{m}}(Z_j, Z_k, Z_l)\Big\rangle\, Z_i \cdot Z_j \cdot Z_k \cdot Z_l \cdot \psi \\
& +\tfrac{1}{8}\Big(\sum_{i,j}||[Z_i, Z_j]||_{\mathfrak{h}} + \tfrac{3(1-t)^2}{4}\sum_{i,j}||[Z_i, Z_j]||_{\mathfrak{m}}\Big)\psi
\end{aligned}
$$

Notation:

- $\mathcal{J}_{\mathfrak{m}}(X, Y, Z) := [X, [Y, Z]_{\mathfrak{m}}]_{\mathfrak{m}} +$ cyclic
- $\mathcal{J}_{\mathfrak{h}}(X, Y, Z) := [X, [Y, Z]_{\mathfrak{h}}] +$ cyclic
- Q : the unique $\mathrm{Ad}\, G$-invariant continuation of $\langle\ ,\ \rangle$ to $\mathfrak{g}$. It satisfies:

$$\mathfrak{h} \perp \mathfrak{m}, \quad Q\big|_{\mathfrak{m}} = \langle\ ,\ \rangle\,, \quad Q\big|_{\mathfrak{h}} \text{ not degenerate}$$

The Kostant-Parthasarathy formula for $t = 1/3$

Thm [Kostant, '99 / IA, '01]. For $n \geq 5$ and $t = 1/3$:

$$(\not{D}^{1/3})^2\psi \;=\; \Omega_G(\psi) + \frac{1}{8}(*)\,\psi\,,$$

where $(*)$ denotes the scalar

$$(*) = \sum_{i,j} ||[Z_i, Z_j]||_{\mathfrak{h}} + \frac{1}{3}\sum_{i,j} ||[Z_i, Z_j]||_{\mathfrak{m}}.$$

It can be rewritten as

$$(*) = Q(\varrho_G, \varrho_G) - Q_{\mathfrak{h}}(\varrho_H, \varrho_H)$$

and is thus *always* strictly positive.

First applications

Corollary. If ψ satisfies $\nabla^t\psi = 0$ and $T^t \cdot \psi = 0$ on $M = G/H$, then $t = 0$ and ∇^t is the LC connection.

. . . purely mathematical applications:

Corollary. On $M = G/H$, there exists a G-invariant differential operator of first order which has no symmetric counterpart:

$$\mathcal{D}(\psi) \ := \ \sum_{i,j,k} \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k \rangle \, Z_i \cdot Z_j \cdot Z_k(\psi) \,.$$

Corollary. If the Casimir operator is non negative, the first eigenvalue $\lambda^{1/3}$ satisfies $(\lambda^{1/3})^2 \geq (*)/8$. In particular, $D\!\!\!/^{1/3}$ has then no kernel.

N.B. Character formulas generalize, too $\longrightarrow$ splitting of H-representations into families with similar properties

[$>$ 1999: Kostant, Sternberg, Ramond, Brink. . .]

• Realisation of infinite dimensional representations for G non compact inside kernels of twisted Dirac operators [> 2003, Zierau-Mehdi . . .]

• Computation of the spectrum of $(\not{D}^{1/3})^2$

N.B. Consider lift of isotropy representation, $\widetilde{\mathrm{Ad}} : H \to \mathrm{Spin}(\mathfrak{m})$:

$$\begin{array}{ccc} & & \mathrm{Spin}(\mathfrak{m}) \\ & \overset{\widetilde{\mathrm{Ad}}}{\nearrow} & \downarrow \lambda \\ H & \xrightarrow{\mathrm{Ad}} & \mathrm{SO}(\mathfrak{m}) \end{array}$$

Assume that it contains the trivial representation. Any such spinor induces a section of the spinor bundle $S = G \times_{\kappa(\widetilde{\mathrm{Ad}})} \Delta_n$ if viewed as a constant map $G \to \Delta_n$.

These are exactly the *parallel spinors of the canonical connection*!

