26^d **Winter School in Geometry and Physics**

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Special geometries, holonomy and string theory

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Geometric structures with parallel characteristic torsion

- Naturally reductive space $(G/H, \nabla^c, \mathrm{T}^c)$:

$$\nabla^c \mathrm{T}^c \ = \ 0, \quad \nabla^c \mathrm{R}^c \ = \ 0 \,.$$

A larger category:

$(M^n, g, \mathcal{R}, \nabla^c)$ – Riemannian manifolds with a geometric structure admitting a characteristic connection such that

$$\nabla^c \mathrm{T}^c \ = \ 0 \,.$$

- The holonomy of the connection ∇^c preserves not only the geometric structure, but also a non-trivial 3-form T^c.

- The condition $\nabla^c \mathrm{T}^c = 0$ implies

$$\delta(\mathrm{T}^c) \ = \ 0 \,, \quad d\mathrm{T}^c \ = \ 2 \cdot \sigma_{\mathrm{T}^c} \ = \ \sum_{i=1}^{n} (e_i \lrcorner \mathrm{T}^c) \wedge (e_i \lrcorner \mathrm{T}^c) \,.$$

• The formula for the Casimir operator of the tuple $(M^n, g, \mathcal{R}, \nabla^c)$ simplifies,

$$\begin{aligned}\Omega &= (D^{1/3})^2 - \frac{1}{16}\big(2\,\mathrm{Scal}^g + \|\mathrm{T}^c\|^2\big) \\ &= \Delta_{\mathrm{T}^c} + \frac{1}{16}\big(2\,\mathrm{Scal}^g + \|\mathrm{T}^c\|^2\big) - \frac{1}{4}(\mathrm{T}^c)^2 \\ &= \Delta_{\mathrm{T}^c} + \frac{1}{8}\big(2\,d\mathrm{T}^c + \mathrm{Scal}\big)\,.\end{aligned}$$

• Ω and $(D^{1/3})^2$ commute with the endomorphism T^c,

$$\Omega \circ \mathrm{T}^c = \mathrm{T}^c \circ \Omega\,, \quad (D^{1/3})^2 \circ \mathrm{T}^c = \mathrm{T}^c \circ (D^{1/3})^2\,.$$

In the compact case, T^c preserves the kernel of $D^{1/3}$.

First example:

$(M^{2k+1}, g, \eta, \xi, \varphi)$ – Sasakian manifold. It admits a characteristic connection and

$$\mathrm{T}^c = \eta \wedge d\eta\,, \quad \nabla^c \mathrm{T}^c = 0.$$

Second example:

Any nearly parallel G_2-manifold (M^7, g, ω^3) admits a characteristic connection with $\nabla^c \mathrm{T}^c = 0$.

Third example: (Matsumoto/Takamatsu/Gray/Kirichenko, 1970-1978)

Any nearly Kähler manifold admits a characteristic connection with $\nabla^c \mathrm{T}^c = 0$.

In dimension $n = 6$, this result implies:

Any nearly Kähler M^6 is Einstein, is a spin manifold and the first Chern class vanishes, $c_1(M^6) = 0$.

Counterexamples: G_2-structure of type $\mathcal{W}_3$

Consider a G_2-manifold (M^7, g, ω^3) of pure type $\mathcal{W}_3$,

$$d * \omega^3 \;=\; 0\,, \quad (d\omega^3\,, *\,\omega^3) \;=\; 0\;.$$

- The torsion T^c and the parallel spinor Ψ_0:

$$\mathrm{T}^c \;=\; -\,*\,d\omega^3\,, \quad \mathrm{Scal}^g \;=\; -\frac{1}{2}\,||\mathrm{T}^c||^2\,,$$

$$\nabla^c\Psi_0 \;=\; 0\,, \quad \mathrm{T}^c\cdot\Psi_0 \;=\; 0\,, \quad \omega^3\cdot\Psi_0 \;=\; 0\,.$$

- In general we have $\nabla^c\mathrm{T}^c \neq 0$,

$$\delta(\mathrm{T}^c) \;=\; 0\,, \quad d\mathrm{T}^c - 2\cdot\sigma_{\mathrm{T}^c} \;\neq\; 0\;.$$

