Besov Regularity for the Stokes System in Polyhedral Cones

Frank Eckhardt
Philipps-Universität Marburg
Fachbereich Mathematik und Informatik
Hans-Meerweinstr., Lahnberge
35032 Marburg

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Abstract

In this paper we study the regularity of solutions to the Stokes system in polyhedral domains contained in $\mathbb{R}^3$. We consider the scale $B^s_\tau(L_\tau)$, $1/\tau = s/3 + 1/2$ of Besov spaces which arise in connection with adaptive numerical shemes. The proof of the main result is performed by combining regularity results in weighted Sobolev spaces with characterizations of Besov spaces by wavelet expansions.

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1 Introduction

In this paper we consider the $3D$-Stokes System on $\mathcal{K}$:

$$\begin{aligned} -\Delta u + \nabla p &= f \quad \text{in } \mathcal{K} \\ \text{div } u &= g \quad \text{in } \mathcal{K} \\ u &= 0 \quad \text{on } \Gamma_j,\ j = 1, ..., d, \end{aligned}$$

where $\mathcal{K}$ is a polyhedral cone in $\mathbb{R}^3$ (see [15] for a definition of polyhedral cones). Here $\Delta := \sum_{k=1}^{3} \frac{\partial^2}{\partial^2 x_i}$ is the Laplace operator and by $\nabla := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ we denote the gradient. As usual, $u(\cdot) = (u_1(\cdot), u_2(\cdot), u_3(\cdot))$ is the velocity field and p stands for the pressure field.

Our aim is to prove regularity results for each component of the solution (u, p) in the specific scale of Besov spaces $B^s_\tau(L_\tau(\mathcal{K}_0))$, $1/\tau = s/3 + 1/2$, where $\mathcal{K}_0$ is a truncated cone. This specific scale comes into play when studying the convergence rate of adaptive numerical shemes. We will explain the relationship very briefly in the following. Assuming $g = 0$ the weak formulation of the Stokes problem is given by

$$a(u,v) + b(p,v) = f(v) := \int_{\mathcal{K}_0} \langle f, v\rangle \, \mathrm{d}x \text{ for all } v \in [H^1_0(\mathcal{K}_0)]^3,$$

$$b(q,u) = 0 \text{ for all } q \in L_{2,0}(\mathcal{K}_0),$$

with

$$a(u,v) := \int_{\mathcal{K}_0} \sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j}\frac{\partial v_j}{\partial x_j} \mathrm{dx}$$

and

$$b(p,v) := -\int_{\mathcal{K}_0} p(x)(\mathrm{div} v)(x)\mathrm{d}x.$$

$H^1_0(\mathcal{K}_0)$ is the closure of $\mathcal{C}^\infty_0(\mathcal{K}_0)$ with respect to the $H^1(\mathcal{K}_0)$-Sobolev norm and $L_{2,0}(\mathcal{K}_0) := \{p \in L_2(\mathcal{K}_0) : \int_{\mathcal{K}_0} p(x)\mathrm{d}x = 0\}$. To treat the equation numerically we use the Galerkin approach, i.e. we consider a nested sequence $\{S_j \times \tilde{S}_j\}_{j\geq 0}$ of finite dimensional linear subspaces of $[H^1_0(\mathcal{K}_0)]^3 \times L_{2,0}(\mathcal{K}_0)$ such that the union is dense in $[H^1_0(\mathcal{K}_0)]^3 \times L_{2,0}(\mathcal{K}_0)$. This leads to the problems

$$a(u_j,v) + b(p_j,v) = f(v) \text{ for all } v \in S_j,$$

$$b(q,u_j) = 0 \text{ for all } q \in \tilde{S}_j,$$

In many cases, the approximation is performed by means of uniform grid refinement strategy. This kind of approximation is called *linear approximation*. It is well known that the performance usually depends on the Sobolev regularity of the solution. For details see [11]. However, in practice, due to singularities at the boundary of the domain, this Sobolev regularity might not be very high and therefore the approximation rate of uniform schemes drop down. In this setting the use of adaptive strategies seems to be reasonable. Roughly speaking an adaptive scheme corresponds to nonuniform grid refinement where the underlaying space is only refined in regions where the current approximation is still far away from the exact solution. In this paper we are in particular interested in adaptive wavelet algorithms. In this setting, an adaptive scheme can be interpreted as a nonlinear approximation scheme, and for that reason best n-term approximation serves as a benchmark for adaptive strategies (see [1],[3] for further details): Instead of linear spaces one can also use a nonlinear manifold $\mathcal{M}_n$ of all functions

$$S = \sum_{\lambda\in\Lambda} c_\lambda \psi_\lambda, \quad |\Lambda| \leq n,$$

where $\{\psi_\lambda\}_{\lambda\in\mathcal{J}}$ is a suitable wavelet basis. We define the approximation error

$$\sigma_n(u)_{L_2} := \inf_{S\in\mathcal{M}_n} ||u - S||_{L_2} \sim ||u - u_n||_{L_2}.$$

It is known that

$$u_n = \sum_{\lambda\in\Lambda_n} c_\lambda \psi_\lambda,$$

where Λ_n is the set of the n biggest wavelet coefficients. In contrast to linear approximation schemes, the order of convergence for best n-term approximation does not depend on the Sobolev regularity, but on the Besov smoothness, i.e.