Another application: Construction of Lie algebras

Kostant's work was based on the following extension idea for Lie algebras. We formulate his work geometrically:

Let M^n be an *Ambrose-Singer manifold,* i. e.. a Riemannian manifold with a connection ∇ with antisymmetric torsion T s. t.

$$\nabla T \;=\; 0\,, \quad \nabla \mathcal{R} \;=\; 0\,.$$

Assumption: Universal cover of G_T is compact.

$\Rightarrow$ M^n is regular and locally isometric to a homogeneous space G/G_T. The Lie algebra of G is $\mathfrak{g} := \mathfrak{g}_T \oplus \mathbb{R}^n$ and its commutator has to be

[Cleyton/Swann, 2002]

$$\big[A+X, B+Y\big] \;:=\; \big([A,B] - \mathcal{R}(X,Y)\big) + \big(AY - BX - T(X,Y)\big).$$

Bianchi I $\Rightarrow$ $\mathcal{R}$ is *unique*:

Lemma. The curvature of ∇ is proportional to the orthogonal projection onto $\mathfrak{g}_T$,

$$\mathcal{R} : \Lambda^2(\mathbb{R}^n) = \mathfrak{so}(n) \longrightarrow \mathfrak{g}_T, \quad \mathcal{R}(X,Y) = 4\,\mathrm{pr}_{\mathfrak{g}_T}(X \wedge Y).$$

Choose an ONF of 2-forms ω_i for $\mathfrak{g}_T$.

Lemma. The commutator above defines an extension of $\mathfrak{g}_T$ iff

$$T^2 + 4\sum \omega_i^2$$

is a scalar in the Clifford algebra of $\mathbb{R}^n$. [a priori: parts of degree 4 + scalar]

– this identity can be understood as a Kostant-Parthasarathy type formula for the symbol of the operator $\not{D}^{1/3}$.

Construction of naturally reductive spaces

General construction:

Consider $M = G/H$ with restriction of the Killing form to $\mathfrak{m}$:

$$\beta(X,Y) := -\mathrm{tr}(X^tY),\ \langle X,Y\rangle = \beta(X,Y) \text{ for } X,Y \in \mathfrak{m}.$$

Suppose that $\mathfrak{m}$ is an orthogonal sum $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ such that

$$[\mathfrak{h}, \mathfrak{m}_2] = 0,\ [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{m}_2\,.$$

Then the new metric, depending on a parameter $s > 0$

$$\langle X,Y\rangle_s = \begin{cases} 0 & \text{for } X \in \mathfrak{m}_1, Y \in \mathfrak{m}_2 \\ \langle X,Y\rangle & \text{for } X,Y \in \mathfrak{m}_1 \\ s\cdot\langle X,Y\rangle & \text{for } X,Y \in \mathfrak{m}_2 \end{cases}$$

is naturally reductive for $s \neq 1$ w. r. t. the realisation as

$$M = (G\times M_2)/(H\times M_2) =: \overline{G}/\overline{H}\,.$$

[Chavel, 1969; Ziller / D'Atri, 1979]

Jensen metrics on the Stiefel manifold

$M^5 = G/H$ with $G = SO(4)$, $H = \mathrm{SO}(2)$ and embed H in G as $\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \mathrm{SO}(2) \end{array}\right]$. Then $\mathfrak{so}(4) = \mathfrak{so}(2) + \mathfrak{m}$ with $(a \in \mathbb{R},\ X \in \mathcal{M}_{2,2}(\mathbb{R}))$

$$\mathfrak{m} = \left\{ \left[\begin{array}{cc|c} 0 & -a & \\ a & 0 & -X^t \\ \hline & & 0 \quad 0 \\ & X & 0 \quad 0 \end{array}\right] =: (a, X) \right\}.$$

Set $\mathfrak{m}_1 := \{(0, X)\}$ and $\mathfrak{m}_2 := \{(a, 0)\}$ $\Rightarrow$ new metric

$$\langle (a, X), (b, Y) \rangle_s = \frac{1}{2}\beta(X, Y) + \frac{s}{2} a \cdot b\,.$$

Properties: • Two ∇^0-parallel spinors for $s = 1$, none for other values of t and s;

• $\mathrm{Ric}^0 = (2 - s)\mathrm{diag}(0, 1, 1, 1, 1)$, Ricci-flat only for $s = 2$ und $t = 0$.

