High-resolution quantization and entropy coding for fractional Brownian motion

by

S. Dereich and M. Scheutzow

Technische Universität Berlin

Summary. We derive a high-resolution formula for the quantization and entropy coding approximation quantities for fractional Brownian motion, respective to the supremum norm and $L^p[0,1]$-norm distortions. We show that all moments in the quantization problem lead to the same asymptotics. Using a general principle, we conclude that entropy coding and quantization coincide asymptotically. Under supremum-norm distortion, our proof uses an explicit construction of efficient codebooks based on a particular entropy constrained coding scheme. This procedure can be used to construct close to optimal high resolution quantizers.

Keywords. High-resolution quantization; complexity; stochastic process; entropy; distortion rate function.

2000 Mathematics Subject Classification. 60G35, 41A25, 94A29.

1 Introduction

Functional quantization and entropy coding concern the finding of “good” discrete approximations to a non-discrete random signal in a Banach space of functions. Such discrete approximations may serve as evaluation points for quasi Monte Carlo methods or as an information reduction of the original to allow storage on a computer or transmission over some channel with finite capacity. In the past years, research in this field has been very active, which resulted in numerous new results. Previous research addressed, for instance, the problem of constructing good approximation schemes, the evaluation of the theoretically best approximation under an information constraint, existence of optimal approximation
schemes and regularity properties of the paths of optimal approximations. The above questions are treated for Gaussian measures in Hilbert spaces by Luschgy and Pagès ([11], [12]) and by the first-named author in [3]. For Gaussian originals in Banach spaces, these problems have been addressed by the authors and collaborators in [6], [7], [3], [4] and by Graf, Luschgy and Pagès in [9]. For general accounts of quantization and coding theory in finite dimensional spaces, see [8] and [1] (see also [10]).

In this article, we consider the asymptotic coding problem of fractional Brownian motion for the supremum and $L^p[0,1]$-norm distortions. We derive the asymptotic quality of optimal approximations. In particular, it is shown that efficient entropy constrained quantizers can be used to construct close to optimal quantizers when considering the supremum norm. Moreover, for one of the above norm-based distortions, all moments and both information constraints lead to the same asymptotic approximation quality. In particular, quantization is asymptotically just as efficient as entropy coding. The main impetus to the present work was provided by the necessity to understand the coding complexity of Brownian motion in order to solve the quantization (resp. entropy constrained coding) problem for diffusions (see [5]).

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, let $H \in (0,1)$ and let $X = (X_t)_{t \geq 0}$ denote fractional Brownian motion with Hurst index $H$ on $(\Omega, \mathcal{A}, \mathbb{P})$, i.e. $(X_t)_{t \geq 0}$ is a centered continuous Gaussian process with covariance kernel

$$K(t, s) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}), \quad t, s \geq 0.$$ 

We need some more notation. In the sequel, $C[0,a], a > 0$, and $D[0,a]$ denote the space of continuous real-valued functions on the interval $[0,a]$ and the space of càdlàg functions on $[0,a]$, respectively. Both spaces are endowed with the supremum norm $\| \cdot \|_{[0,a]}$. Moreover, we let $(L^p[0,a], \| \cdot \|_{L^p[0,a]})$ denote the standard $L^p$-space of real-valued functions defined on $[0,a]$. Finally, $\| \cdot \|_q, q \in (0,\infty]$, denotes the $L^q$-norm induced by the probability measure $\mathbb{P}$ on the set of real-valued random variables.

Let us briefly introduce the main objectives of quantization and entropy coding. Let $E$ and $\hat{E}$ denote measurable spaces, and let $d : E \times \hat{E} \to [0,\infty)$ be a product measurable function. For a given $E$-valued r.v. $Y$ (original) and moment $q > 0$, the aim is to minimize

$$\|d(Y, \pi(Y))\|_q$$

over all measurable functions $\pi : E \to \hat{E}$ with discrete image (strategy) that satisfy a particular information constraint parameterized by the rate $r \geq 0$.

Entropy coding (also known as entropy constrained quantization in the literature) concerns the minimization of (1) over all strategies $\pi$ having entropy $\mathbb{H}(\pi(Y))$ at most $r$. 

2
Recall that the entropy of a discrete r.v. \( Z \) with probability weights \((p_w)\) is defined as
\[
\mathbb{H}(Z) = -\sum_w p_w \log p_w = \mathbb{E}[-\log p_Z].
\]
In the quantization problem, one is considering strategies \( \pi \) satisfying the range constraint: \(|\text{range}(\pi(Y))| \leq e^r\). The corresponding approximation quantities are the entropy-constrained quantization error
\[
D^{(e)}(r|Y, E, \hat{E}, d, q) := \inf_\pi \|d(Y, \pi(Y))\|_q,
\]
where the infimum is taken over all strategies \( \pi \) with entropy rate \( r \geq 0 \), and the quantization error
\[
D^{(q)}(r|Y, E, \hat{E}, d, q) := \inf_\pi \|d(Y, \pi(Y))\|_q,
\]
the infimum being taken over all strategies \( \pi \) having quantization rate \( r \geq 0 \). Often, all or some of the parameters \( Y, E \) are clear from the context. Then we omit these parameters in the quantities \( D^{(e)} \) and \( D^{(q)} \). The quantization information constraint is more restrictive, so that the quantization error always dominates the entropy coding error. Moreover, the coding error increases with the moment under consideration.

Unless otherwise stated, we choose as original \( Y = X \) and as original space \( E = C[0,1] \). We are mainly concerned with two particular choices for \( \hat{E} \) and \( d \). In the first sections, we treat the case where \( \hat{E} = \mathbb{D}[0,1] \) and \( d(f, g) = \| f - g \|_{[0,1]} \). In this setting we find:

**Theorem 1.1.** There exists a constant \( \kappa = \kappa(H) \in (0, \infty) \) such that for all \( q_1 \in (0, \infty) \) and \( q_2 \in (0, \infty) \),
\[
\lim_{r \to \infty} r^H D^{(e)}(r|q_1) = \lim_{r \to \infty} r^H D^{(q)}(r|q_2) = \kappa.
\]

**Remark 1.2.** In the above theorem, general càdlàg functions are allowed as reconstructions. Since the original process is continuous, it might seem more natural to use continuous functions as approximations. The following argument shows that, for a finite moment \( q > 0 \), the space \( \hat{E} = \mathbb{D}[0,1] \) can be replaced by \( \hat{E} = C[0,1] \) without changing \( D^{(q)} \) and \( D^{(e)} \). Let \( \pi : C[0,1] \to \mathbb{D}[0,1] \) be an arbitrary strategy and let \( \tau_n : \mathbb{D}[0,1] \to C[0,1] \) denote the linear operator mapping \( f \) to its piecewise linear interpolation with supporting points \( 0, \frac{1}{n}, \frac{2}{n}, \ldots, 1 \). Then
\[
\|\|X - \tau_n \circ \pi(X)\|_{[0,1]}\|_q \leq \|\|\tau_n(X) - \tau_n \circ \pi(X)\|_{[0,1]}\|_q + \|\|X - \tau_n(X)\|_{[0,1]}\|_q
\]
\[
\leq \|\|X - \pi(X)\|_{[0,1]}\|_q + \|\|X - \tau_n(X)\|_{[0,1]}\|_q.
\]
Note that the second term vanishes when \( n \) tends to infinity and that \( \tau_n \circ \pi \) satisfies the same information constraint as \( \pi \).
In the last section we conclude the article with a discussion of the case where $\hat{E} = L^p[0,1]$ and $d(f,g) = \|f - g\|_{L^p[0,1]}$ for some $p \geq 1$. In this case, one has the following analog to Theorem 1.1:

Theorem 1.3. For every $p \geq 1$ there exists a constant $\kappa = \kappa(H,p) \in (0, \infty)$ such that for all $q \in (0, \infty)$,

$$\lim_{r \to \infty} r^H D^{(e)}(r|q) = \lim_{r \to \infty} r^H D^{(g)}(r|q) = \kappa.$$

Remark 1.4. It is again possible to replace the space $\hat{E} = L^p[0,1]$ by $\hat{E} = C[0,1]$ without changing $D^{(g)}$ and $D^{(e)}$. Indeed, for $\varepsilon > 0$, let $h_\varepsilon : \mathbb{R} \to [0, \infty)$ denote a smooth function supported on $[-\varepsilon, \varepsilon]$ with $\int h_\varepsilon = 1$, and define $\tau_\varepsilon : L^p[0,1] \to C[0,1]$ through $\tau_\varepsilon(f)(t) = \int_0^1 f(s) h(t-s) \, ds$. Then for a given strategy $\pi : C[0,1] \to L^p[0,1]$ one obtains

$$\|X - \tau_\varepsilon \circ \pi(X)\|_{L^p[0,1]} \leq \|\tau_\varepsilon(X) - \tau_\varepsilon \circ \pi(X)\|_{L^p[0,1]} + \|X - \tau_\varepsilon(X)\|_{L^p[0,1]}$$

where the last inequality is a consequence of Young’s inequality. Now for $\varepsilon \downarrow 0$ the second term converges to 0.