$$\sum_{n=1}^{\infty}[n^{s/d}\sigma_n(u)_{L_2}]^\tau \frac{1}{n} < \infty \Longleftrightarrow u \in B^s_\tau(L_\tau),\ 1/\tau = s/d + 1/2,$$

see [9] for further details. As suggested above this shows that it is profitable to use adaptive shemes if the Besov regularity of the solution in this specific scale is higher than the Sobolev regularity. It is known that in smooth domains the Sobolev regularity of the solution increases if the Sobolev regularity of f and g increase (see [12] for details). If the domain is only Lipschitz this conclusion is no longer true: As mentioned above in non smooth domains it is hard to obtain a high Sobolev regularity but we can still hope to achieve a high Besov regularity. That is what we try to do here. Remember that we consider the problem on a truncated cone with singularities in the origin and on the edges of this cone. To prove regularity results we need certain weighted Sobolev spaces which take these singularities into account. We denote these spaces with $W^{l,2}_{\beta,\vec{\delta}}$, for details see Section 2. In this paper we establish a result which shows that under certain technical conditions the Besov regularity to the solution of the Stokes problem is higher than the Sobolev regularity if additionally the parameter l is not so small: For suitable values of l the Besov regularity is at least $3/2$ times higher than the Sobolev regularity. For details see Theorem 2.1.

There are already some positive results for diverse linear equations: In [2] it was shown that the Besov regularity of the 2D-Stokes system in a polygonal domain is under some technical conditions higher than the Sobolev regularity. In [4] the Besov regularity of the solution to the Dirichlet problem for harmonic functions and for the Poisson equation in Lipschitz domains was investigated. A result which is similiar to our main statement was proven in [5] for Poisson equation. Similiar to the investigation in this paper, the results are proven by using the caracterization of Besov spaces by means of weighted sequence norms of coefficients related to the wavelet decomposition of the solution. Furthermore there are also results for nonlinear partial differential equations, see [6].

This paper is organized as follows: In the second section we recall the definition of the weighted Sobolev spaces. Next we state and prove the main result of this paper. In the third section we discuss regularity results which play a fundamental role in the proof of our theorem: As mentioned above we need weighted Sobolev estimations. Moreover we need that the solution has a certain Sobolev regularity in order to ensure Besov regularity near the boundary of the cone. In the last section we recall the definition of Besov and Sobolev Spaces and state the fundamental result which explains the connection between the Besov regularity of a distribution and its wavelet decomposition.

2 Besov Regularity for the Stokes System

Let

$$\mathcal{K} = \{x \in \mathbb{R}^3 : x = \rho \cdot \omega,\ \ 0 < \rho < \infty, \omega \in \Omega\} \tag{2.1}$$

be a polyhedral cone with vertex at the origin where Ω is a curvilinear polygon on the unit sphere bounded by the arcs $\gamma_1, ..., \gamma_d$. Suppose that the boundary $\partial\mathcal{K}$ consists of the vertex $x = 0$, the edges $M_1, ..., M_d$ and the faces $\Gamma_j := \{x : x/|x| \in \gamma_j\}$, $j = 1, ..., d$. The angle at edge M_j will be denoted by θ_j. Furthermore we define for $x \in \mathcal{K}$ the function $r_j(x) := \mathrm{dist}(x, M_j)$. We consider the stationary Stokes system

$$\begin{aligned} -\Delta u + \nabla p &= f \text{ in } \mathcal{K} \\ \mathrm{div}\ u &= g \text{ in } \mathcal{K} \\ u &= 0 \text{ on } \Gamma_j,\ j = 1, ..., d. \end{aligned} \tag{2.2}$$

To prove our regularity results we need that the solution (u, p) of (2.2) is contained in a certain weighted Sobolev space which was introduced by Maz'ya and Rossmann, see [15]. These spaces are defined as follows: Let l be a nonnegative integer, $\beta \in \mathbb{R}$ and $\vec{\delta} = (\delta_1, ..., \delta_d) \in \mathbb{R}^d$, $\delta_j > -1$ for $j = 1, ..., d$. We define the space $W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})$ as the closure of the set $\mathcal{C}_0^\infty(\overline{\mathcal{K}}\backslash\{0\})$ with respect to the norm

$$||u||_{W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})} := \left(\int_{\mathcal{K}} \sum_{|\alpha|\le l} \rho(x)^{2(\beta-l+|\alpha|)} \prod_{k=1}^{d} \left(\frac{r_k(x)}{\rho(x)}\right)^{2\delta_k} |D^\alpha u(x)|^2 \mathrm{d}x\right)^{1/2}.$$

By $\mathcal{K}_0$ we denote an arbitrary truncated cone, i.e. there exists a positive real number r_0 such that

$$\mathcal{K}_0 = \{x \in \mathcal{K} : |x| < r_0\}.$$

Now we can formulate the main result of this paper.

Theorem 2.1. *Fix an integer $l \geq 2$ and a real number $0 < \alpha_0 < 0.5$. It exists a countable set $E \subset \mathbb{C}$ such that for all $\beta \in \mathbb{R}$, $\vec{\delta} \in (\mathbb{R}\backslash\mathbb{Z})^d$ with*

$$\begin{aligned} \beta &< l - 1, \\ Re\lambda &\neq l - \beta - \frac{3}{2} \quad \textit{for all } \lambda \in E \end{aligned} \tag{2.3}$$

and

$$\max\left(0, l - 1 - \frac{\pi}{\theta_k}\right) < \delta_k < l - 1, \quad k = 1, ..., d$$

the following holds: If $(f, g) \in \left[W^{l-2,2}_{\beta,\vec{\delta}}(\mathcal{K}) \cap L_2(\mathcal{K}_0)\right]^3 \times W^{l-1,2}_{\beta,\vec{\delta}}(\mathcal{K}) \cap H^{\alpha_0}(\mathcal{K}_0)$ and g fulfills the compatibility condition