The square of the Dirac operator

Extend this to non-homogeneous mnfds!

- (M^n, g, ∇) – Riemannian spin mnfd, $T \in \Lambda^3(M^n)$ torsion of ∇,

$$\nabla_X Y \ := \ \nabla^g_X Y + \frac{1}{2} T(X, Y, -) \, .$$

- Lift into spinor bundle: $\nabla_X \psi \ := \ \nabla^g_X \psi + \frac{1}{4}(X \lrcorner T) \cdot \psi$

- A first order differential operator: $\mathcal{D}\psi \ := \ \sum_{k=1}^{n} (e_k \lrcorner T) \cdot \nabla_{e_k} \psi$

- A 4-form from T [appeared in Bianchi I]: $\sigma_{\mathrm{T}} \ := \ \frac{1}{2} \sum_{k=1}^{n} (e_k \lrcorner T) \wedge (e_k \lrcorner T)$

- $D\!\!\!/$: Dirac operator of connection ∇ with torsion T

- $D\!\!\!/^{1/3}$: Dirac operator of connection with torsion $T/3$.

“Classical” Weitzenböck Formula:

$$D^2 = \Delta_T + \frac{3}{4}d\mathrm{T} - \frac{1}{2}\sigma_T + \frac{1}{2}\delta T - \mathcal{D} + \frac{1}{4}\mathrm{Scal}^{\nabla}.$$

Rescaled Weitzenböck Formula:

$$(D^{1/3})^2 = \Delta_T + \frac{1}{4}dT + \frac{1}{4}\mathrm{Scal}^g - \frac{1}{8}||T||^2.$$

[1/3-Rescaling: Slebarski ('87), Bismut ('89), Kostant ('99), IA ('02), IA & TF ('03)]

A Vanishing Thm. Let (M^n, g, T) be a compact Riemannian spin mnfd with $\mathrm{Scal}^g \leq 0$, and suppose dT acts on spinors as a negative endomorphism. If there exists a spinor $\psi \neq 0$ in the kernel of Δ_T, then $T = 0 = \mathrm{Scal}^g$, and ψ is parallel w.r.t. the LC connection.

Corollary. On a Calabi-Yau or Joyce mnfd $(\mathrm{Scal}^g = 0)$, a metric connection with closed torsion $(dT = 0)$ can have parallel spinors only for $\mathrm{T} = 0$.

$\Rightarrow$ “Rigidity” of vacuum solutions under deformation of the connection

N.B. Different situation if M^n is not compact:

Consider solvmanifolds $Y^7 = N \times \mathbb{R}$, $\mathfrak{n}$: nilpotent 6-dim. Lie algebra $(\neq \mathfrak{h}_3 \oplus \mathfrak{h}_3) \Rightarrow$ [Chiossi/Fino, 2004]

1) N carries "half flat" $\mathrm{SU}(3)$ structure,

2) Y carries a G_2 structure (ω, g) with antisymmetric torsion,

3) Y carries – after a conformal change of the metric – an *integrable* G_2 structure $(\tilde{\omega}, \tilde{g})$. In particular, $\tilde{g}$ is Ricci flat und admits (at least) one LC-parallel spinor.

Thm ('05). For $\mathfrak{n} \cong (0, 0, e_{15}, e_{25}, 0, e_{12})$, there exists on $(Y, \tilde{\omega}, \tilde{g})$ a oneparametric family $(T_h, \psi_h) \in \Lambda^3(Y) \times S(Y)$ s. t. every connection ∇^h with torsion T_h satisfies:

$$\nabla^h \psi_h = 0.$$

For $h = 1$: $T_h = 0, \nabla^h = \nabla^g$ und ψ_h coincides with the LC-parallel spinor.