- The Casimir operator (general formula)

$$\begin{aligned} \Omega &= (D^{1/3})^2 + \frac{1}{8}(d\mathrm{T}^c - 2\,\sigma_{\mathrm{T}^c}) \\ &= \Delta_{\mathrm{T}^c} + \frac{1}{8}(3\,d\mathrm{T}^c - 2\,\sigma_{\mathrm{T}^c} - 2\,||\mathrm{T}^c||^2)\ . \end{aligned}$$

Explicite counterexample: On the manifold $N(1,1) = \mathrm{SU}(3)/\mathrm{S}^1$ there exist G_2-structures of pure type $\mathcal{W}_3$ such that the operators

$$\Omega - (D^{1/3})^2\,, \quad \Omega - \Delta_{\mathrm{T}^c}$$

are negative or positive.

(Ref. Agricola/Friedrich, *Math. Ann.* and *Journ. Geom. Phys.*, 2004)

A second explicite counterexample:

Consider the 3-dimensional complex solvable group N^6 as well as $M^7 := N^6 \times \mathbb{R}^1$. There exists a left invariant metric and a left invariant G_2-structure on M^7 such that the structure equations are:

$$de_1 \;=\; de_2 \;=\; de_7 \;=\; 0$$
$$de_3 \;=\; e_1 \wedge e_3 - e_2 \wedge e_4, \quad de_4 \;=\; e_2 \wedge e_3 + e_1 \wedge e_4$$
$$de_5 \;=\; -e_1 \wedge e_5 + e_2 \wedge e_6, \quad de_6 \;=\; -e_2 \wedge e_5 - e_1 \wedge e_6 \,.$$

The G_2-structure is of pure type $\mathcal{W}_3$ and we obtain:

- $\mathrm{T}^c = 2 \cdot e_{256} - 2 \cdot e_{234}, \quad \delta(\mathrm{T}^c) = 0, \quad d\mathrm{T}^c - 2 \cdot \sigma_{\mathrm{T}^c} \neq 0.$
- $\mathrm{Scal}^{\nabla^c} = -16\,, \quad \mathrm{div}\big(\mathrm{Ric}^{\nabla^c}\big) \;=\; 0.$
- There are two ∇^c-parallel spinors and any of these satisfies the equation $\mathrm{T}^c \cdot \Psi = 0$.

General working program:

Fix a compact Lie group $\mathrm{G} \subset \mathrm{SO}(n)$ Study the class of n-dimensional Riemannian manifolds (M^n, g) equipped with a G-structure $\mathcal{R} \subset \mathcal{F}(M^n)$ such that the G-structure admits a characteristic connection ∇^c with parallel torsion form,

$$\nabla^c \mathrm{T}^c \ = \ 0 \ .$$

Results:
The case $n = 6\, , \mathrm{G} = \mathrm{U}(3) \subset \mathrm{SO}(6)$:
B. Alexandrov/Th.Friedrich/N. Schoemann, Journ. Geom. Phys. 2005
N. Schoemann, PhD 2006.

The case $n = 7\, , \mathrm{G} = \mathrm{G}_2 \subset \mathrm{SO}(7)$: Friedrich, preprint in 2006.

Almost hermitian geometries with parallel characteristic torsion

- (M^6, g, J): almost hermitian 6-manifold, $\mathfrak{so}(6) = \mathfrak{u}(3) \oplus \mathfrak{m}^6$
- $\Gamma \in \mathbb{R}^6 \otimes \mathfrak{m}^6 =_{\mathrm{U}(3)} \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$
- $\mathcal{W}_1$ – nearly Kähler manifolds, $\dim_{\mathbb{R}}(\mathcal{W}_1) = 2$
- $\mathcal{W}_3 \oplus \mathcal{W}_4$ – hermitian manifolds $(\mathrm{N}_J = 0)$, $\dim_{\mathbb{R}}(\mathcal{W}_3) = 12$, $\dim_{\mathbb{R}}(\mathcal{W}_4) = 6$

Thm. An almost hermitian manifold admits a characteristic connection if and only if it is of type $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$. This condition is equivalent to the condition that the Nijenhuis tensor N_J is totally skew symmetric. The characteristic torsion is given by

$$\mathrm{T}^c = J(d\,\Omega) + \mathrm{N}_J \, .$$

First case:
The $\mathcal{W}_4$-part of Γ does not vanish. Since it is basically a vector field, it induces an action of the abelian group $\mathbb{C}$.