$$\int_{\mathcal{K}_0} g(x)\, dx = 0,$$

then the unique solution (u,p) *of problem (2.2) satisfies*

$$u \in \left[B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{K}_0))\right]^3, \quad \frac{1}{\tau_1} = \frac{s_1}{3} + \frac{1}{2}, \;\; s_1 < \min\left(l, \frac{3}{2}\cdot(\alpha_0+1), 3\cdot(l-|\vec{\delta}|)\right), \tag{2.4}$$

$$p \in B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{K}_0)), \quad \frac{1}{\tau_2} = \frac{s_2}{3} + \frac{1}{2}, \;\; s_2 < \min\left(l-1, \frac{3}{2}\cdot\alpha_0, 3\cdot(l-(|\vec{\delta}|+1))\right) \tag{2.5}$$

Remark 2.2. (i) In the proof of Theorem 2.1 we will show that for $\beta < l-1$ we get

$$||u||_{B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{K}_0))^3} + ||p||_{B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{K}_0))} \lesssim ||u||_{W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})^3} + ||u||_{H^{\alpha_0+1}(\mathcal{K}_0)^3} + ||p||_{W^{l-1,2}_{\beta,\vec{\delta}}(\mathcal{K})} + ||p||_{H^{\alpha}_0(\mathcal{K}_0)}.$$

From Remark 3.3 we obtain

$$||u||_{B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{K}_0)^3} + ||p||_{B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{K}_0))} \lesssim$$

$$\left(||F||_{H^*_{l-1-\beta}} + ||g||_{V^{0,2}_{\beta-l+1}} + ||f||_{W^{l-2,2}_{\beta,\delta}(\mathcal{K})^3} + ||g||_{W^{l-1,2}_{\beta,\delta}(\mathcal{K})} + ||f||_{L_2(\mathcal{K}_0)^3} + ||g||_{H^{\alpha}(\mathcal{K}_0)}\right)$$

(ii) The set E in the theorem is the set of eigenvalues of the operator pencil related to (2.2). It is known that E consists of isolated points, see [13] for details. Therefore by a minor modification of β, condition (2.3) is satisfied and our arguments in the proof below also work with this minor modification. That shows that condition (2.3) is not as restrictive as it seems to be.

(iii) If we achieve a Sobolev regularity $\beta_0 > \alpha_0$ we would get the condition $s_1 < \min(l-1, 3/2(\beta_0+1), 3(l-|\vec{\beta}), s_2 < \min(l-1, 3/2\beta_0, 3(l-(|\vec{\beta}+1))$. As a consequence we need a higher value of l to justify the use of adaptive shemes. But if l is high enough the Besov regularity is also higher (compared to the case when the Sobolev regularity is only α_0) so we obtain a better convergence rate.

Proof of Theorem 2.1

Proof. : To prove the theorem we will concentrate on each component of the solution $(u,p) = (u_1,u_2,u_3,p)$ to (2.2) separatly. Let v be one of the functions u_1,u_2,u_3 respectively p. Moreover we define the natural number

$$\mu := \begin{cases} l & v = u_i \text{ for } i = 1,2 \text{ or } 3 \\ l-1 & v = p \end{cases}.$$

The proof uses the characterizations of Besov spaces by wavelet expansions. Therefore we estimate the wavelet coefficients of v in order to show that the equivalent quasi-norm in Proposition 4.1 is bounded. We make the following agreements concerning the wavelet characterization of Besov spaces on $\mathbb{R}^3$: For the sake of simplicity we associate to each dyadic cube $I := 2^{-j}k + 2^{-j}[0,1]^3$ the functions

$$\eta_I := \tilde{\psi}_{i,j,k}, \quad j \in \mathbb{N}_0, \;\; k \in \mathbb{Z}^3, \; i = 1,...,7,$$

that means we disregard the dependence on i. By η_I^* we denote the corresponding element of the dual basis. Because the supports of the wavelets are assumed to be compact there exists a cube Q centered at the origin such that

$$Q(I) := 2^{-j}k + 2^{-j}Q$$

contains the support of η_I and η_I^* for all I. Since $f \in [L_2(\mathcal{K}_0)]^3$ and $g \in H^{\alpha_0}(\mathcal{K}_0)$ we have $(u,p) \in [H^{\alpha_0+1}(\mathcal{K}_0)]^3 \times H^{\alpha_0}(\mathcal{K}_0)$, see Proposition 3.1. We will prove the result in three steps: In a first step we will estimate the coefficients $|\langle v, \eta_I \rangle|$ for with $Q(I)$ is contained in the truncated cone and the distance from $Q(I)$ to the origin is not so small. We will specifiy this later. In a second step we look for the coefficients for which $Q(I)$ is contained in $\mathcal{K}_0$ but $Q(I)$ can be located arbitrarily close zu the origin. In the last step we consider the coefficients for which the intersection of $Q(I)$ and the boundary of $\mathcal{K}_0$ is not empty.

step 1: We start by estimating the coefficients $|\langle v, \eta_I \rangle|$ with $Q(I) \subset \mathcal{K}_0$. We define

$$\rho_I := \operatorname{dist}(Q(I), 0)$$

and

$$r_I := \min_{j=1,\dots,d} \min_{x \in Q(I)} r_j(x).$$

For $j \in \mathbb{N}_0$ consider the set of indices:

$$\Lambda_j := \{I : Q(I) \subset \mathcal{K}_0,\ \ 2^{-3j} \le |I| \le 2^{-3j+2}\}.$$