Parallel spinors in oneparametric families

Define the family of connections

$$\nabla^{s}_{X}\psi \;:=\; \nabla^{g}_{X}\psi + s\cdot(X\lrcorner\mathrm{T})\cdot\psi\,.$$

Q: For which values of s can there be parallel spinors ?

Example. G a simple Lie group, g its biinvariant metric and the torsion form

$$T(X,Y,Z) := g([X,Y],Z)\,.$$

The connections $\nabla^{\pm 1/4}$ are flat. In particular, there exist $\nabla^{\pm 1/4}$-parallel spinor fields.

Thm. Assume M compact. Every ∇^{s}-parallel spinor ψ satisfies the eq.

$$64\,s^{2}\int_{M^n}\langle\sigma_{\mathrm{T}}\cdot\psi\,,\,\psi\rangle \;+\; \int_{M^n}\mathrm{Scal}^{s}\cdot||\psi||^{2} \;=\; 0\,.$$

If the mean values $\langle \sigma_{\mathrm{T}} \cdot \psi \,,\, \psi \rangle$ does not vanish, the parameter s is given by

$$s \;=\; \frac{1}{8} \int_{M^n} \langle d\mathrm{T} \cdot \psi \,,\, \psi \rangle \Big/ \int_{M^n} \langle \sigma_{\mathrm{T}} \cdot \psi \,,\, \psi \rangle .$$

If $\langle \sigma_{\mathrm{T}} \cdot \psi \,,\, \psi \rangle = 0$, the parameter s depens only on the Riemannian scalar curvature and the length of T,

$$0 \;=\; \int_{M^n} \mathrm{Scal}^s \;=\; \int_{M^n} \mathrm{Scal}^g \;-\; 24 s^2 \int_{M^n} ||\mathrm{T}||^2 \,.$$

Corollary. If the 4-forms $d\mathrm{T}$ and σ_{T} are proportional, there exist at most three parameter values with ∇^s-parallel spinors.

• On the Aloff-Wallach spaces $M^7 := N(1,1) = \mathrm{SU}(3)/\mathrm{SU}(2)$, there exist examples of G_2-structures such that s and $-s$ admit parallel spinors.

Sometimes, the value of s is fixed through the geometry of M:

Thm. On a 5-dimensional Sasaki mnfd, only the characteristic connection can have parallel spinors.

The Casimir operator of a characteristic connection

(M^n, g, ∇, T): Riemannian manifold with torsion.

Dfn. The **Casimir operator** acting on spinor fields is defined by

$$\begin{aligned}\Omega \quad &:= \quad (D^{1/3})^2 + \frac{1}{8}(dT - 2\sigma_T) + \frac{1}{4}\delta(T) - \frac{1}{8}\mathrm{Scal}^g - \frac{1}{16}||T||^2 \\ &= \quad \Delta_T + \frac{1}{8}(3dT - 2\sigma_{\mathrm{T}} + 2\delta(\mathrm{T}) + \mathrm{Scal}).\end{aligned}$$

Motivation: For a naturally reductive space and its canonical connection, Ω coincides with the usual Casimir operator.

Example: For the Levi-Civita connection $(T = 0)$, we obtain:

$$\Omega \; = \; (D^g)^2 - \frac{1}{8}\mathrm{Scal}^g \; = \; \Delta^g + \frac{1}{8}\mathrm{Scal}^g$$

Proposition. The kernel of the Casimir operator contains all ∇-parallel spinors.

The case $\nabla T = 0$: Ω then simplifies,

$$\begin{aligned}\Omega &= (D^{1/3})^2 - \frac{1}{16}\big(2\,\mathrm{Scal}^g + \|T\|^2\big) \\ &= \Delta_T + \frac{1}{16}\big(2\,\mathrm{Scal}^g + \|T\|^2\big) - \frac{1}{4}T^2 \\ &= \Delta_T + \frac{1}{8}\big(2\,dT + \mathrm{Scal}\big)\,.\end{aligned}$$

Proposition. (M^n, g, ∇) compact, $\nabla T = 0$. If

$$2\,\mathrm{Scal}^g \leq -\|T\|^2 \quad \text{or} \quad 2\,\mathrm{Scal}^g \geq 4\,T^2 - \|T\|^2$$

holds, the Casimir operator is non-negative.