Thm. A compact, regular hermitian manifold M^6 with ∇^c-parallel characteristic torsion T^c and a nontrivial $\mathcal{W}_4$-part of Γ is a T^2-bundle over a 4-dimensional compact Kähler manifold X^4. The bundle is defined by two parallel, anti-self dual forms Ω_1, Ω_2 on X^4 such that

$$2 \cdot \Omega_2 \ , \ 2 \cdot \Omega + 2 \cdot \Omega_1 \in \mathrm{H}^2(X^4; \mathbb{Z}) \ .$$

- The admissible Kähler surfaces X^4 are products $\Sigma_1^2 \times \Sigma_2^2$ of 2-dimensional manifolds.

Second case:

M^6 is of type $\mathcal{W}_3$. Then J is integrable and

$$d\,\Omega \;=\; -\,*\,\mathrm{T}^c\;,\;\delta\,\Omega\;=\;0.$$

The 3-form T^c belongs to the 12-dimensional representation $\mathcal{W}_3$ defined by

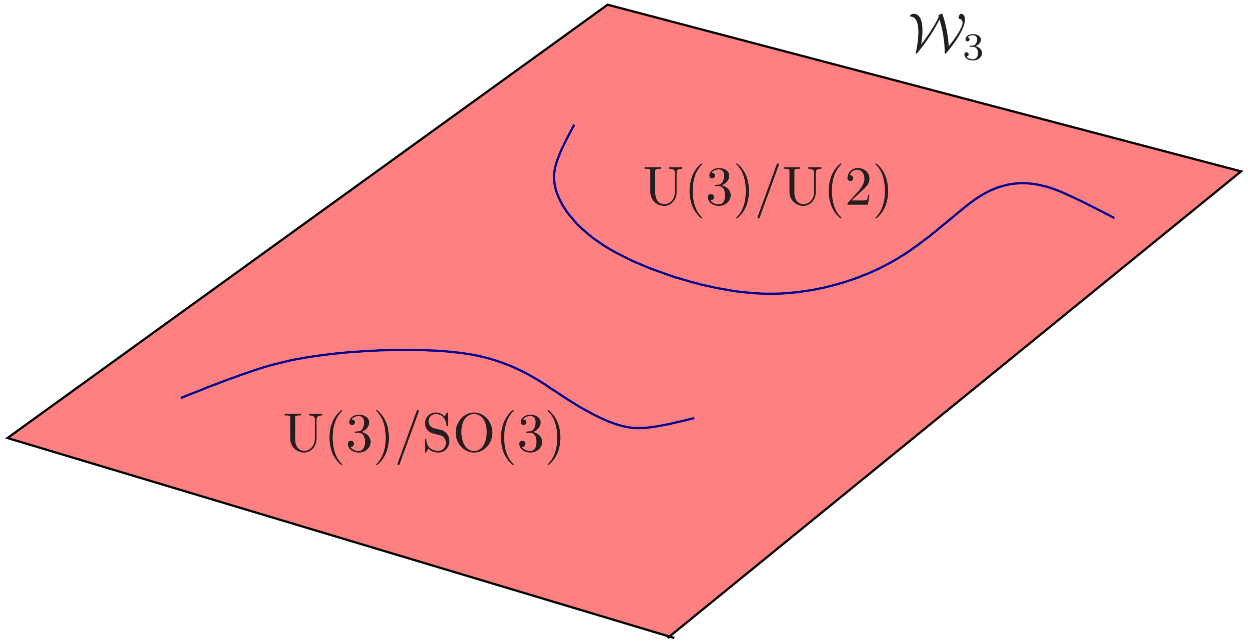
$$J(\mathrm{T}) \;=\; *\mathrm{T}, \quad \tau(\mathrm{T}) = -\,\mathrm{T},$$

where τ is the action of the central element $\Omega \in \mathfrak{u}(3)$ on 3-forms.