Then we define a subset of Λ_j for $k \in \mathbb{N}$:

$$\Lambda_{j,k} := \{I \in \Lambda_j : k2^{-j} \le \rho_I < (k+1)2^{-j}\}.$$

Further we put for $m \in \mathbb{N}$

$$\Lambda_{j,k,m} := \{I \in \Lambda_{j,k} : m2^{-j} \le r_I < (m+1)2^{-j}\}.$$

We recognize the following facts:

- There exists a general number C such that

$$Q(I) \cap \mathcal{K}_0 = \emptyset \text{ if } I \in \Lambda_{j,k},\ \ k > C2^j. \tag{2.6}$$

- For the cardinality $|\Lambda_{j,k}|$ of $\Lambda_{j,k}$ holds

$$|\Lambda_{j,k}| \lesssim k^2,\ \ k \in \mathbb{N}. \tag{2.7}$$

- It holds

$$|\Lambda_{j,k,m}| \lesssim m,\ \ m \in \mathbb{N}. \tag{2.8}$$

In every case the constant is independent of j,k and m. We put

$$|v|_{W^{\mu}(L_2(Q(I)))} := \left(\int_{Q(I)} |\nabla^{\mu} v(x)|^2 \mathrm{d}x\right)^{1/2},$$

which is well defined because of Proposition 3.2. The vector space of polynomials of order at most μ is finite-dimensional so there exists a polynomial P_I such that

$$||v - P_I||_{L_2(Q(I))} = \inf\left\{||v-P||_{L_2(Q(I))} : P \text{ is a polynomial of degree } \leq \mu\right\}.$$

The vanishing moment property of wavelets, see subsection 4.3, Hölders inequality and a classical Whitney-estimate (see [10, Theorem 3.4]) lead to

$$\begin{aligned} |\langle v, \eta_I \rangle| &\leq ||v - P_I||_{L_2(Q(I))} ||\eta_I||_{L_2(Q(I))} \\ &\lesssim |I|^{\mu/3} \cdot |v|_{W^{\mu}(L_2(Q(I)))}. \end{aligned}$$

For $I \in \Lambda_j$ we obtain

$$|\langle v, \eta_I \rangle| \lesssim 2^{-\mu j} |v|_{W^{\mu}(L_2(Q(I)))}.$$

Let $0 < \tau < 2$. Summing up over $I \in \Lambda_{j,k}$ yields

$$\begin{aligned} \sum_{I\in\Lambda_{j,k}} |\langle v, \eta_I\rangle|^{\tau} &\lesssim \sum_{I\in\Lambda_{j,k}} 2^{-\mu j \tau} \left(\int_{Q(I)} |\nabla^{\mu} v(x)|^2 \mathrm{d}x\right)^{\tau/2} \\ &\lesssim \sum_{I\in\Lambda_{j,k}} 2^{-\mu j\tau} r_I^{-\tau|\vec{\delta}|} \rho_I^{-\tau(\beta-|\vec{\delta}|)} \left(\int_{Q(I)} \rho^{2(\beta-|\vec{\delta}|)} \left(\prod_{\nu=1}^{d} r_{\nu}^{\delta_{\nu}}\right)^2 |\nabla^{\mu} v(x)|^2 \mathrm{d}x\right)^{\tau/2}. \end{aligned}$$

We define

$$v_I := \int_{Q(I)} \rho^{2(\beta-|\vec{\delta}|)} \left(\prod_{\nu=1}^{d} r_{\nu}^{\delta_{\nu}}\right)^2 |\nabla^{\mu} v(x)|^2 \mathrm{d}x.$$

Now we focus on the coefficients belonging to $\Lambda_{j,k,m}$. We now have to consider the cases $\beta > |\vec{\delta}|$ and $|\vec{\delta}| \geq \beta$ separatly. If $\beta - |\vec{\delta}| > 0$ we can conclude $\rho_I^{-\tau(\beta-|\vec{\delta}|)} \lesssim (k2^{-j})^{-\tau(\beta-|\vec{\delta}|)}$. Otherwise we get $\rho_I^{-\tau(\beta-|\vec{\delta}|)} \lesssim ((k+1)2^{-j})^{-\tau(\beta-|\vec{\delta}|)}$. Without loss of generality we assume $\beta > |\vec{\delta}|$. The second case can be treated analogously. Using Hölder's inequality with $q = 2/\tau$, $q' = 2/(2-\tau)$ results in

$$\begin{aligned} \sum_{I\in\Lambda_{j,k,m}} |\langle v, \eta_I\rangle|^{\tau} &\lesssim 2^{-\mu\tau j} (k2^{-j})^{-\tau(\beta-|\vec{\delta}|)} \left(\sum_{I\in\Lambda_{j,k,m}} r_I^{-\tau|\vec{\delta}|\frac{2}{2-\tau}}\right)^{\frac{2-\tau}{2}} \cdot \left(\sum_{I\in\Lambda_{j,k,m}} v_I\right)^{\frac{\tau}{2}} \\ &\lesssim 2^{-\mu\tau j} (k2^{-j})^{-\tau(\beta-|\vec{\delta}|)} \left(\sum_{I\in\Lambda_{j,k,m}} (m2^{-j})^{-\tau|\vec{\delta}|\frac{2}{2-\tau}}\right)^{\frac{2-\tau}{2}} \left(\sum_{I\in\Lambda_{j,k,m}} v_I\right)^{\frac{\tau}{2}} \end{aligned}$$