Proposition. If $\nabla T = 0$, Ω and $(D^{1/3})^2$ commute with T,

$$\Omega \circ T = T \circ \Omega\,, \quad (D^{1/3})^2 \circ T = T \circ (D^{1/3})^2\,.$$

In the compact case, T preserves the kernel of $D^{1/3}$.

5-Dimensional Sasakian Manifolds

- M^5: a 5-dimensional Sasakian manifold, η its contact structure.
- Consider characteristic connection with torsion T:

$$\begin{aligned} \nabla T &= 0\,, \quad T = \eta \wedge d\eta = 2\,(e_{12} + e_{34}) \wedge e_5\,, \\ T^2 &= 8 - 8\,e_{1234}\,, \quad T = \mathrm{diag}(4\,,0\,,0\,,-4)\,. \end{aligned}$$

$\Rightarrow$ the Casimir operator splits into $\Omega = \Omega_0 \oplus \Omega_4 \oplus \Omega_{-4}$,

$$\Omega_0 = \Delta_T + \frac{1}{8}\mathrm{Scal}^g + \frac{1}{2} = (D^{1/3})^2 - \frac{1}{8}\mathrm{Scal}^g - \frac{1}{2},$$

$$\Omega_{\pm 4} = \Delta_T + \frac{1}{8}\mathrm{Scal}^g - \frac{7}{2} = (D^{1/3})^2 - \frac{1}{8}\mathrm{Scal}^g - \frac{1}{2}.$$

- If $\mathrm{Scal}^g \neq -4$, $\mathrm{Ker}(\Omega_0) = 0$.
- If $\mathrm{Scal}^g < -4$ or $\mathrm{Scal}^g > 28$, $\mathrm{Ker}(\Omega_{\pm 4}) = 0$.
- The interesting cases: $-4 \leq \mathrm{Scal}^g \leq 28$.

If $\mathrm{Scal}^g = -4$: $\quad \Omega_0 = \Delta_T = (D^{1/3})^2, \quad \Omega_{\pm 4} = \Delta_T - 4 = (D^{1/3})^2.$

• The kernel of Ω_0 coincides with the space of ∇-parallel spinors ψ such that $T \cdot \psi = 0$. [Examples: TF/Ivanov, 2002]

• Spinors in both kernels $\mathrm{Ker}(\Omega_0)$ and $\mathrm{Ker}(\Omega_{\pm 4})$ exist on the 5-dimensional Heisenberg group

$$
\begin{aligned}
e_1 &= dx_1/2, \quad e_2 = dy_1/2, \quad e_3 = dx_2/2, \quad e_4 = dy_2/2, \\
e_5 &= \eta \ := \ \big(dz - y_1\, dx_1 - y_2\, dx_2\big)/2.
\end{aligned}
$$

• Spinors in the kernel of $\Omega_{\pm 4}$ occur on Sasakian η-Einstein manifolds of type $\mathrm{Ric}^g = -\,2 \cdot g \, + \, 6 \cdot \eta \otimes \eta$ [Examples: TF/Kim, 2000]

If $\mathrm{Scal}^g = 28$: $\Omega_0 = \Delta_T + 4 = (D^{1/3})^2 - 4, \quad \Omega_{\pm 4} = \Delta_T = (D^{1/3})^2 - 4.$

• The kernel of $\Omega_{\pm 4}$ coincides with the space of ∇-parallel spinors ψ such that $T \cdot \psi = \pm 4\psi$.

Einstein-Sasaki manifolds, $\mathrm{Scal}^g = 20$**:**

$$\Omega_0 = \Delta_T + 3\,, \quad \Omega_{\pm 4} = \Delta_T - 1 = (D^{1/3})^2 - 3\,.$$

Thm. The Casimir operator of a compact 5-dimensional Einstein-Sasaki manifold has trivial kernel.