- If $\nabla^c \mathrm{T}^c = 0$, then the orbit type of $\mathrm{T}^c \in \mathcal{W}_3$ is an invariant of the hermitian manifold M^6.

Consequence: We need the whole orbit type structure of the 12-dimensional representation $\mathcal{W}_3$ under the action of the 9-dimensional group $\mathrm{U}(3)$.

Basic Theorem for the Classification: There are exactly two orbits in $\mathcal{W}_3$ with a non-abelian isotropy group.



Thm: Let M^6 be a compact hermitian manifold of type $\mathcal{W}_3$ such that

$$\nabla^c \mathrm{T}^c \;=\; 0, \quad G_{\mathrm{T}^c} \;=\; \mathrm{U}(2).$$

Then M^6 is the twistor space of a 4-dimensional compact selfdual Einstein space with positive scalar curvature. J is the standard complex structure of the twistor space, but the metric is the unique, non-Kählerian Einstein metric of the twistor space.

Remark. There are only two such spaces,

$$M^6 \;=\; \mathbb{CP}^3, \quad \mathbb{F}(1,2).$$

Thm. Let M^6 be a compact hermitian manifold of type $\mathcal{W}_3$ such that

$$\nabla^c \mathrm{T}^c \;=\; 0 \;,\; G_{\mathrm{T}^c} \;=\; \mathrm{SO}(3) \;.$$

Then M^6 is is locally isomorphic to $\mathrm{SL}(2,\mathbb{C})$ equipped with a left-invariant hermitian structure.

Strominger equations on this space:
There exists a spinor field Ψ on M^6 such that

$$\mathrm{Ric}^{\nabla^c} \;=\; -\frac{1}{3}\cdot ||\mathrm{T}^c||^2 \cdot \mathrm{Id} \quad , \quad \delta\, \mathrm{T}^c \;=\; 0 \;,$$

$$\nabla^c \Psi \;=\; 0 \quad , \quad \mathrm{T}^c \cdot \Psi \;=\; 0 \;.$$

Geometric structures of vectorial type

Ref. Agricola/Friedrich, math.dg/0509147

- $(M^n, g, \mathcal{R})$ – Riemannian manifold with a geometric structure,
- $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$ – the decomposition of the Lie algebra,
- $\Gamma \in \mathbb{R}^n \otimes \mathfrak{m}$ – the intrinsic torsion.
- A universal embedding $\mathbb{R}^n \longrightarrow \mathbb{R}^n \otimes \mathfrak{m}$

$$\Theta_1 \, : \, \mathbb{R}^n \longrightarrow \mathbb{R}^n \otimes \mathfrak{m} \, , \quad \Theta_1(\Gamma) \; = \; \sum_{i=1}^{n} e_i \otimes \mathrm{pr}_{\mathfrak{m}}(e_i \wedge \Gamma) \; .$$

Definition: Let M^n be an oriented Riemannian manifold and denote by $\mathcal{F}(M^n)$ its frame bundle. A geometric structure $\mathcal{R} \subset \mathcal{F}(M^n)$ is called of vectorial type if its intrinsic torsion belongs to $\Gamma \in \mathbb{R}^n \subset \mathbb{R}^n \otimes \mathfrak{m}$.

Remark: These geometric structures are usually called $\mathcal{W}_4$-structures .

Proposition: If a G-structure is of vectorial type, then there exists a unique metric connection ∇^{vec} of vectorial type in the sense of Cartan and preserving the G-structure. The formula is

$$\nabla^{\mathrm{vec}}_X Y \;=\; \nabla^g_X Y \,-\, g(X\,,Y)\cdot\Gamma \,+\, g(Y\,,\Gamma)\cdot X\,.$$

Conversely, if a G-structure $\mathcal{R}$ admits a connection of vectorial type in the sense of Cartan, then $\mathcal{R}$ is of vectorial type in our sense.