Together with (2.8) we obtain

$$\sum_{I\in\Lambda_{j,k,m}} |\langle v,\eta_I\rangle|^\tau \lesssim 2^{\tau j(\beta-\mu)} k^{-\tau(\beta-|\vec{\delta}|)} m^{-\tau|\vec{\delta}|+\frac{2-\tau}{2}} \left(\sum_{I\in\Lambda_{j,k,m}} v_I\right)^{\frac{\tau}{2}}.$$

We continue by using the fact that there are of order k sets $\Lambda_{j,k,m}$ in each layer $\Lambda_{j,k}$. Together with Hölders inequality, this gives

$$\sum_{I\in\Lambda_{j,k}} |\langle v,\eta_I\rangle|^\tau \lesssim 2^{j\tau(\beta-\mu)} k^{-\tau(\beta-|\vec{\delta}|)} \left(\sum_{m=1}^{Ck} m^{-\tau|\vec{\delta}|\frac{2}{2-\tau}+1}\right)^{\frac{2-\tau}{2}} \left(\sum_{I\in\Lambda_{j,k}} v_I\right)^{\frac{\tau}{2}}. \tag{2.9}$$

Note, that the constant C only depends on $\mathcal{K}_0$. Together with

$$\left(\sum_{m=1}^{Ck} m^{-\tau|\vec{\delta}|\frac{2}{2-\tau}+1}\right)^{\frac{2-\tau}{2}} \lesssim \begin{cases} k^{-\tau|\vec{\delta}|+2-\tau} & 2>\tau(1+|\vec{\delta}|),\\ (\log(1+k))^{\frac{2-\tau}{2}} & 2=\tau(1+|\vec{\delta}|),\\ 1 & 2<\tau(1+|\vec{\delta}|). \end{cases}$$

we obtain from (2.9)

$$\sum_{I\in\Lambda_{j,k}} |\langle v,\eta_I\rangle|^\tau \lesssim 2^{j\tau(\beta-\mu)} \left(\sum_{I\in\Lambda_{j,k}} v_I\right)^{\frac{\tau}{2}}$$
$$\times \begin{cases} k^{-\tau(\beta+1)+2} & 2>\tau(1+|\vec{\delta}|),\\ k^{-\tau(\beta-|\vec{\delta}|)}(\log(1+k))^{\frac{2-\tau}{2}} & 2=\tau(1+|\vec{\delta}|),\\ k^{-\tau(\beta-|\vec{\delta}|)} & 2<\tau(1+|\vec{\delta}|). \end{cases}$$

To simplify notation we denote these functions of k in the second line by a_k. Employing (2.6) and Hölder's inequality we get

$$\sum_{I\in\Lambda_j} |\langle v,\eta_I\rangle|^\tau \lesssim 2^{j\tau(\beta-\mu)} \left(\sum_{k=1}^{C2^j} a_k^{\frac{2}{2-\tau}}\right)^{\frac{2-\tau}{2}} \left(\sum_{I\in\Lambda_j} v_I\right)^{\frac{\tau}{2}}.$$

From Proposition 3.2 we conclude that the last factor is bounded. To complete the estimate we have to sum with respect to $j\in\mathbb{N}_0$: We first consider the sum $\sum_{k=1}^{C2^j} a_k^{\frac{2}{2-\tau}}$ and derive estimates depending on β. Then we study the convergence of

$$\sum_{j\geq 0}\left(2^{j\tau(\beta-\mu)}\left(\sum_{k=1}^{C2^j} a_k^{\frac{2}{2-\tau}}\right)^{\frac{2}{2-\tau}}\right).$$

More detailed we get the following cases:

$$\begin{aligned}
3\left(\tfrac{1}{\tau}-\tfrac{1}{2}\right) &< \mu && \text{if } \tau(1+|\vec{\delta}|)<2 \quad \text{and} \quad \beta < 3\left(\tfrac{1}{\tau}-\tfrac{1}{2}\right),\\
\beta &< \mu && \text{if } \tau(1+|\vec{\delta}|)<2 \quad \text{and} \quad \beta \geq 3\left(\tfrac{1}{\tau}-\tfrac{1}{2}\right),\\
\tfrac{3}{2}|\vec{\delta}| &< \mu && \text{if } \tau(1+|\vec{\delta}|)=2 \quad \text{and} \quad \beta < \tfrac{3}{2}|\vec{\delta}|,\\
\beta &< \mu && \text{if } \tau(1+|\vec{\delta}|)=2 \quad \text{and} \quad \beta \geq \tfrac{3}{2}|\vec{\delta}|,\\
\tfrac{1}{\tau}-\tfrac{1}{2} &< \mu-|\vec{\delta}| && \text{if } \tau(1+|\vec{\delta}|)>2 \quad \text{and} \quad \tfrac{1}{\tau}-\tfrac{1}{2} > \beta-|\vec{\delta}|,\\
\beta &< \mu && \text{if } \tau(1+|\vec{\delta}|)>2 \quad \text{and} \quad \tfrac{1}{\tau}-\tfrac{1}{2} \leq \beta-|\vec{\delta}|.
\end{aligned}$$