For ease of notation, the article is restricted to the analysis of 1-dimensional processes. However, when replacing $(X_t)$ by a process $(X_t^{(1)}, \ldots, X_t^{(d)})$ consisting of $d$ independent fractional Brownian motions, the proofs can be easily adapted, and one obtains analogous results. In particular, it is possible to prove analogs of the above theorems for a multi dimensional Brownian motion.

Let us summarize some of the known estimates for the constant $\kappa$ in the case where $X$ is standard Brownian motion, i.e. $H = 1/2$.

- When $\hat{E} = \mathbb{D}[0,1]$ and $d(f,g) = \|f - g\|_{[0,1]}$, the relationship between the small ball function and the quantization problem (see [6]) leads to

$$\kappa \in \left[\frac{\pi}{\sqrt{8}}, \pi\right].$$

- For $\hat{E} = L^p[0,1]$, $p \geq 1$, and $d(f,g) = \|f - g\|_{L^p[0,1]}$, $\kappa$ may again be estimated via a connection to the small ball function. Indeed, letting

$$\lambda_1 = \inf \left\{ \int_{-\infty}^{\infty} |x|^p \varphi^2(x) \, dx + \frac{1}{2} \int_{-\infty}^{\infty} (\varphi'(x))^2 \, dx \right\},$$

where the infimum is taken over all weakly differentiable $\varphi \in L^2(\mathbb{R})$ with unit norm, one has

$$\kappa \in [c, \sqrt{8}\, \varepsilon]$$

4
for \( c = 2^{1/p} \sqrt{p} (\frac{1-p}{2+p})^{(2+p)/2p} \).

In the case where \( p = 2 \), the constant \( \kappa \) is known explicitly: \( \kappa = \frac{\sqrt{2}}{\pi} \) (see [12] and [3]).

The article is outlined as follows. In Sections 2 to 5 we consider the approximation problems under the supremum norm. We start in Section 2 by introducing a coding scheme which plays an important role in the sequel. In Section 3, we use the construction of Section 2 and the self similarity of \( X \) to establish a polynomial decay for \( D^{(c)}(\cdot|\infty) \). In the following section, the asymptotics of the quantization error are computed. The proof relies on a concentration property for the entropies of “good” coding schemes (Proposition 4.4). In Section 5, we use the equivalence of moments in the quantization problem to establish a lower bound for the entropy coding problem. In the last section, we treat the case where the distortion is based on the \( L^p[0,1] \)-norm, i.e. \( d(f,g) = \| f - g \|_{L^p[0,1]} \); we introduce the distortion rate function and prove Theorem 1.3 with the help of Shannon’s source coding Theorem.

It is convenient to use the symbols \( \sim \), \( \lesssim \) and \( \simeq \). We write \( f \sim g \) iff \( \lim \frac{f}{g} = 1 \), while \( f \lesssim g \) stands for \( \limsup \frac{f}{g} \leq 1 \). Finally, \( f \simeq g \) means

\[
0 < \liminf \frac{f}{g} \leq \limsup \frac{f}{g} < \infty.
\]

## 2 The coding scheme

This section is devoted to the construction of strategies \( \pi^{(n)} : \mathbb{C}[0,n] \to \mathbb{D}[0,n] \) which we will need later in our discussion. The construction depends on three parameters: \( M \in \mathbb{N} \setminus \{1\} \), \( d > 0 \) and a strategy \( \pi : \mathbb{C}[0,1] \to \mathbb{D}[0,1] \).

We define the maps by induction. Let \( w \in \mathbb{C}[0,\infty) \) and set \( (w^{(n)}_t)_{t \in [0,1]} := (w_{t+n} - w_n)_{t \in [0,1]} \) and \( \hat{w}_t := \pi(w^{(0)})(t) \) for \( t \in [0,1] \). Assume that \( (\hat{w}_t)_{t \in [0,n)} \) \( n \in \mathbb{N} \) has already been defined. Then we choose \( \xi_n \) to be the smallest number in \( \{-d + 2kd/(M-1) : k = 0, \ldots, M-1\} \) minimizing

\[
|w_n - (\hat{w}_{n-} + \xi_n)|,
\]

and extend the definition of \( \hat{w} \) on \([n,(n+1)]\) by setting

\[
\hat{w}_{n+t} := \hat{w}_{n-} + \xi_n + \pi(w^{(n)})(t), \quad t \in [0,1).
\]

Note that \( (\hat{w}_t)_{t \in [0,n]} \) depends only upon \( (w_t)_{t \in [0,n]} \), so that the above construction induces strategies

\[
\pi^{(n)} : \mathbb{C}[0,n] \to \mathbb{D}[0,n], \quad w \mapsto (\hat{w}^{(n)}_t)_{t \in [0,n]}.
\]
where $\hat{w}_t^{(n)} = \hat{w}_t$ for $t \in [0, n)$ and $\hat{w}_n = \hat{w}_{n-}$. Moreover, we can write
\[ (\hat{w}_t)_{t \in [0, n]} = \pi^{(n)}(w) = \varphi_n(\pi(w^{(0)}), \ldots, \pi(w^{(n-1)}), \xi_1, \ldots, \xi_{n-1}) \] (4)
for an appropriate measurable function $\varphi_n : (\mathbb{D}[0, n])^n \times \mathbb{R}^{n-1} \to \mathbb{D}[0, n]$.

The main motivation for this construction is the following property. If one has, for some $(w_t) \in \mathbb{D}[0, 1)$ and $n \in \mathbb{N}$,
\[ \|w - \pi^{(n)}(w)\|_{[0, n]} \leq \frac{M}{M-1}d \]
and $\|w^{(n)} - \pi(w^{(n)})\|_{[0, 1]} \leq d$, then
\[ |w_n - (\hat{w}_{n-} + \xi_n)| \leq \frac{d}{M-1}, \]
whence,
\[ \|w - \hat{w}\|_{[n, n+1]} = \|w_n + w_t^{(n)} - (\hat{w}_{n-} + \xi_n + \pi(w^{(n)}(t)))\|_{[0, 1]} \]
\[ \leq |w_n - (\hat{w}_{n-} + \xi_n)| + \|w^{(n)} - \pi(w^{(n)})\|_{[0, 1]} \]
\[ \leq \frac{d}{M-1} + d = \frac{M}{M-1}d. \]
In particular, if $\pi : \mathbb{C}[0, 1] \to \mathbb{D}[0, 1]$ satisfies
\[ \|X - \pi(X)\|_{[0, 1]} \leq d, \]
then for any $n \in \mathbb{N}$,
\[ \|X - \pi^{(n)}(X)\|_{[0, n]} \leq \frac{M}{M-1}d. \] (5)

3 Polynomial decay of $D^{(e)}(r|\infty)$

The objective of this section is to prove the following theorem.

**Theorem 3.1.** There exists a constant $\kappa = \kappa(H) \in (0, \infty)$ such that
\[ \lim_{r \to \infty} r^H D^{(e)}(r|\infty) = \kappa. \] (6)

Thereafter, $\kappa = \kappa(H)$ will always denote the finite constant defined via equation (6).

In order to simplify notations, we abridge $\| \cdot \| = \| \cdot \|_{[0, 1]}$. 

6
Remark 3.2. It was found in [3] (see Theorem 3.5.2) that for finite moments \( q \geq 1 \) the entropy coding error is related to the asymptotic behavior of the small ball function of the Gaussian measure. In particular, for fractional Brownian motion, one obtains that

\[
D^{(c)}(r|q) \approx \frac{1}{r^H}, \quad r \to \infty.
\]

In order to show that \( D^{(c)}(r|x) \) is of the order \( r^{-H} \), we still need to prove an appropriate upper bound. We prove a stronger statement which will be useful later on.

Lemma 3.3. There exist strategies \( \pi(r) : \mathbb{C}[0,1] \to \mathbb{C}[0,1], r \geq 0, \) and probability weights \( (p_n^{(r)})_{n \in \mathbb{N}} \) such that for any \( q \geq 1 \),

\[
\|X - \pi(r)(X)\|_\infty \leq \frac{1}{r^H} \quad \text{and} \quad \mathbb{E}[-\log p^{(r)}_\pi(X)]^{q^{1/q}} \approx r. \tag{7}
\]

In particular, \( D^{(c)}(r|x) \approx r^{-H} \).

The proof of the lemma is based on an asymptotic estimate for the mass concentration in randomly centered small balls, to be found in [7]. Let \( \hat{X}_1 \) denote a fractional Brownian motion that is independent of \( X \) with \( L(X) = L(\hat{X}_1) \). Then, for any \( q \geq 1 \), one has

\[
\mathbb{E}[-\log \mathbb{P}(\|X - \hat{X}_1\| \leq \varepsilon|X)]^{q^{1/q}} \approx -\log \mathbb{P}(\|X\| \leq \varepsilon) \approx \varepsilon^{-1/H} \tag{8}
\]

as \( \varepsilon \downarrow 0 \) (see [7], Theorem 4.2 and Corollary 4.4).