• Consider a conformal change of $g^* := e^{2f}g$ and define a new G-structure $\mathcal{R}^* \subset \mathcal{F}(M^n, g^*)$ by

$$\mathcal{R}^* \;=\; \Big\{(e^{-f}\cdot e_1\,,\; e^{-f}\cdot e_2\;\ldots\;,\; e^{-f}\cdot e_n)\;:\;(e_1\,,\;e_2\,,\;\ldots\;,\;e_n)\in\mathcal{R}\Big\}\,.$$

The intrinsic torsion changes by the element $df \in \mathbb{R}^n \subset \mathbb{R}^n \otimes \mathfrak{m}$,

$$\Gamma^* \;=\; \Gamma + df\,,\quad d\Gamma \;=\; d\Gamma^*\,.$$

On the other side, starting with an arbitrary geometric structure on a compact manifold, the equation

$$0 \;=\; \delta^{g^*}(\Gamma^*) \;=\; \delta^g(\Gamma) + \Delta(f) + (n-2)\cdot\big((df,\Gamma) + ||df||^2\big)$$

has a unique solution $f = -\,\Delta^{-1}(\delta^g(\Gamma))$.

Proposition: An arbitrary geometric structure of vectorial type on a compact manifold admits a conformal change such that the new 1-form is coclosed, $\delta^{g^*}(\Gamma^*) = 0$.

Proposition: Let $\mathrm{G} \subset \mathrm{SO}(n)$ be a subgroup such that

1. there exists a G-invariant differential form Ω^k of some degree k, and
2. the multiplication $\Omega^k : \Lambda^2(\mathbb{R}^n) \to \Lambda^{k+2}(\mathbb{R}^n)$ is injective.

Then, for any G-structure of vectorial type, the 1-form Γ is closed, $d\Gamma = 0$.

Remark: The groups $\mathrm{G}_2 \subset \mathrm{SO}(7)$ and $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$ satisfy the conditions of the Proposition. Consequently, it generalizes results of Cabrera (1995, 1996) . Moreover, there are other groups satisfying the conditions, namely $\mathrm{U}(n) \subset \mathrm{SO}(2n)$ for $n > 2$ and $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$.

Remark: $\mathrm{SO}(3) \subset \mathrm{SO}(5)$ (the irreducible representation) does not admit any invariant differential form. $\mathrm{SO}(n-1) \subset \mathrm{SO}(n)$ and $\mathrm{U}(2) \subset \mathrm{SO}(4)$ admit invariant forms, but the second condition of the Proposition is not satisfied. In these geometries the condition $d\Gamma = 0$ is an additional requirement on the geometric structure of vectorial type.

Example: Consider the subgroup $\mathrm{G} = \mathrm{SO}(n-1) \subset \mathrm{SO}(n)$. A G-structure on (M^n, g) is a vector field Ω (a 1-form) of length one. The geometric structure is of vectorial type if and only if there exists a vector field Γ (a 1-form) such that

$$0 \;=\; \nabla^{\mathrm{vec}}_X \Omega \;=\; \nabla^g_X \Omega \,-\, g(X\,,\,\Omega)\,\Gamma \,+\, g(\Omega\,,\,\Gamma)\,X$$

holds. This condition implies that Ω defines a codimension one foliation on M^n,

$$d\Omega \;=\; \Omega \wedge \Gamma\ .$$

Moreover, the second fundamental form of any leave $F^{n-1} \subset M^n$ is given by the formula $\mathrm{II}(X) = -\,g(\Omega\,,\,\Gamma) \cdot X\,, \quad X \in TF^{n-1}$. Therefore, the leaves are umbilic. In consequence,

$\mathrm{SO}(n-1)$-structures of vectorial type coincide with umbilic foliations of codimension one.

Γ satisfies the condition $\Omega \wedge d\Gamma = 0$, but in general it does not have to be closed.