Now we want to derive from these six cases sufficient conditions for $s := 3\left(\frac{1}{\tau}-\frac{1}{2}\right)$ such that

$$v^* := \sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{N}}\sum_{I\in\Lambda_{j,k}} \langle v,\eta_I\rangle\, \eta_I^*$$

belongs to $B^s_\tau(L_\tau(\mathbb{R}^3))$. We find that $\beta<\mu$ is necessary in all six cases. First we consider the case $|\vec{\delta}| < \frac{2}{3}\mu$. If we require

$$s < \mu$$

we can conclude (depending on the value of $\tau(1+|\vec{\delta}|)$) from the first, the third respectively the fifth case the convergence ot the above series if additionaly $s>\beta$ is fulfilled. For the regularity result this is no relevant restriction. Next we look for the case $|\vec{\delta}| \geq \frac{2}{3}\mu$. From the fifth case again we conclude

$$\beta - |\vec{\delta}| < s < 3(\mu - |\vec{\delta}|).$$

Since we have already found $s<\mu$ we actually obtain from $|\vec{\delta}| \geq \frac{2}{3}\mu$ the condition $s < \frac{3}{2}|\vec{\delta}|$. But $|\vec{\delta}| \geq \frac{2}{3}\mu$ implies $3(\mu-|\vec{\delta}|) \leq 3/2|\vec{\delta}|$. All in all we have found the second restriction in (2.4), (2.5).

step 2: In the next step we have to estimate the coefficients in

$$\Lambda_{j,0} := \{I\in\Lambda_j : 0<\rho_I<2^{-j}\}. \tag{2.10}$$

If $\Lambda_{j,0}$ is empty, there is nothing to do. Otherwise we argue as follows. From the Lipschitz character of $\mathcal{K}_0$ follows

$$|\Lambda_{j,0}| \lesssim 2^{2j}, \quad j\in\mathbb{N}_0.$$

Hence for $0<q<2$ we obtain together with Hölder's inequality:

$$\sum_{I\in\Lambda_{j,0}} |\langle v,\eta_I\rangle|^q \lesssim 2^{j2(1-q/2)} \left(\sum_{I\in\Lambda_{j,0}} |\langle v,\eta_I\rangle|^2\right)^{\frac{q}{2}}.$$

Summing up over $j\in\mathbb{N}_0$ yields

$$\begin{aligned}
\sum_{j\geq 0} 2^{j(s+3(1/2-1/q))q} \sum_{I\in\Lambda_{j,0}} |\langle v,\eta_I\rangle|^q &\lesssim \sum_{j\geq 0} 2^{j(s+3(1/2-1/q))q} 2^{j(2/q-1)q} \left(\sum_{I\in\Lambda_{j,0}} |\langle v,\eta_I\rangle|^2\right)^{\frac{q}{2}}\\
&\lesssim ||v||^q_{B_q^{s+\frac{1}{2}-\frac{1}{q}}(L_2(\mathbb{R}^3))}
\end{aligned}$$

$$\lesssim ||v||^q_{B_2^{s+\frac{1}{2}-\frac{1}{q}}(L_2(\mathbb{R}^3))}.$$

The last step is a consequence of $q < 2$, see [16, Chapter 2.2.1]. Define

$$\alpha := \begin{cases} \alpha_0 + 1 & v = u_i \text{ for } i = 1, 2 \text{ or } 3 \\ \alpha_0 & v = p \end{cases}.$$

We choose s and q such that

$$s := \frac{3\alpha}{2} \quad \text{and} \quad \frac{1}{q} := \frac{s}{3} + \frac{1}{2}, \quad \text{i.e. } s = 3\left(\frac{1}{q} - \frac{1}{2}\right).$$

We obtain $\alpha = \frac{2}{q} - 1$, i.e. $\alpha > 0$ is insured. Additionally we get $\alpha = s + \frac{1}{2} - \frac{1}{q}$. That means $||v||^q_{B_2^{s+\frac{1}{2}-\frac{1}{q}}(L_2(\mathbb{R}^3))} < \infty$. We get that

$$v^{**} := \sum_{j\geq 0} \sum_{I\in\Lambda_{j,0}} \langle v, \eta_I\rangle \, \eta_I^* \tag{2.11}$$

belongs to $B_q^{3/2\alpha}(L_q(\mathbb{R}^3))$.

step 3: Finally we have to estimate the coefficients for which the support of the appendant wavelets intersects with the boundary of the truncated cone. More precisely we consider the set

$$\Lambda_j^{\#} := \{I|\, Q(I) \cap \partial\mathcal{K}_0 \neq \emptyset,\, 2^{-3j} \leq |I| \leq 2^{-3j+2}\}, \quad j \in \mathbb{N}_0.$$

Since $\mathcal{K}_0$ is a bounded Lipschitz domain there exists a linear and bounded extension operator

$$\mathcal{E} : H^\alpha(\mathcal{K}_0) \to H^\alpha(\mathbb{R}^3).$$

which is simultaneously a bounded operator from $B_q^s(L_p(\mathcal{K}_0))$ to $B_q^s(L_p(\mathbb{R}^3))$ not depending on s, p and q. See [17] for details. We define

$$v^{\#} := \sum_{j=0}^{\infty} \sum_{I\in\Lambda_j^{\#}} \langle \mathcal{E}v, \eta_I\rangle \, \eta_I^*.$$

We recognize that

$$|\Lambda_j^{\#}| \lesssim 2^{2j}, \quad j \in \mathbb{N}_0.$$

So we can argue as in step 2, this yields:

$$||v^{\#}||^q_{B_q^{3/2\alpha}(L_q(\mathbb{R}^3))} \lesssim ||\mathcal{E}v||^q_{B_2^\alpha(L_2(\mathbb{R}^3))} \lesssim ||v||^q_{B_2^\alpha(L_2(\mathcal{K}_0))} \lesssim ||v||^q_{H^\alpha(\mathcal{K}_0)}.$$

We end with summing up the functions v^*, v^{**} and $v^{\#}$ and obtain a function belonging to $B_\tau^s(L_\tau(\mathbb{R}^3))$ where $s < (\mu, 3/2 \cdot \alpha, 3 \cdot (\mu - |\vec{\delta}|))$ and $1/\tau = s/3 + 1/2$. This shows that $v \in B_\tau^s(L_\tau(\mathcal{K}_0))$.