Example: Stiefel manifold $\mathrm{V}_{4,2} = \mathrm{SO}(4)/\mathrm{SO}(2)$ with its Einstein-Sasaki metric. There exist Riemannian Killing spinors. The Casimir operator is equivalent to the operators

$$\Omega_0 = -3\sum_{\alpha=1}^{5} X_\alpha^2 + 3, \quad \Omega_{\pm 4} = -3\sum_{\alpha=1}^{5} X_\alpha^2 - \frac{3}{4} \pm \sqrt{3}i \cdot X_5$$

acting on functions $f : \mathrm{SO}(4) \to \mathbb{C}$ satisfying the quasi-periodicity conditions $E_{34}(f) = \pm\, i\, f$ and $E_{34}(f) = 0$, respectively.

6-Dimensional nearly Kähler manifolds

- (M^6, g, J): 6-dimensional nearly Kähler manifold, Kähler form Ω.
- M^6 is Einstein, $\mathrm{Ric}^g = \frac{5a}{2} g, \quad a > 0$.
- Consider its characteristic connection with torsion $T = N/4$:

$$\nabla T = 0, \quad \mathrm{Ric}^{\nabla} = 2a\, g, \quad 2\sigma_T = dT = a\,\Omega \wedge \Omega, \quad ||T||^2 = 2a.$$

- We compute

$$2\, dT + \mathrm{Scal} \; = \; 16\, a \cdot \mathrm{diag}(\,0\,,\,0\,,\,1\,,\,1\,,\,1\,,\,1\,,\,1\,,\,1).$$

$$\Omega \; = \; \Delta_T + \frac{1}{8}(2\, dT + \mathrm{Scal}) \; = \; (D^{1/3})^2 - 2\, a$$

- If M^6 is compact, then $\mathrm{Ker}(\Omega) = \mathrm{Ker}(\nabla) = \{\mathrm{Killingspinors}\}$ and

$$(D^{1/3})^2 \; \geq \; \frac{2}{15} \mathrm{Scal}^g \; = \; 2 \cdot a \; > \; 0\,.$$

7-Dimensional G_2-manifolds

• (M^7, g, ω) cocalibrated G_2-manifold (type $W_3 \oplus W_4 \Leftrightarrow d * \omega = 0$), and suppose that $(d\omega\,,\, *\,\omega)$ is constant.

• Its characteristic connection:

$$T \;=\; -\,*\,d\omega + \frac{1}{6}(d\omega\,,\,*\omega)\cdot\omega\,,\quad \delta(T) \;=\; 0\,.$$

• Main difference to the previous examples: $\nabla T \neq 0,\ dT \neq 2\sigma_T$.

• Scalar curvature: $\mathrm{Scal}^g = 2(T,\omega)^2 - \frac{1}{2}||T||^2$.

• The parallel spinor ψ_0 corresponding to ω satisfies

$$\nabla\psi_0 \;=\; 0\,,\quad T\cdot\psi_0 \;=\; -\frac{1}{6}(d\omega\,,\,*\omega)\cdot\psi_0\,.$$

- Casimir operator:

$$\Omega \;=\; (D^{1/3})^2 - \frac{1}{4}(T\,,\,\omega)^2 + \frac{1}{8}(dT \;-\; 2\,\sigma_T)$$
$$\;=\; \Delta_T + \frac{1}{4}(T\,,\,\omega)^2 + \frac{1}{8}(3\,dT - 2\,\sigma_T - 2\,||T||^2).$$

Nearly parallel G_2-structures (type W_1): $d\omega = -\,a\,(*\omega)$.

$$\Omega \;=\; (D^{1/3})^2 \;-\; \frac{49}{144}\,a^2.$$

Thm. Let (M^7, g, ω) be a compact, nearly parallel G_2-manifold and denote by ∇ its characteristic connection. The kernel of the Casimir operator of the triple (M^7, g, ∇) coincides with the space of ∇-parallel spinors,

$$\mathrm{Ker}(\Omega) = \{\psi \,:\, \nabla\psi \;=\; 0,\; T\cdot\psi \;=\; \frac{7}{6}\,a\cdot\psi\} = \mathrm{Ker}(\nabla).$$