Theorem: Let $\mathrm{G} \subset \mathrm{SO}(n)$ be a subgroup lifting into the spin group and suppose that there exists a G-invariant spinor $0 \neq \Psi \in \Delta_n$. Moreover, suppose that $n \geq 5$ is at least five. Then Γ is closed, $d\Gamma = 0$. The Ricci tensor is given by

$$\mathrm{Ric}^g(X) \ = \ (n-2)\,\nabla^g_X \Gamma \ - \ \delta^g(\Gamma) \cdot X \ + \ A(X\,,\,\Gamma)\,.$$

where the vector $A(X, \Gamma)$ is defined by

$$A(X\,,\,\Gamma) \ := \begin{cases} 0 & \text{if } X \text{ and } \Gamma \text{ are proportional} \\ (n-2)||\Gamma||^2 \cdot X & \text{if } X \text{ and } \Gamma \text{ are orthogonal} \end{cases}$$

The scalar curvature Scal^g can be expressed by Γ,

$$\mathrm{Scal}^g \ = \ 2\,(1 \ - \ n)\,\delta^g(\Gamma) \ + \ (n \ - \ 1)(n \ - \ 2)\,||\Gamma||^2.$$

Remark: The conditions of the latter theorem are satisfied for the groups $\mathrm{G}_2 \subset \mathrm{SO}(7)$ and $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$. The subgroups $\mathrm{U}(n) \subset \mathrm{SO}(2n)$ or $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$ do *not* satisfy the conditions, there are no invariant spinors.

Corollary: Suppose that the subgroup $\mathrm{G} \subset \mathrm{SO}(n)$ lifts into the spin group and admits an invariant spinor $0 \neq \Psi \in \Delta_n$. Then, for any G-structure of vectorial type, we have

$$g(\mathrm{Ric}^g(\Gamma)\,,\,\Gamma) \;=\; \frac{(n-2)}{2} \cdot \Gamma(||\Gamma||^2) \;-\; \delta^g(\Gamma) \cdot ||\Gamma||^2 .$$

If the manifold M^n is compact, then

$$\int_{M^n} g(\mathrm{Ric}^g(\Gamma)\,,\,\Gamma) \;=\; \frac{(n-4)}{2} \cdot \int_{M^n} \delta^g(\Gamma) \cdot ||\Gamma||^2 .$$

Corollary: Let $\mathrm{G} \subset \mathrm{SO}(n)$ be a subgroup that can be lifted into the spin group and suppose that there exists a spinor G-invariant $0 \neq \Psi \in \Delta_n$ $(n \geq 5)$. Consider a G-structure of vectorial type on a compact manifold and suppose that $\delta^g(\Gamma) = 0$ holds. Then we have

1. $\nabla^g \Gamma = 0$.
2. $\mathrm{Ric}^g(\Gamma) = 0.$
3. If X is orthogonal to Γ, then $\mathrm{Ric}^g(X) = (n-1) \cdot ||\Gamma||^2 \cdot X$.
4. The scalar curvature is positive

$$\mathrm{Scal}^g \;=\; (n-1)(n-2)||\Gamma||^2 \; > 0\,.$$

5. The universal covering $\tilde{M}^n = Y^{n-1} \times \mathbb{R}^1$ splits into $\mathbb{R}$ and an Einstein manifold Y^{n-1} with positive scalar curvature admitting a real Riemannian Killing spinor.

Remark: For $\mathrm{G} = \mathrm{G}_2$ ($n = 7$) and $\mathrm{G} = \mathrm{Spin}(7)$ ($n = 8$) the latter result has been obtained by Ivanov/Parton/Piccinni, math.dg/0509038 .

Geometric structures of vectorial type admitting a characteristic connection

• $\mathcal{R} \subset \mathcal{F}(M^n)$ – a structure of vectorial type admitting a characteristic connection.

• We have two connections ∇^{vec} and ∇^c preserving the G-structure.

• The link between Γ and T^c: $\quad 2 \cdot (X \wedge \Gamma) + X \lrcorner \mathrm{T}^c \in \mathfrak{g}$.

• In the sense of G-representations, $\mathbb{R}^n \subset \Lambda^3(\mathbb{R}^n)$ is a necessary condition !

Example: For the subgroups $\mathrm{G} = \mathrm{SO}(3) \subset \mathrm{SO}(5)$, $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$ or $\mathrm{G} = \mathrm{F}_4 \subset \mathrm{SO}(26)$ this condition is not satisfied.

Example: In dimensions $n = 7, 8$ any G_2- or $\mathrm{Spin}(7)$-structure of vectorial type admits a characteristic connection.