Proof. For a given \( \mathbb{D}[0,1] \)-valued sequence \( (\tilde{w}_n)_{n \in \mathbb{N} \cup \{\infty\}} \), we consider the following coding strategy \( \pi^{(r)} : (\tilde{w}_n) : \) let

\[
T^{(r)}(w) := T^{(r)}(w|\tilde{w}_n)) := \inf\{n \in \mathbb{N} : \|w - \tilde{w}_n\| \leq 1/r^H\},
\]

with the convention that the infimum of the empty set is \( \infty \), and set

\[
\pi^{(r)}(w) := \pi^{(r)}(w|\tilde{w}_n)) := \tilde{w}_{T(r)(w)}. \tag{9}
\]

Moreover, let \( (p_n)_{n \in \mathbb{N}} \) denote the sequence of probability weights defined as

\[
p_n = \frac{6}{\pi^2} \frac{1}{n^2}, \quad n \in \mathbb{N},
\]

and set \( p_{\infty} := 0 \).

Now we let \( (\hat{X}_n)_{n \in \mathbb{N} \cup \{\infty\}} \) denote independent FBM's that are also independent of \( X \), and analyze the random coding strategies \( \pi^{(r)}(\cdot) := \pi^{(r)}(\cdot|(\hat{X}_n)) \). With \( T^{(r)} := T^{(r)}(X|(\hat{X}_n)) \) we obtain

\[
\hat{X}^{(r)} := \pi^{(r)}(X) = \hat{X}_{T^{(r)}},
\]
and
\[ \mathbb{E}[-\log p_{T(r)}]^{1/q} \leq 2\mathbb{E}[(\log T(r)]^{1/q} + \log \frac{\pi^2}{6}. \]  
(9)

Given \( X \), the random time \( T(r) \) is geometrically distributed with parameter \( \mathbb{P}(||X - \tilde{X}_1|| \leq 1/r^H|X) \), and due to Lemma A.2 there exists a universal constant \( c_1 = c_1(q) < \infty \) for which
\[ \mathbb{E}[(\log T(r)]^{1/q}|X]^{1/q} \leq c_1 [1 + \log \mathbb{P}(||X - \tilde{X}_1|| \leq 1/r^H|X)]. \]

Consequently,
\[ \mathbb{E}[(\log T(r)]^{1/q} = \mathbb{E}\left[ \mathbb{E}[(\log T(r)]^{1/q}|X] \right]^{1/q} \]
\[ \leq c_1 \mathbb{E}[(1 + \log \mathbb{P}(||X - \tilde{X}_1|| \leq 1/r^H|X)]^{1/q} \]
\[ \leq c_1 (1 + \mathbb{E}(-\log \mathbb{P}(||X - \tilde{X}_1|| \leq 1/r^H|X)]^{1/q}). \]  
(10)

Due to (8), one has
\[ \mathbb{E}[-\log \mathbb{P}(||X - \tilde{X}_1|| \leq 1/r^H|X)]^{1/q} \approx r, \]
so that (9) and (10) imply that \( \mathbb{E}[-\log p_{T(r)}]^{1/q} \leq c_2 r \) for some appropriate constant \( c_2 < \infty \). In particular, for any \( r \geq 0 \), we can find a \( \mathbb{C}[0, 1] \)-valued sequence \( (\tilde{w}^{(r)})_{n \in \mathbb{N}} \) of pairwise different elements such that
\[ \mathbb{E}[-\log p_{T(r)}(X|\tilde{w}^{(r)}_n)]^{1/q} \leq \mathbb{E}[-\log p_{T(r)}]^{1/q} \leq c_2 r. \]

Now the strategies \( \pi^{(r)}(\cdot|\tilde{w}^{(r)}_n) \) with associated probability weights \( p^{(r)}_{\tilde{w}^{(r)}_n} := p_n \) \( (n \in \mathbb{N}) \) satisfy (7). Moreover, \( D(e)(r|\infty) \approx r^{-H} \) follows since
\[ \mathbb{H}(\pi^{(r)}(X|\tilde{w}^{(r)}_n))) \leq \mathbb{E}[-\log p^{(r)}_{\pi^{(r)}(X|\tilde{w}^{(r)}_n)}]. \]

Let us now use the coding scheme of Section 2 to prove

**Lemma 3.4.** Let \( n \in \mathbb{N} \), \( r \geq 0 \) and \( \Delta r \geq 1 \). Then
\[ D(e)(n(r + \Delta r)|\infty) \leq n^{-H} \frac{e^{\Delta r}}{e^{\Delta r} - 2} D(e)(r|\infty). \]  
(11)

**Proof.** Fix \( \varepsilon > 0 \) and let \( \pi : \mathbb{C}[0, 1] \rightarrow \mathbb{D}[0, 1] \) be a strategy satisfying
\[ \|X - \pi(X)\|_{[0, 1]}\|_{\infty} \leq (1 + \varepsilon) D(e)(r|\infty) =: d \]
and

\[ \mathbb{H}(\pi(X)) \leq r. \]

Choose \( M := [e^{\Delta r}] \) and let \( \pi^{(n)} \) be as in Section 2. Note that \( \Delta r \geq 1 \) guarantees that \( M \geq e^{\Delta r} - 1 \geq e^{\Delta r}/2 \), so that

\[ \|X - \pi^{(n)}(X)\|_{[0,n]} \leq \frac{M}{M-1} d \leq \frac{e^{\Delta r}}{e^{\Delta r} - 2} (1 + \varepsilon) D^{(e)}(r|\infty). \]

We let \( (X_i)_{t \in [0,1]} = (X_{i+t} - X_i)_{t \in [0,1]} \) for \( i = 1, \ldots, n \), and \( (\xi_i)_{i=1,\ldots,n-1} \) be as in Section 2 for \( w = X \). Observe that, due to the representation (4),

\[ \begin{align*}
\mathbb{H}(\pi^{(n)}(X)) &\leq \mathbb{H}(\pi(X^{(0)}), \ldots, \pi(X^{(n-1)}), \xi_1, \ldots, \xi_{n-1}) \\
&\leq \mathbb{H}(\pi(X^{(0)})) + \cdots + \mathbb{H}(\pi(X^{(n-1)})) + \mathbb{H}(\xi_1, \ldots, \xi_{n-1}) \\
&\leq nr + \log |\text{range } (\xi_1, \ldots, \xi_{n-1})| \leq nr + n \log M \\
&\leq n(r + \Delta r).
\end{align*} \tag{12} \]

Now let \( \alpha_n : \mathbb{D}[0,1] \to \mathbb{D}[0,n], \ f \mapsto \alpha_n(f)(s) = n^H f(s/n) \) and consider the strategy

\[ \tilde{\pi} : \mathbb{C}[0,1] \to \mathbb{D}[0,1], \ f \mapsto \alpha_n^{-1} \circ \pi^{(n)} \circ \alpha_n(f). \]

Since \( \alpha_n(X) \) is again a fractional Brownian motion on \([0,n]\), it follows that, a.s.

\[ \|X - \tilde{\pi}(X)\|_{[0,1]} = n^{-H} \|\alpha_n(X) - \pi^{(n)}(\alpha_n(X))\|_{[0,n]} \leq (1 + \varepsilon) n^{-H} \frac{e^{\Delta r}}{e^{\Delta r} - 2} D^{(e)}(r|\infty). \]

Moreover,

\[ \mathbb{H}(\tilde{\pi}(X)) = \mathbb{H}(\alpha_n^{-1} \circ \pi^{(n)}(\alpha_n(X))) = \mathbb{H}(\pi^{(n)}(X)) \leq r. \]

Since \( \varepsilon > 0 \) is arbitrary, the proof is complete. \( \square \)

**Proof of Theorem 3.1.** For \( r \geq 0, \Delta r \geq 1 \) and \( n \in \mathbb{N} \), Lemma 3.4 yields

\[ D^{(e)}(n(r + \Delta r)|\infty) \leq \frac{1}{nH} \frac{e^{\Delta r}}{e^{\Delta r} - 2} D^{(e)}(r|\infty). \]

Now set \( \kappa := \lim_{r \to \infty} r^H D^{(e)}(r|\infty) \) which lies in \((0, \infty)\) due to Lemma 3.3. Let \( \varepsilon \in (0, 1/2) \) be arbitrary, and choose \( r_0, \Delta r \geq 1 \) such that

\[ \begin{align*}
&\left\{ r^H_0 D^{(e)}(r_0|\infty) \leq (1 + \varepsilon) \kappa, \\
&\Delta r \leq \varepsilon r_0 \quad \text{and} \\
e^{-\Delta r} \leq \varepsilon.
\end{align*} \]

Hence, we have

\[ \mathbb{H}(\pi^{(n)}(X)) \leq n(r + \Delta r) \leq nH \frac{e^{\Delta r}}{e^{\Delta r} - 2} D^{(e)}(r|\infty) \leq \mathbb{H}(\tilde{\pi}(X)) \leq \mathbb{H}(\pi^{(n)}(X)), \]

since \( \mathbb{H}(\cdot) \) is monotonous. Finally, note that

\[ \|X - \tilde{\pi}(X)\|_{[0,1]} \leq n^{-H} \|\alpha_n(X) - \pi^{(n)}(\alpha_n(X))\|_{[0,n]} \leq (1 + \varepsilon) n^{-H} \frac{e^{\Delta r}}{e^{\Delta r} - 2} D^{(e)}(r|\infty). \]