□

3 Appendix A - Regularity of Solutions of the Stokes System

In this section we state two results which play a fundamental role in the proof of the main theorem. First of all we recall a result of Dauge, see [8, Theorem 9.20].

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a polyhedral domain. Consider problem (2.2). Assume that $(f, g) \in [L_2(\Omega)]^3 \times H^{\alpha_0}(\Omega)$ for $\alpha_0 < 0.5$. Furthermore let g fulfill the compatibility condition*

$$\int_\Omega g(x)dx = 0. \tag{3.1}$$

Then there exists a unique solution $(u, p) \in [H^{\alpha_0+1}(\Omega)]^3 \times H^{\alpha_0}(\Omega)$.

Next we cite a regularity result in weighted Sobolev spaces, see [15, Theorem 10.3.2].

Proposition 3.2. *Let $\mathcal{K}$ be a polyhedral cone as defined in (2.1). Suppose $(f, g) \in \left[W^{l-2,2}_{\beta,\vec{\delta}}(\mathcal{K})\right]^3 \times W^{l-1,2}_{\beta,\vec{\delta}}(\mathcal{K})$ where $l \geq 2$ is an integer. Then there exists a countable set $E \subset \mathbb{C}$ such that the following holds. If $\beta \in \mathbb{R}$ and the vector $\vec{\delta} \in (\mathbb{R}\backslash\mathbb{Z})^d$ are chosen such that*

$$Re\ \lambda \neq l - \beta - \frac{3}{2} \quad \text{for all } \lambda \in E$$

and

$$\max\left(0, l - 1 - \frac{\pi}{\theta_k}\right) < \delta_k < l - 1, \quad k = 1, ..., d,$$

then there exists a uniquely determined solution of (2.2)

$$(u, p) \in \left[W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})\right]^3 \times W^{l-1,2}_{\beta,\vec{\delta}}(\mathcal{K}) \tag{3.2}$$

Remark 3.3. Following Maz'ya and Rossmann (see [15, Chapter 10.2, 10.3]) we use the notation $V^{l,2}_\beta(\mathcal{K}) := W^{l,2}_{\beta,0}(\mathcal{K})$ and define the space

$$\mathcal{H}_\beta := \left\{u \in \left[V^{1,2}_\beta(\mathcal{K})\right]^3 : u = 0 \text{ on } \Gamma_j,\ j = 1, ..., d\right\}.$$

If the assumptions from Proposition 3.2 are fulfilled the functional

$$F(v) := \int_\mathcal{K} (f + \nabla g) \cdot v \mathrm{d}x$$

defines a linear and continuous mapping on $\mathcal{H}_{l-1-\beta}$. The solution (u, p) found in Proposition 3.2 fulfills

$$||u||^2_{W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})^3} + ||p||^2_{W^{l-1,2}_{\beta,\vec{\delta}}(\mathcal{K})} \lesssim \left(||F||^2_{\mathcal{H}^*_{l-1-\beta}} + ||g||^2_{V^{0,2}_{\beta-l+1}} + ||f||^2_{W^{l-2,2}_{\beta,\delta}} + ||g||^2_{W^{l-1,2}_{\beta,\delta}(\mathcal{K})}\right).$$

Moreover we obtain from [8] the estimate

$$||u||_{H^{\alpha_0+1}(\mathcal{K}_0)^3} + ||p||_{H^{\alpha_0}(\mathcal{K}_0)} \lesssim ||f||_{L_2(\mathcal{K}_0)^3} + ||g||_{H^{\alpha_0}(\mathcal{K}_0)}.$$

4 Appendix B - Function Spaces

We assume that the definition of Besov and Sobolev spaces on $\mathbb{R}^n$ are known. For details see [18]. We now recall the definition of the Besov and Sobolev spaces on a bounded open nonempty set $\Omega \subset \mathbb{R}^n$.

4.1 Besov Spaces on Domains

Let $\Omega \subset \mathbb{R}^n$ be a bounded open nonempty set. Then we define $B^s_q(L_p(\Omega))$ to be the collection of all distributions $f \in D'(\Omega)$ such that there exists a tempered distribution $g \in B^s_q(L_p(\mathbb{R}^n))$ satisfying

$$f(\varphi) = g(\varphi) \text{ for all } \varphi \in D(\Omega),$$

i.e. $g|\Omega = f$ in $\mathbb{D}'(\Omega)$. We put

$$||f||_{B^s_q(L_p(\Omega))} := \inf ||g||_{B^s_q(L_p(\mathbb{R}^n))},$$

where the infimum is taken with respect to all distributions g as above.

4.2 Sobolev Spaces on Domains

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. As usual with $H^m(\Omega)$ we denote the collection of all functions f such that the distributional derivatives D^α of order $|\alpha| \leq m$ belong to $L_2(\Omega)$. We put

$$||f||_{H^m(\Omega)} := \sum_{|\alpha| \leq m} ||D^\alpha f||_{L_2(\Omega)}.$$

It holds $H^m(\mathbb{R}^n) = B^m_2(L_2(\mathbb{R}^n))$ in the sense of equivalent norms (see e.g. [18]) and because of the existense of a bounded and linear extension operator for Sobolev spaces on bounded Lipschitz domains (see [17]) we also know

$$H^m(\Omega) = B^m_2(L_2(\Omega))$$

in the sense of equivalent norms. For fractional $s > 0$ we introduce the Sobolev Spaces by complex interpolation. For $0 < s < m$, $m \in \mathbb{N}$, $s \notin \mathbb{N}$ we put

$$H^s(\Omega) := [H^m(\Omega), L_2(\Omega)]_\theta\,, \quad \theta = 1 - \frac{1}{m},$$

see [14] for details. This definition does not depend on m in the sense of equivalent norms, cf. [19]. Moreover it can be shown that

$$H^s(\Omega) = B^s_2(L_2(\Omega)),$$

cf. [19], [20] for details.