Theorem: Let $\mathrm{G} \subset \mathrm{SO}(n)$ be a subgroup lifting into the spin group and suppose that there exists a G-invariant spinor $0 \neq \Psi \in \Delta_n$. Consider a G-structure of vectorial type that admits a characteristic connection. Then we have

$$\begin{aligned}
\big(\Gamma \lrcorner \mathrm{T}^c\big) \cdot \Psi &= 0\,, \quad \delta(\mathrm{T}^c) \cdot \Psi = 0\,, \quad \mathrm{T}^c \cdot \Psi = \frac{2}{3}(n-1)\,\Gamma \cdot \Psi\,, \\
(\mathrm{T}^c)^2 \cdot \Psi &= \frac{4}{9}(n-1)^2\,||\Gamma||^2 \cdot \Psi\,, \\
d\mathrm{T}^c \cdot \Psi &= \frac{1}{3}\Big(||\mathrm{T}^c||^2 - \frac{4}{9}(n-1)^2\,||\Gamma||^2 - \mathrm{Scal}^{\nabla^{\mathrm{T}^c}}\Big) \cdot \Psi\,, \\
2\,(n-1)\,\delta^g(\Gamma) &= 2\Big(\frac{4}{9}(n-1)^2\,||\Gamma||^2 - ||\mathrm{T}^c||^2\Big) - \mathrm{Scal}^{\nabla^{\mathrm{T}^c}}\,.
\end{aligned}$$

Generalized Hopf structures

• $\mathcal{R} \subset \mathcal{F}(M^n)$ – a structure of vectorial type admitting a characteristic connection.

• The condition $\nabla^{\mathrm{vec}}\Gamma = 0$ or $\nabla^{\mathrm{vec}}\mathrm{T}^{\mathrm{c}} = 0$ is very restrictive. Indeed, it implies that

$$\delta^g(\Gamma) \;=\; (n\;-\;1)\cdot||\Gamma||^2.$$

• The conditions $\nabla^{\mathrm{c}}\Gamma = 0$ or $\nabla^{\mathrm{c}}\mathrm{T}^{\mathrm{c}} = 0$ are more interesting.

• In complex geometry, a hermitian manifold of vectorial type such that its characteristic torsion T^{c} is ∇^{c}-parallel is called a generalized Hopf manifold. These $\mathcal{W}_4$-manifolds have been studied by Vaisman.

Definition: A G-structure $\mathcal{R} \subset \mathcal{F}(M^n)$ of vectorial type and admitting a characteristic connection is called a *generalized Hopf G-structure* if $\nabla^{\mathrm{c}}\Gamma = 0$ holds.

Theorem: Suppose that $\Theta : \Lambda^3(\mathbb{R}^n) \longrightarrow \mathbb{R}^n \otimes \mathfrak{m}$ is injective and let $\mathcal{R}$ be a G-structure of vectorial type admitting a characteristic connection. If $\nabla^{\mathrm{c}}\Gamma = 0$, then

$$\delta^g(\Gamma) \ = \ 0\,, \quad \delta^g(\mathrm{T}^{\mathrm{c}}) \ = \ 0\,, \quad d\Gamma \ = \ \Gamma \lrcorner \mathrm{T}^{\mathrm{c}}\,, \quad 2 \cdot \nabla^g\Gamma \ = \ d\Gamma\ .$$

In particular, Γ is a Killing vector field.

Remark: The vector field Γ of a Hopf G-structure is a Killing vector field. Γ is ∇^g-parallel if and only if $d\Gamma = 0$ holds. We discussed sufficient conditions that the vector field of any Hopf G-structure is ∇^g-parallel. This situation occurs for the standard geometries of the groups $\mathrm{G} = \mathrm{G}_2, \mathrm{Spin}(7)$ and for $\mathrm{U}(n)\,,\, n \geq 3$.

Interesting problem: Investigate subgroups $\mathrm{G} \subset \mathrm{SO}(n)$ and Hopf G-structures ($\nabla^{\mathrm{c}}\Gamma = 0$) with a non ∇^g-parallel vector field.