Since \( \varepsilon > 0 \) is arbitrary, the proof is complete. \( \square \)
Then
\[
D^{(e)}((1 + \varepsilon)nr_0|\infty) \leq \frac{1}{nH} \frac{1}{1 - 2\varepsilon} D^{(e)}(r_0|\infty)
\]
\[
\leq \frac{1}{((1 + \varepsilon)nr_0)^H} \frac{1}{1 - 2\varepsilon} (1 + \varepsilon)^{1+H} \kappa
\]
and we obtain that
\[
\limsup_{n \to \infty} ((1 + \varepsilon)nr_0)^H D^{(e)}((1 + \varepsilon)nr_0|\infty) \leq \frac{(1 + \varepsilon)^{1+H}}{1 - 2\varepsilon} \kappa.
\]

Let now \( r \geq (1 + \varepsilon)r_0 \) and introduce \( \bar{r} = \bar{r}(r) = \min\{(1 + \varepsilon)nr : n \in \mathbb{N}, r \leq (1 + \varepsilon)nr_0\} \)
as well as \( \underline{r} = \underline{r}(r) = \max\{(1 + \varepsilon)nr : n \in \mathbb{N}, (1 + \varepsilon)nr_0 \leq r\} \). Using the monotonicity of \( D^{(e)}(r|\infty) \), we conclude that
\[
\limsup_{r \to \infty} r^H D^{(e)}(r|\infty) \leq \limsup_{r \to \infty} \bar{r}^H D^{(e)}(\bar{r}|\infty)
\]
\[
\leq \limsup_{r \to \infty} (\underline{r} + (1 + \varepsilon)r_0)^H D^{(e)}(\underline{r}|\infty)
\]
\[
\leq \frac{(1 + \varepsilon)^{1+H}}{1 - 2\varepsilon} \kappa.
\]
Noticing that \( \varepsilon > 0 \) is arbitrary finishes the proof.

\[\square\]

4 The quantization problem

Theorem 4.1. One has for any \( q \in (0, \infty) \),
\[
D^{(q)}(r|q) \sim \kappa \frac{1}{r^H}, \quad r \to \infty.
\]

We need some preliminary lemmas for the proof of the theorem.

Lemma 4.2. There exist strategies \( (\pi^{(r)})_{r \geq 0} \) and probability weights \( (p^{(r)}_\pi) \) such that
\[
\|X - \pi^{(r)}(X)\|_\infty \leq \kappa \frac{1}{r^H} \quad \text{and} \quad -\log p^{(r)}_\pi(X) \lesssim r, \quad \text{in probability}.
\]

Proof. Let \( \varepsilon > 0 \) and choose \( r_0 \geq 2 \) such that
\[
\left( \frac{r_0 + 1}{r_0 - 1} \right)^{1/H} \leq 1 + \frac{\varepsilon}{2}
\]
By Theorem 3.1,
\[
D^{(e)}((1 + \varepsilon/2)r|\infty) \lesssim \kappa \frac{r_0 - 1}{r_0 + 1} \frac{1}{r^H}
\]
10
In particular, there exists $r_1 \geq r_0 \vee \frac{2}{3} \log(r_0 + 1)$ and a map $\pi : \mathbb{C}[0,1] \to \mathbb{D}[0,1]$ such that

$$\|\|X - \pi(X)\|_{[0,1]}\|_\infty \leq \kappa \frac{r_0 - 1}{r_0} \frac{1}{r_1^H} =: d \quad \text{and} \quad \mathbb{H}(\pi(X)) \leq (1 + \varepsilon/2)r_1.$$ 

For $n \in \mathbb{N}$, let $\pi^{(n)}$ and $\varphi_n$ be as in Section 2 for $M = [r_0]$, $d$ and $\pi$. Then by (5)

$$\|\|X - \pi^{(n)}(X)\|_{[0,n]}\|_\infty \leq \kappa \frac{(r_0 - 1)M}{r_0(M - 1)} \frac{1}{r_1^H} \leq \kappa \frac{1}{r_1^H}. \quad (13)$$

For $\hat{w}^{(0)}, \ldots, \hat{w}^{(n-1)} \in \text{im}(\pi)$ and $k_1, \ldots, k_{n-1} \in \{-d + \frac{2kd}{M-1} : k = 0, \ldots, M - 1\}$, let $p^{(n)}$ be defined as

$$p^{(n)}_{\varphi_n(\hat{w}^{(0)}, \ldots, \hat{w}^{(n-1)}, k_1, \ldots, k_{n-1})} = \frac{1}{M^{n-1}} \prod_{i=0}^{n-1} \mathbb{P}((X) = \hat{w}^{(i)}).$$

The $(p^{(n)}_{\hat{w}})$ define probability weights on the image of $\varphi_n$. Moreover,

$$- \log p^{(n)}_{(X_t)_{t \in [0,n]}} = (n-1) \log M - \sum_{i=0}^{n-1} \log p_{\pi(X^{(i)})}$$

and the ergodic theorem implies

$$\lim_{n \to \infty} - \frac{1}{n} \log p^{(n)}_{(X_t)_{t \in [0,n]}} = \log M + \mathbb{H}(\pi(X)), \quad \text{a.s.}$$

Note that $\log M + \mathbb{H}(\pi(X)) \leq (1 + \varepsilon)r_1$.

Just as in the proof of Lemma 3.4, we use the self similarity of $X$ to translate the strategy $\pi^{(n)}$ into a strategy for encoding $(X_t)_{t \in [0,1]}$. For $n \in \mathbb{N}$, let

$$\alpha_n : \mathbb{D}[0,1] \to \mathbb{D}[0,n], \quad f \mapsto (\alpha_n f)(t) = n^H f(t/n)$$

and consider $\tilde{p}^{(n)}_{\hat{w}} := p^{(n)}_{\alpha_n(\hat{w})}$ and $\tilde{\pi}^{(n)}(w) := \alpha_n^{-1} \circ \pi^{(n)} \circ \alpha_n(w)$. Then

$$- \log p^{(n)}_{\tilde{\pi}^{(n)}(X)} = - \log p^{(n)}_{\pi^{(n)}(\alpha_n(X))} \lesssim (1 + \varepsilon)n r_1, \quad \text{in probability}$$

and by (13)

$$\|\|X - \tilde{\pi}^{(n)}(X)\|_{[0,1]}\|_\infty = \|\|\alpha_n^{-1}(\alpha_n(X) - \pi^{(n)}(\alpha_n(X)))\|_{[0,1]}\|_\infty$$

$$= \frac{1}{n^H} \|\|\alpha_n(X) - \pi^{(n)}(\alpha_n(X))\|_{[0,n]}\|_\infty$$

$$= \frac{1}{n^H} \|\|X - \pi^{(n)}(X)\|_{[0,n]}\|_\infty \leq \kappa \frac{1}{(nr_1)^H}.$$
By choosing $\pi^{(r)} = \tilde{\pi}^{(n)}$ and $(\tilde{p}^{(r)}) = (\tilde{p}^{(n)})$ for $r \in ((n - 1)r_1, nr_1]$, one obtains a coding scheme satisfying

$$\|\|X - \pi^{(r)}(X)\|\|_\infty \leq \kappa \frac{1}{rH}$$

and

$$-\log p^{(r)}_{\pi^{(r)}(X)} \lesssim (1 + \varepsilon)r, \quad \text{in probability},$$

so that the assertion follows by a diagonalization argument. \hfill \Box

**Remark 4.3.** In the above proof, we have constructed a high resolution coding scheme based on a strategy $\pi : \mathbb{C}[0, 1] \to \mathbb{D}[0, 1]$, using the identity $\tilde{\pi}_n = \alpha_n^{-1} \circ \pi^{(n)} \circ \alpha_n$. This coding scheme leads to a coding error which is at most

$$M \frac{M}{M - 1} \|\|X - \pi(X)\|\|_{[0, 1]} \|\|_\infty n^{-H}. \quad (14)$$

Moreover, the ergodic theorem implies that, for large $n$, $\tilde{\pi}_n(X)$ lies with probability almost one in the typical set $\{w \in \mathbb{D}[0, 1] : -\log p^{(n)}(w) \leq n(H(\pi(X)) + \log M + \varepsilon)\}$, where $\varepsilon > 0$ is arbitrarily small. This set is of size $\exp\{n(H(\pi(X)) + \log M + \varepsilon)\}$, and will serve as a close to optimal high resolution codebook. It remains to control the case where $\tilde{\pi}_n(X)$ is not in the typical set. We will do this in the proof of Theorem 4.1 at the end of this section (see (19)).

**Proposition 4.4.** For $q \geq 1$ there exist strategies $(\pi^{(r)})_{r \geq 0}$ and probability weights $(p^{(r)}_w)$ such that

$$\|\|X - \pi^{(r)}(X)\|\|_\infty \leq \kappa \frac{1}{rH} \quad \text{and} \quad \lim_{r \to \infty} \mathbb{E}[(-\log p^{(r)}_{\pi^{(r)}(X)})^{q}]^{1/q} = 1. \quad (15)$$

In addition, for any $\varepsilon > 0$ one has

$$\lim_{r \to \infty} \sup_{\pi, (p_w)} \mathbb{P}\{-\log p_\pi(X) \leq (1 - \varepsilon)r, \|X - \pi(X)\| \leq \kappa \frac{1}{rH}\} = 0, \quad (16)$$

where the supremum is taken over all strategies $\pi : \mathbb{C}[0, 1] \to \mathbb{D}[0, 1]$ and over all sequences of probability weights $(p_w)$.