4.3 Besov Spaces and Wavelets

In this section we impose the notations concerning the wavelets. Moreover we state the result which provides a characerization of the Besov spaces by the coefficients of the wavelet expansion. For the construction of wavelets see, e.g., [7]. Let φ be a compactly supported scaling function of sufficiently high regularity and let ψ, $i = 1, ..., 2^n - 1$ be corresponding wavelets. More detailed we require for some $N > 0$ and $r \in \mathbb{N}$:

- $\operatorname{supp} \varphi, \operatorname{supp} \psi_i \subset [-N, N]$, $i = 1, ..., 2^n - 1$.
- $\varphi, \psi_i \in \mathcal{C}^r(\mathbb{R}^n)$, $i = 1, ..., 2^n - 1$.
- The wavelets have the vanishing moments property:

  $$\int x^\alpha \psi_i(x) dx = 0$$

  for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq r$, $i = 1, ..., 2^n - 1$.
- We use the standard abbreviations $\varphi_k(x) := \varphi(x - k)$ and $\psi_{i,j,k}(x) := 2^{jn/2}\psi_i(2^j x - k)$. We assume that

  $$\{\varphi_k,\ \psi_{i,j,k} : (i, j, k) \in \{1, ..., 2^n - 1\} \times \mathbb{N}_0 \times \mathbb{Z}^n\}$$

  is a Riesz basis in $L_2(\mathbb{R}^n)$.

Further, the dual Riesz basis should fulfill the same requirements, i.e. there exist functions $\tilde{\varphi}$ and $\tilde{\psi}_i$, $i = 1, ..., 2^n - 1$, such that

- $\langle \tilde{\varphi}_k, \psi_{i,j,k} \rangle = \left\langle \tilde{\psi}_{i,j,k}, \varphi_k \right\rangle = 0$,
- $\langle \tilde{\varphi}_k, \varphi_l \rangle = \delta_{k,l}$,
- $\left\langle \tilde{\psi}_{i,j,k}, \psi_{u,v,l} \right\rangle = \delta_{i,u}\delta_{j,v}\delta_{k,l}$,
- $\operatorname{supp} \tilde{\varphi}, \operatorname{supp} \tilde{\psi}_i \subset [-N, N]$, $i = 1, ..., 2^n - 1$.
- $\tilde{\varphi}, \tilde{\psi}_i \in \mathcal{C}^r(\mathbb{R}^n)$, $i = 1, ..., 2^n - 1$.
- $\int x^\alpha \tilde{\psi}_i(x) dx = 0$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq r$, $i = 1, ..., 2^n - 1$.

Next we state a result which allows to prove Besov regularity by estimating the coefficients of the wavelet expansion. More detailed, it holds:

Proposition 4.1. *Let $s \in \mathbb{R}$ and $0 < p, q \leq \infty$. Suppose*

$$r > \max\left(s, n \max\left(0, \frac{1}{p} - 1\right) - s\right). \tag{4.1}$$

Then $B^s_q(L_p(\mathbb{R}^n))$ is the collection of all tempered distributions f such that f is representable as

$$f = \sum_{k\in\mathbb{Z}^n} a_k\varphi_k + \sum_{i=1}^{2^n-1}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} a_{i,j,k}\psi_{i,j,k}$$

with

$$||f||^*_{B^s_q(L_p(\mathbb{R}^n))} := \left(\sum_{k\in\mathbb{Z}^n}|a_k|^p\right)^{1/p} + \left(\sum_{i=1}^{2^n-1}\sum_{j=0}^{\infty} 2^{j(s+n(1/2-1/p))q}\left(\sum_{k\in\mathbb{Z}^n}|a_{i,j,k}|^p\right)^{q/p}\right)^{1/q} < \infty$$

if $q < \infty$ *and*

$$||f||^*_{B^s_\infty(L_p(\mathbb{R}^n))} := \left(\sum_{k\in\mathbb{Z}^n}|a_k|^p\right)^{1/p} + \sup_{i=1,\dots,2^n-1}\sup_{j\geq 0} 2^{j(s+n(1/2-1/p))}\left(\sum_{k\in\mathbb{Z}^n}|a_{i,j,k}|^p\right)^{1/p} < \infty.$$

The representation is unique and

$$a_k = \langle f, \tilde{\varphi}_k\rangle \text{ and } a_{i,j,k} = \left\langle f, \tilde{\psi}_{i,j,k}\right\rangle \tag{4.2}$$

hold. Further $J : f \mapsto \{\langle f, \tilde{\varphi}_k\rangle, \langle f, \tilde{\psi}_{i,j,k}\rangle\}$ *is an isomorphic map of* $B^s_q(L_p(\mathbb{R}^n))$ *onto the sequence space (equipped with the quasi-norm* $||\cdot||^*_{B^s_q(L_p(\mathbb{R}^n))}$*), i.e.* $||\cdot||^*_{B^s_q(L_p(\mathbb{R}^n))}$ *may serve as an equivalent quasi-norm on* $B^s_q(L_p(\mathbb{R}^n))$*.*

A proof of this proposition can be found in [21].

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