**Proof.** Let $q > 1$ and let $\pi_1^{(r)} (r \geq 0)$ be a strategy and $(p^{(r,1)}_w)$ a sequence of probability weights as in Lemma 4.2. Moreover, let $\pi_2^{(r)} (r \geq 0)$ and $(p^{(r,2)}_w)$ be as in Lemma 3.3 for $2q$. We consider the maps $\kappa_1^{(r)}(w) := -\log p^{(r,1)}_{\pi_1^{(r)}(w)}$ and $\kappa_2^{(r)}(w) := -\log p^{(r,2)}_{\pi_2^{(r)}(w)}$, and set

$$\pi^{(r)}(w) := \begin{cases} \pi_1^{(r)}(w) & \text{if } \kappa_1^{(r)}(w) \leq (1 + \delta)r, \\ \pi_2^{(r)}(w) & \text{otherwise}, \end{cases}$$

where the supremum is taken over all strategies $\pi : \mathbb{C}[0, 1] \to \mathbb{D}[0, 1]$ and over all sequences of probability weights $(p_w)$. \hfill \Box
for some fixed $\delta > 0$. Then one obtains, for $p_{w}^{(r)} = \frac{1}{2}(p_{w}^{(r,1)} + p_{w}^{(r,2)})$ and $T_{r} := \{w \in \mathbb{C}[0,1] : \rho_{1}^{(r)}(w) \leq (1 + \delta)r\}$,

$$\mathbb{E}[-\log 2p_{\pi^{(r)}(X)}^{(r)}]^{1/q} \leq \mathbb{E}[1_{T_{r}}(X)\rho_{1}^{(r)}(X)]^{1/q} + \mathbb{E}[1_{T_{r}^{c}}(X)\rho_{2}^{(r)}(X)]^{1/q}$$

$$\leq (1 + \delta)r + \mathbb{P}(X \in T_{r}^{c})^{1/2q} \mathbb{E}[\rho_{2}^{(r)}(X)]^{1/2q}.$$ 

The definitions of $\pi_{1}^{(r)}$ and $\pi_{2}^{(r)}$ imply that $\lim_{r \to \infty} \mathbb{P}(X \in T_{r}^{c}) = 0$ and $\mathbb{E}[\rho_{2}^{(r)}(X)]^{1/2q} \approx r$. Consequently,

$$\mathbb{E}[-\log p_{\pi^{(r)}(X)}^{(r)}]^{1/q} \leq (1 + \delta)r.$$ 

Since $\delta > 0$ can be chosen arbitrarily small, a diagonalization procedure leads to strategies $\tilde{\pi}^{(r)}$ and probability weights $(\tilde{p}_{w}^{(r)})$ with

$$\|X - \tilde{\pi}^{(r)}(X)\|_{0,1} \leq C \frac{1}{r^{H}} \quad \text{and} \quad \mathbb{E}[-\log \tilde{p}_{\tilde{\pi}^{(r)}(X)}]^{1/q} \lesssim r,$$

which proves the first assertion.

It remains to show that for arbitrary strategies $\pi^{(r)}$, $r \geq 0$, and probability weights $(\tilde{p}_{w}^{(r)})$:

$$\lim_{r \to \infty} \mathbb{P}\left(-\log p_{\pi^{(r)}(X)}^{(r)} \leq (1 - \varepsilon)r, \|X - \pi^{(r)}(X)\| \leq \frac{1}{r^{H}}\right) = 0. \quad (17)$$

Without loss of generality, we can assume that

$$\|X - \tilde{\pi}^{(r)}(X)\|_{0,1} \leq \frac{1}{r^{H}}. \quad (18)$$

Otherwise we modify the map $\tilde{\pi}^{(r)}$ for all $w \in \mathbb{C}[0,1]$ with $\|w - \tilde{\pi}^{(r)}(w)\| > \kappa r^{-H}$ in such a way that (18) be valid. Hereby the probability in (17) increases and it suffices to prove the statement for the modified strategy. Let us consider

$$\pi^{(r)}(w) = \begin{cases} 
\pi^{(r)}(w) & \text{if } \tilde{p}_{\tilde{\pi}^{(r)}(w)}^{(r)} \geq \tilde{p}_{\pi^{(r)}(w)}^{(r)} \\
\tilde{\pi}^{(r)}(w) & \text{else}.
\end{cases}$$

Then the probability weights $p^{(r)} := \frac{1}{2}(\tilde{p}^{(r)} + \tilde{p}(r))$ satisfy

$$\mathbb{E}[-\log 2p_{\pi^{(r)}(X)}^{(r)}]^{1/q} \leq \mathbb{E}[-\log \tilde{p}_{\pi^{(r)}(X)}]^{1/q} \lesssim r.$$ 

Recall that

$$\|X - \pi^{(r)}(X)\|_{0,1} \leq \frac{1}{r^{H}},$$

hence by Theorem 3.1, one has $\mathbb{E}[-\log p_{\pi^{(r)}(X)}^{(r)}] \geq H(\pi^{(r)}(X)) \gtrsim r$. Lemma A.1 thus implies that

$$-\log p_{\pi^{(r)}(X)}^{(r)} \sim r, \quad \text{in probability.}$$
In particular,
\[-\log P_{\pi^{(r)}_n}(X) \geq -\log 2p_{\pi^{(r)}_n}(X) \geq r, \quad \text{in probability,}\]
which implies (17). \(\square\)

**Proof of Theorem 4.1.** We start by proving the lower bound. Fix \(q > 0\), let \(C_r, r \geq 0\), denote arbitrary codebooks of size \(e^r\), and let \(\pi^{(r)} : \mathbb{C}[0, 1] \to C_r\) denote arbitrary strategies. Moreover, let \((p_w^{(r)})\) be the sequence of probability weights defined as \(p_w^{(r)} = 1/|C_r|, w \in C_r\). Then \(-\log p_{\pi^{(r)}_n}(X) \leq r \) a.s., and the above lemma implies that for any \(\varepsilon \in (0, 1),\)
\[
\lim_{r \to \infty} \mathbb{P}\left(\|X - \pi^{(r)}(X)\| \leq \kappa \frac{(1 - \varepsilon)^H}{r^H}\right) = 0.
\]
Therefore,
\[
\mathbb{E}[\|X - \pi^{(r)}(X)\| q^{1/q}] \geq \kappa \frac{(1 - \varepsilon)^H}{r^H} \mathbb{P}\left(\|X - \pi^{(r)}(X)\| \geq \kappa \frac{(1 - \varepsilon)^H}{r^H}\right)^{1/q} \sim \kappa \frac{(1 - \varepsilon)^H}{r^H},
\]
which proves the lower bound.

It remains to show that \(D^{(q)}(r, q) \leq \kappa/r^H\). By Lemma 4.2, there exist strategies \(\pi^{(r)}\) and probability weights \((p_w^{(r)})\) such that
\[
\|X - \pi^{(r)}(X)\|_\infty \leq \frac{1}{r^H} \quad \text{and} \quad -\log p_{\pi^{(r)}_n}(X) \leq r, \quad \text{in probability.}
\]
Furthermore, due to Theorem 4.1 in [6], there exist codebooks \(\tilde{C}_r\) of size \(e^r\) with
\[
\mathbb{E}[\min_{\tilde{w} \in \tilde{C}_r} \|X - \tilde{w}\|^2]^{1/2} \approx \frac{1}{r^H}.
\]
We consider the codebook \(C_r := \tilde{C}_r \cup \{\tilde{w} : -\log p_{\bar{w}}^{(r)} \leq (1 + \varepsilon/2)r\}\). Clearly, \(C_r\) contains at most \(e^r + e^{(1+\varepsilon)/2}\) elements. Moreover,
\[
\mathbb{E}[\min_{\tilde{w} \in \tilde{C}_r} \|X - \tilde{w}\|^q]^{1/q} \leq \mathbb{E}[1_{\tilde{C}_r}(\pi^{(r)}(X))(\kappa \frac{1}{r^H})^{q}]^{1/q} + \mathbb{E}[1_{\tilde{C}_r}(\pi^{(r)}(X)) \min_{\tilde{w} \in \tilde{C}_r} \|X - \tilde{w}\|^q]^{1/q}
\]
\[
\leq \frac{1}{r^H} + \mathbb{P}(\pi^{(r)}(X) \notin C_r)^{1/2q} \mathbb{E}[\min_{\tilde{w} \in \tilde{C}_r} \|X - \tilde{w}\|^2]^{1/2q}.
\]
Since \(\lim_{r \to \infty} \mathbb{P}(\pi^{(r)}(X) \notin C_r) = 0\) and the succeeding expectation is of order \(O(1/r^H)\), the second summand is of order \(o(1/r^H)\). Therefore, for \(r \geq 2/\varepsilon\)
\[
D^{(q)}((1 + \varepsilon)r|q) \leq \mathbb{E}[\min_{\tilde{w} \in \tilde{C}_r} \|X - \tilde{w}\|^q]^{1/q} \leq \frac{1}{r^H}.
\]
By switching from \(r\) to \(\bar{r} = (1 + \varepsilon)r\), we obtain
\[
D^{(q)}(\bar{r}|q) \leq \kappa (1 + \varepsilon)^H \frac{1}{\bar{r}^H}.
\]
Since \(\varepsilon > 0\) was arbitrary, the proof is complete. \(\square\)
5 Implications of the equivalence of moments

In this section we complement Theorem 4.1 by

\textbf{Theorem 5.1.} For arbitrary \( q \in (0, \infty] \), one has

\[
D^{(e)}(r|q) \sim \frac{1}{r^{q_1}}.
\]

The proof of this theorem is based on the following general principle: if the asymptotic quantization error coincides for two different moments \( q_1 < q_2 \), then all moments \( q \leq q_2 \) lead to the same asymptotic quantization error and the entropy coding problem coincides with the quantization problem for all moments \( q \leq q_2 \).

Let us prove this relationship in a general setting. \( E \) and \( \hat{E} \) denoting arbitrary measurable spaces and \( d : E \times \hat{E} \to [0, \infty) \) a measurable function, the quantization error for a general \( E \)-valued r.v. \( X \) under the distortion \( d \) is defined as

\[
D^{(q)}(r|q) = \inf_{C \subset \hat{E}} \mathbb{E} \left[ \min_{\hat{x} \in C} d(X, \hat{x})^q \right]^{1/q},
\]

where the infimum is taken over all codebooks \( C \subset \hat{E} \) with \( |C| \leq e^r \). In order to simplify notations, we abridge

\[
d(x, A) = \inf_{y \in A} d(x, y), \quad x \in E, \ A \subset \hat{E}.
\]

Analogously, we denote the entropy coding error by

\[
D^{(e)}(r|q) = \inf_{\hat{X}} \mathbb{E} \left[ d(X, \hat{X})^q \right]^{1/q},
\]

where the infimum is taken over all discrete \( \hat{E} \)-valued r.v. \( \hat{X} \) with \( \mathbb{H}(\hat{X}) \leq r \).

Then Theorem 5.1 is a consequence of Theorem 4.1 and the following theorem.

\textbf{Theorem 5.2.} Assume that \( f : [0, \infty) \to \mathbb{R}_+ \) is a decreasing, convex function satisfying

\[
\limsup_{r \to \infty} \frac{-r^{q_1} f(r)}{f(r)} < \infty,
\]

and suppose that, for some \( 0 < q_1 < q_2 \),

\[
D^{(q)}(r + \log 2|q_1) \sim D^{(q)}(r|q_2) \geq f(r).
\]

Then for any \( q > 0 \),

\[
D^{(e)}(r|q) \geq f(r).
\]

We need some technical lemmas.
Lemma 5.3. Let $0 < q_1 < q_2$ and $f : [0, \infty) \to \mathbb{R}_+$. If
\[ D^{(q)}(r + \log 2|q_1) \sim D^{(q)}(r|q_2) \sim f(r), \]
then for any $\varepsilon > 0$,
\[ \lim_{r \to \infty} \sup_{C \in E: |C| \leq e^r} \mathbb{P}(d(X, C) \leq (1 - \varepsilon)f(r)) = 0. \]

Proof. For $r \geq 0$, let $C^*_r$ denote codebooks of size $e^r$ with
\[ \mathbb{E}[d(X, C^*_r)^{q_2}]^{1/q_2} \sim f(r). \] (21)
Now let $C_r$ denote arbitrary codebooks of size $e^r$, and consider the codebooks $\tilde{C}_r := C^*_r \cup C_r$. Using (21) and the inequality $q_1 \leq q_2$, it follows that
\[ f(r) \geq \mathbb{E}[d(X, \tilde{C}_r)^{q_2}]^{1/q_2} \geq \mathbb{E}[d(X, C^*_r)^{q_1}]^{1/q_1} \geq D^{(q)}(r + \log 2|q_1) \sim f(r). \]
Hence, Lemma A.1 implies that
\[ d(X, \tilde{C}_r) \sim f(r), \quad \text{in probability}, \]
so that in particular,
\[ d(X, C_r) \gtrsim f(r), \quad \text{in probability}. \]

Lemma 5.4. Assume that $f : [0, \infty) \to \mathbb{R}_+$ is a decreasing, convex function satisfying (20) and
\[ \lim_{r \to \infty} \sup_{C \in E: |C| \leq e^r} \mathbb{P}(d(X, C) \leq f(r)) = 0. \]
Then for any $q > 0$,
\[ D^{(e)}(r|q) \gtrsim f(r). \]

Proof. The result is a consequence of the technical Lemma A.3. Consider the family $F$ consisting of all random vectors
\[ (A, B) = (d(X, \tilde{X})^{q}, -\log p_{\tilde{X}}), \]
where $\tilde{X}$ is an arbitrary discrete $E$-valued r.v. and $(p_w)$ is an arbitrary sequence of probability weights on the range of $\tilde{X}$. Let $\tilde{f}(r) = f(r)^q$, $r \geq 0$. Then for any choice of $\tilde{X}$ and $(p_w)$ and an arbitrary $r \geq 0$, the set $C := \{w \in E : -\log p_w \leq r\}$ contains at most $e^r$ elements. Consequently,
\[ \mathbb{P}(d(X, \tilde{X})^q \leq \tilde{f}(r), -\log p_{\tilde{X}} \leq r) = \mathbb{P}(d(X, \tilde{X}) \leq f(r), \tilde{X} \in C) \leq \mathbb{P}(d(X, C) \leq f(r)). \]
By assumption the right hand side converges to 0 as $r \to \infty$, independently of the choice of $\hat{X}$ and $(p_w)$. Since $\hat{f}$ satisfies condition (27), Lemma A.3 implies that

$$D^{(e)}(r|q) = \inf_{\hat{X}: \mathbb{H}(\hat{X}) \leq r} \mathbb{E}[d(X, \hat{X})^q]^{1/q} = \inf_{A \in \mathcal{F}_r} \mathbb{E}[A]^{1/q} \gtrsim \hat{f}(r)^{1/q} = f(r),$$

where $\mathcal{F}_r = \{ A : (A, B) \in \mathcal{F}, EB \leq r \}$. $\square$

Theorem 5.2 is now an immediate consequence of Lemma 5.3 and Lemma 5.4.

6 Coding with respect to the $L^p[0, 1]$-norm distortion

In this section, $p \in [1, \infty)$ is fixed. In contrast to the previous sections, we consider entropy coding and quantization of $X$ in $L^p[0, 1]$, i.e. $\hat{E} = L^p[0, 1]$ and $d(f, g) = \|f - g\|_{L^p[0, 1]}$. In order to treat these approximation problems, we need to introduce Shannon’s distortion rate function. It is defined as

$$D(r|q) = \inf \| X - \hat{X} \|_{L^p[0, 1]},$$

where the infimum is taken over all $\hat{E}$-valued r.v.’s $\hat{X}$ satisfying the mutual information constraint $I(X; \hat{X}) \leq r$. Here and elsewhere $I$ denotes the Shannon mutual information, defined as

$$I(X; \hat{X}) = \begin{cases} \int \log \frac{d\mathbb{P}_{X, \hat{X}}}{d\mathbb{P}_{X, \hat{X}}} d\mathbb{P}_{X, \hat{X}} & \text{if } \mathbb{P}_{X, \hat{X}} \ll \mathbb{P}_X \otimes \mathbb{P}_{\hat{X}} \\ \infty & \text{else.} \end{cases}$$

The objective of this section is to prove

**Theorem 6.1.** The following limit exists

$$\kappa_p = \kappa_p(H) = \lim_{r \to \infty} r^H D(r|p) \in (0, \infty), \quad (22)$$

and for any $q > 0$, one has

$$D^{(q)}(r|q) \sim D^{(e)}(r|q) \sim \kappa_p \frac{1}{r^H}. \quad (23)$$

We will first prove that statement (23) is valid for

$$\kappa_p := \lim_{r \to \infty} r^H D(r|p).$$

Since $D(r|p)$ is dominated by $D^{(q)}(r|p)$, the existence of the limit in (22) then follows immediately. Due to Theorem 1.2 in [4], the distortion rate function $D(\cdot|p)$ has the same weak asymptotics as $D^{(q)}(\cdot|p)$. In particular, $D(r|p) \approx r^{-H}$ and $\kappa_p$ lies in $(0, \infty)$. 

17
We proceed as follows: decomposing $X$ into the two processes

$$X^{(1)} = (X_t - X_{[t]})_{t \geq 0} \quad \text{and} \quad X^{(2)} = (X_{[t]})_{t \geq 0},$$

we consider the coding problem for $X^{(1)}$ and $X^{(2)}$ in $L^p[0, n]$ ($n \in \mathbb{N}$ being large). We control the coding complexity of the first term via Shannon’s Source Coding Theorem (SCT) and use a limit argument in order to show that the coding complexity of $X^{(2)}$ is asymptotically negligible. We recall the SCT in a form which is appropriate for our discussion; for $n \in \mathbb{N}$, let

$$d_p(f, g) = \left( \int_0^1 |f(t) - g(t)|^p \, dt \right)^{1/p}$$

and

$$d_{n,p}(f, g) = \left( \int_0^n |f(t) - g(t)|^p \, \frac{dt}{n} \right)^{1/p}.$$  

Then $\tilde{d}_n(f, g) = d_{n,p}(f, g)^p, n \in \mathbb{N}$, is a single letter distortion measure, when interpreting the function $f|_{[0,n)}$ as the concatenation of the “letters” $f^{(0)}, \ldots, f^{(n-1)}$, where $f^{(i)} = (f(i+t))_{t \in [0,1)}$. Analogously, the process $X^{(1)}$ corresponds to the letters $X^{(1,i)} := (X_{i+t})_{t \in [0,1)}, i \in \mathbb{N}_0$. Since $(X^{(1,i)})_{i \in \mathbb{N}_0}$ is an ergodic stationary $\mathcal{C}[0,1)$-valued process, the SCT implies that for fixed $r > 0$ and $\varepsilon > 0$ there exist codebooks $C_n \subset L^p[0, n], n \in \mathbb{N}$, with at most

$$\exp \left\{ \left( 1 + \varepsilon \right) nr \right\} \text{ elements such that }$$

$$\lim_{n \to \infty} \mathbb{P}(\tilde{d}_n(X^{(1)}, C_n) \leq (1 + \varepsilon)D(r|p)^p) = 1.$$  

A proof of this statement can be carried out by using the asymptotic equipartition property as stated in [2] (Theorem 1). The proof is standard and therefore omitted. For further details concerning the distortion rate function one can consult [1] or [2].

First we prove a lemma which will later be used to control the coding complexity of $X^{(2)}$.

**Lemma 6.2.** Let $(Z_i)_{i \in \mathbb{N}}$ be an ergodic stationary sequence of real-valued r.v.’s and let

$$S_n = \sum_{i=1}^n Z_i, \quad n \in \mathbb{N}_0. \quad \text{Then there exist codebooks } C_n \subset \mathbb{R}^n \text{ of size } \exp \left\{ n \varepsilon \log(|Z_1|/2\varepsilon + 2) + nc \right\} \text{ satisfying}$$

$$\lim_{n \to \infty} \mathbb{P} \left( \min_{\hat{S} \in C} \| S^n_\hat{S} - S^n \|_{\mathbb{R}^n} \leq \varepsilon \right) = 1,$$

where $S^n_\hat{S}$ denotes $(S_i)_{i=1,\ldots,n}, c$ is a universal constant and $\| \cdot \|_{\mathbb{R}^n}$ denotes the maximum norm on $\mathbb{R}^n$.

**Proof.** Let $c > 0$ be such that $(p_n)_{n \in \mathbb{Z}}$ defined through

$$p_n = c^{-c} \frac{1}{(|n| + 1)^2}$$

18
is a sequence of probability weights. For a given sequence \((s_n)_{n \in \mathbb{N}}\), we define a reconstruction \((\hat{s}_n)\) recursively. The construction depends on a parameter \(\varepsilon > 0\). Let \(\hat{s}_0 = 0\) and suppose that \(\hat{s}_n^{(i)} = (\hat{s}_i)_{i=0,...,n}\) is already defined. Then we choose a \(\xi_{n+1} \in 2\varepsilon \mathbb{R}\) minimizing the distance

\[ |s_{n+1} - (\hat{s}_n + \xi_{n+1})| \]

and set \(\hat{s}_{n+1} := \hat{s}_n + \xi_{n+1}\). This defines maps \(\pi_n : \mathbb{R}^n \to \mathbb{R}^n, s_n^{(i)} \mapsto \pi_n(s_n^{(i)}) := \hat{s}_n^{(i)}\). We equip the range of \(\pi_n\) with a sequence of probability weights via

\[ p_{s_1^{(n)}}^{(n)} = \prod_{i=1}^{n} p_{\xi_i/2\varepsilon}. \]

Then

\[ - \log p_{s_1^{(n)}}^{(n)} \leq 2 \sum_{i=1}^{n} \log(|\xi_i|/2\varepsilon + 1) + n c. \]

Now consider \(\pi_n(S_1^n)\). Let \(\xi_n = \xi_n((S_i))\) be as above when replacing the deterministic argument \((s_n)\) by \((S_n)\). Then

\[ |\xi_n - Z_n| = |\hat{S}_n - \hat{S}_{n-1} - S_n + S_{n-1}| \leq 2\varepsilon \]

and, hence, \(|\xi_n| \leq |Z_n| + 2\varepsilon\). Consequently,

\[ - \frac{1}{n} \log p_{s_1^{(n)}}^{(n)} \leq 2 \frac{1}{n} \sum_{i=1}^{n} \log(|Z_i|/2\varepsilon + 2) + c \to 2\mathbb{E}[\log(|Z_1|/2\varepsilon + 2)] + c, \]

where the convergence follows due to the ergodicity of \((Z_n)\). Therefore the codebooks

\[ C_n := \{ \hat{s}_1^n \in \mathbb{R}^n : - \frac{1}{n} \log p_{s_1^{(n)}}^{(n)} \leq 2\mathbb{E}[\log(|Z_1|/2\varepsilon + 2)] + 2c \} \]

satisfy the required assertion. \(\square\)

We now use the SCT combined with the previous lemma to construct codebooks that guarantee almost optimal reconstructions with a high probability.

**Lemma 6.3.** For any \(\varepsilon > 0\) there exist codebooks \(C_r, r \geq 0\), of size \(e^r\) such that

\[ \lim_{r \to \infty} \mathbb{P}(d_p(X, C_r) \leq (1 + \varepsilon) \kappa_p r^{-H}) = 1. \]

**Proof.** Let \(\varepsilon > 0\) be arbitrary and \(c\) be as in Lemma 6.2. We fix \(r_0 \geq \left(\frac{4\varepsilon \kappa_p}{\mathbb{E}[|X_1|]}\right)^{1/H}\) such that

\[ \varepsilon \kappa_p r^{-H} \geq e^{-\varepsilon r_0 + c + \log \mathbb{E}[|X_1|]} \]  

(25)
for all $r \geq r_0$. Then choose $r_1 \geq r_0$ with

$$D(r_1|p) \leq (1 + \varepsilon)\kappa_p r_1^{-H}.$$  

We decompose $X$ into the two processes

$$X^{(1)}_t = X_t - X_{[t]} \quad \text{and} \quad X^{(2)}_t = X_{[t]}.$$  

Due to the SCT (24), there exist codebooks $C^{(1)}_n \subset L^p[0, n]$ of size $\exp\{(1 + \varepsilon)nr_1\}$ satisfying

$$\lim_{n \to \infty} \mathbb{P}(d_{n,p}(X^{(1)}_t, C^{(1)}_n) \leq (1 + 2\varepsilon)\kappa_p r_1^{-H}) = 1.$$  

We apply Lemma 6.2 for $\varepsilon' := \varepsilon \kappa_p r_1^{-H}$. Note that

$$\mathbb{E} \log \left( \frac{|X_1|}{2\varepsilon'} + 2 \right) + c \leq \log \left( \frac{\mathbb{E}|X_1|}{2\varepsilon'} + 2 \right) + c$$

Since $r_1^H \geq \frac{4\kappa_p}{\varepsilon', |X_1|}$, it follows that $\frac{\mathbb{E}|X_1|}{2\varepsilon'} = r_1^H \frac{\mathbb{E}|X_1|}{2\kappa_p} \geq 2$, so that

$$\mathbb{E} \log \left( \frac{|X_1|}{2\varepsilon'} + 2 \right) + c \leq \log \left( \frac{\mathbb{E}|X_1|}{\varepsilon'} + 1 \right) + c$$

$$= -\log(\varepsilon \kappa_p r_1^{-H}) + c + \log \mathbb{E}|X_1| \leq r,$$

due to (25). Hence, there exist codebooks $C^{(2)}_n \subset L^p[0, n]$ of size $\exp\{\varepsilon nr_1\}$ with

$$\lim_{n \to \infty} \mathbb{P}(d_{n,p}(X^{(2)}_t, C^{(2)}_n) \leq \varepsilon \kappa_p \frac{1}{r_1^H}) = 1.$$  

Let now $\tilde{C}_n := C^{(1)}_n + C^{(2)}_n$ denote the Minkowski sum of the sets $C^{(1)}_n$ and $C^{(2)}_n$. Then $|\tilde{C}_n| \leq \exp\{(1 + 2\varepsilon)nr_1\}$, and one has

$$\mathbb{P}(d_{n,p}(X, \tilde{C}_n) \leq (1 + 3\varepsilon)\kappa_p r_1^{-H}) \geq \mathbb{P}(d_{n,p}(X^{(1)}_t, C^{(1)}_n) \leq (1 + 2\varepsilon)\kappa_p r_1^{-H} \text{ and}$$

$$d_{n,p}(X^{(2)}_t, C^{(2)}_n) \leq \varepsilon \kappa_p r_1^{-H}) \to 1.$$  

Consider the isometric isomorphism

$$\beta_n : L^p[0, 1] \to (L^p[0, n], d_{n,p}), \ f \mapsto f(nt),$$

and the codebooks $C_n \subset L^p[0, 1]$ given by

$$C_n = \{n^{-H} \beta^{-1}_n(\hat{w}) : \hat{w} \in \tilde{C}_n\}$$

Then $\tilde{X}^{(n)} = n^{-H} \beta^{-1}_n(X)$ is a fractional Brownian motion and one has

$$d_{p}(\tilde{X}^{(n)}, C_n) = d_{n,p}(\beta_n(\tilde{X}^{(n)}), \beta_n(C_n)) = n^{-H}d_{n,p}(X, \tilde{C}_n).$$
Hence, the codebooks $C_n$ are of size \( \exp\{(1 + 2\varepsilon)nr_1\} \) and satisfy
\[
\mathbb{P}(d_p(X, C_n) \leq (1 + 3\varepsilon)\kappa_p(nr_1)^{-H}) = \mathbb{P}(d_n, p(X, \tilde{C}_n) \leq (1 + 3\varepsilon)\kappa_p r_1^{-H}) \to 0
\]
as \( n \to \infty \). Now the general statement follows by an interpolation argument similar to that used at the end of the proof of Theorem 3.1. \( \square \)

**Proof of Theorem 6.1.** Let \( q \geq 1 \) be arbitrary, let \( C^{(1)}_r \) be as in the above lemma for some fixed \( \varepsilon > 0 \). Moreover, we let \( C^{(2)}_r \) denote codebooks of size \( e^r \) with
\[
\mathbb{E}[d_p(X, C^{(2)}_r)^{2q}]^{1/(2q)} \approx \frac{1}{r^H}.
\]
Then the codebooks \( C_r := C^{(1)}_r \cup C^{(2)}_r \) contain at most \( 2e^r \) elements and satisfy, in analogy to the proof of Theorem 4.1 (see (19)),
\[
\mathbb{E}[d_p(X, C_r)^q]^{1/q} \lesssim (1 + \varepsilon)\kappa_p \frac{1}{r^H}, \quad r \to \infty.
\]
Since \( \varepsilon > 0 \) is arbitrary, it follows that
\[
D^{(q)}(r|q) \lesssim \kappa_p \frac{1}{r^H}.
\]
For \( q \geq p \) the quantization error is greater than the distortion rate function \( D(r|p) \), so that the former inequality extends to
\[
\lim_{r \to \infty} r^H D^{(q)}(r|q) = \kappa_p.
\]
In particular, we obtain the asymptotic equivalence of all moments \( q_1, q_2 \) greater or equal to \( p \). Next, an application of Theorem 5.2 with \( d(f, g) = d_p(f, g)^q \) implies that for any \( q > 0 \),
\[
D^{(q)}(r|q) \gtrsim \kappa_p \frac{1}{r^H},
\]
which establishes the assertion. \( \square \)

**Appendix**

**Lemma A.1.** For \( r \geq 0 \), let \( A_r \) denote \([0, \infty)\)-valued r.v.’s. If one has, for \( 0 < q_1 < q_2 \) and some function \( f : [0, \infty) \to \mathbb{R}_+ \),
\[
\mathbb{E}[A_r^{q_1}]^{1/q_1} \sim \mathbb{E}[A_r^{q_2}]^{1/q_2} \sim f(r), \quad (26)
\]
then
\[
A_r \sim f(r), \text{ in probability.}
\]
Proof. Consider
\[ \tilde{A}_r := A^{q_1}_r / \mathbb{E}[A^{q_1}_r], \]
and \( \tilde{q}_2 = q_2 / q_1 \). Then (26) implies that
\[ \mathbb{E}[\tilde{A}^{\tilde{q}_2}]^{1/\tilde{q}_2} \sim \mathbb{E}[\tilde{A}_r] = 1 \]
Denoting \( \Delta \tilde{A}_r := \tilde{A}_r - 1 \) and \( g(x) := x^{\tilde{q}_2} \), we obtain
\[ \mathbb{E}[\tilde{A}^{\tilde{q}_2}] = \mathbb{E}[1 + \Delta \tilde{A}_r g'(1) + g(1 + \Delta \tilde{A}_r) - (1 + \Delta \tilde{A}_r g'(1))] \]
Due to the strict convexity of \( g \), for arbitrary \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[ g(x + 1) \geq 1 + xg'(1) + \delta, \text{ for } x \in [-1, 1 - \varepsilon] \cup [1 + \varepsilon, \infty). \]
Consequently,
\[ \mathbb{E}[\tilde{A}^{\tilde{q}_2}] \geq 1 + \delta \mathbb{P}(|\Delta \tilde{A}_r| \geq \varepsilon). \]
Since \( \lim_{r \to \infty} \mathbb{E}[\tilde{A}^{\tilde{q}_2}] = 1 \), it follows that \( \lim_{r \to \infty} \mathbb{P}(|\Delta \tilde{A}_r| \geq \varepsilon) = 0 \). Hence,
\[ A_r = \mathbb{E}[A^{q_1}_r]^{1/q_1} \tilde{A}_r^{1/q_1} \sim \mathbb{E}[A^{q_1}_r]^{1/q_1} \sim f(r), \text{ in probability}. \]

Lemma A.2. Let \( q \geq 1 \). There exists a constant \( c = c(q) < \infty \) such that for all \([1, \infty)\)-valued r.v.\'s \( Z \) one has
\[ \mathbb{E}[(\log Z)^q]^{1/q} \leq c[1 + \log \mathbb{E}[Z]]. \]

Proof. Using elementary analysis, there exists a positive constant \( c_1 = c_1(q) < \infty \) such that \( \psi(x) := (\log x)^q + c_1 \log x, x \in [1, \infty) \), is concave. For any \([1, \infty)\)-valued r.v. \( Z \), Jensen’s inequality then yields
\[ \mathbb{E}[(\log Z)^q]^{1/q} \leq \mathbb{E}[\psi(Z)]^{1/q} \leq \psi(\mathbb{E}[Z])^{1/q} \]
\[ \leq \log \mathbb{E}[Z] + c_1^{1/q}(\log \mathbb{E}[Z])^{1/q} \leq c[1 + \log \mathbb{E}[Z]], \]
where \( c = c(q) < \infty \) is an appropriate universal constant. \( \square \)

Lemma A.3. Let \( f : [0, \infty) \to \mathbb{R}_+ \) be a decreasing, convex function satisfying \( \lim_{r \to \infty} f(r) = 0 \) and
\[ \limsup_{r \to \infty} \frac{-r \frac{\partial}{\partial r} f(r)}{f(r)} < \infty, \quad (27) \]

22
and $\mathcal{F}$ be a family of $[0, \infty]^2$-valued random variables for which

$$
\lim_{r \to \infty} \sup_{(A, B) \in \mathcal{F}} \mathbb{P}(A \leq f(r), B \leq r) = 0.
$$

Then the sets of random variables $\mathcal{F}_r$ defined for $r \geq 0$ through

$$
\mathcal{F}_r := \{ A : (A, B) \in \mathcal{F}, \mathbb{E}B \leq r \}
$$

satisfy

$$
\inf_{A \in \mathcal{F}_r} \mathbb{E}A \gtrsim f(r)
$$
as $r \to \infty$.

**Proof.** Fix $R > 0$, positive integers $I$ and $N$, and define $\lambda := -\frac{d}{dr} f(R)$, $r_i := \frac{i + N}{N} R$, $i = -N, -N + 1, \ldots$.

For $(A, B) \in \mathcal{F}_R$, we define

$$
T_{A, B} := \{ \# i \in \{-N + 1, \ldots, I\} \text{ such that } A \leq f(r_i) \text{ and } B \leq r_i \}.
$$

Then we have

$$
\mathbb{E}[A + \lambda B] \geq \sum_{i=-N}^{I-1} \mathbb{E}[1_{T_{A, B}}(r_i, r_{i+1})(B)(A + \lambda r_i)]
$$

$$
\geq \sum_{i=-N}^{I-1} \mathbb{E}[1_{T_{A, B}}(r_i, r_{i+1})(B)(f(r_{i+1}) + \lambda r_i)]
$$

$$
= \sum_{i=-N}^{I-1} \mathbb{E}[1_{T_{A, B}}(r_i, r_{i+1})(B)(f(r_{i+1}) + \lambda r_{i+1} - \frac{R}{N})]
$$

$$
\geq \sum_{i=-N}^{I-1} \mathbb{E}[1_{T_{A, B}}(r_i, r_{i+1})(B)(f(R) + \lambda R - \frac{R}{N})],
$$

where the last inequality follows from the fact that

$$
f(R) + \lambda R = \inf_{r \geq 0} [f(r) + \lambda r]
$$

by the definition of $\lambda$ and the convexity of $f$. Now, fix $\varepsilon > 0$ and pick $N \geq 1/\varepsilon$, $I \geq 2N/\varepsilon$ and $R_0$ so large that

$$
\mathbb{P}(T_{A, B}) \geq 1 - \frac{\varepsilon}{2} \text{ for all } R \geq R_0 \text{ and all } (A, B) \in \mathcal{F}_R.
$$
Using Chebychev’s inequality, we then obtain for $R \geq R_0$,

$$\mathbb{E}[A + \lambda B] \geq (1 - \varepsilon)(f(R) + \lambda R) \left(1 - \mathbb{P}(T^c) - \mathbb{P}\left(B \geq R \frac{I}{N}\right)\right) \geq (1 - \varepsilon)(f(R) + \lambda R) \left(1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2}\right).$$

Hence,

$$\lambda R + \mathbb{E} A \geq (1 - \varepsilon)^2 (f(R) + \lambda R)$$

and therefore

$$\mathbb{E} A \geq (1 - \varepsilon)^2 f(R) + \lambda R \left((1 - \varepsilon)^2 - 1\right).$$

Using the definition of $\lambda$ and (27), as well as the fact that $\varepsilon > 0$ is arbitrary, the conclusion follows.

References


