# Analysis III, SoSe 2011 – Lösung Blatt 4

**4.1.** Beweise folgende Formel der n-dimensionalen Polarkoordinaten:

$$\int_{B_{r,R}(0)} f(y)\,dy = \int_r^R \int_{-\pi}^{\pi} \int_{[0,\pi]^{n-2}} f(x_1,\dots,x_n)\, s^{n-1} \prod_{j=1}^{n-2} (\sin\Theta_j)^j \, ds\, d\phi\, d(\Theta_1,\dots,\Theta_{n-2})$$

mit $x_1 = s\cos\phi \prod_{j=1}^{n-2} \sin\Theta_j$ , $x_2 = s\sin\phi \prod_{j=1}^{n-2}\sin\Theta_j$ und für $K>2$

$x_K = s\cos\Theta_{K-2} \prod_{j=K-1}^{n-2} \sin\Theta_j$ .

Beweis:

Setze

- $\Psi_n : \mathbb{R}_{>0} \times \mathbb{R}^{n-2} \times (0,\pi) \longrightarrow \mathbb{R}_{>0} \times \mathbb{R}^{n-2} \times \mathbb{R}$

  $(r, x_2, \dots, x_{n-1}, \Theta) \longmapsto (r\sin\Theta, x_2, \dots, x_{n-1}, r\cos\Theta)$

- $Z_n : \mathbb{R}_{>0} \times (-\pi,\pi) \times (0,\pi)^{n-3} \times \mathbb{R} \to \mathbb{R}^2 \setminus (-\infty,0] \times \mathbb{R}^{n-2}$

  $(r, \phi, \Theta_1, \dots, \Theta_{n-3}, x_n) \longmapsto (P_{n-1}(r,\phi,\Theta_1,\dots,\Theta_{n-3}), x_n)$

- $P_n : \mathbb{R}_{>0} \times (-\pi,\pi) \times (0,\pi)^{n-2} \longrightarrow \mathbb{R}^2 \setminus (-\infty,0] \times \mathbb{R}^{n-2}$

$P_2(r,\phi) := (r\cos\phi, r\sin\phi)$ , $P_n := Z_n \circ \Psi_n$ .

$\overset{\text{Vor.}}{\Rightarrow}$ Alle Abbildungen sind bijektiv.

(i) Behauptung:

$$P_n(r,\phi,\Theta_1,\dots,\Theta_{n-2}) = \begin{pmatrix} r\cos\phi \prod_{j=1}^{n-2}\sin\Theta_j \\ r\sin\phi \prod_{j=1}^{n-2}\sin\Theta_j \\ r\cos\Theta_{3-2} \prod_{j=3-1}^{n-2}\sin\Theta_j \\ \vdots \\ r\cos\Theta_{n-2} \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{wie oben.}$$

Induktion nach n: I.A.: $n=2$. $P_2(r,\phi) = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}$ ✓

I.S.: $(n-1) \to n$ Ablesen in den Definitionen und der I.V. liefert:

$$\left(P_n(r,\phi,\theta_1,\dots,\theta_{n-2})\right)_1 = \left(Z_n \circ \Psi_n(r,\phi,\theta_1,\dots,\theta_{n-2})\right)_1$$

$$= \left(Z_n\left(r\sin\theta_{n-2},\phi,\theta_1,\dots,\theta_{n-3},r\cos\theta_{n-2}\right)\right)_1$$

$$= \left(P_{n-1}\left(r\sin\theta_{n-2},\phi,\theta_1,\dots,\theta_{n-3}\right)\right)_1$$

$$\overset{\text{I.V.}}{=} r\sin\theta_{n-2}\cos\phi\prod_{j=1}^{n-3}\sin\theta_j$$

$$= r\cos\phi\prod_{j=1}^{n-2}\sin\theta_j$$

$$\left(P_n(r,\phi,\theta_1,\dots,\theta_{n-2})\right)_2 \overset{\text{s.o.}}{=} \left(P_{n-1}\left(r\sin\theta_{n-2},\phi,\theta_1,\dots,\theta_{n-3}\right)\right)_2$$

$$\overset{\text{I.V.}}{=} r\sin\theta_{n-2}\sin\phi\prod_{j=1}^{n-3}\sin\theta_j$$

$$= r\sin\phi\prod_{j=1}^{n-2}\sin\theta_j$$

$$\left(P_n(r,\phi,\theta_1,\dots,\theta_{n-2})\right)_n \overset{\text{s.o.}}{=} \left(Z_n\left(r\sin\theta_{n-2},\phi,\theta_1,\dots,\theta_{n-3},r\cos\theta_{n-2}\right)\right)_n$$

$$= r\cos\theta_{n-2}$$

Für $2 < K < n$:

$$\left(P_n(r,\phi,\theta_1,\dots,\theta_{n-2})\right)_K \overset{\text{s.o.}}{=} \left(P_{n-1}\left(r\sin\theta_{n-2},\phi,\theta_1,\dots,\theta_{n-3}\right)\right)_K$$

$$\overset{\text{I.V.}}{=} r\sin\theta_{n-2}\cos\theta_{K-2}\prod_{j=K-1}^{n-3}\sin\theta_j$$

$$= r\cos\theta_{K-2}\prod_{j=K-1}^{n-2}\sin\theta_j .$$

$\Rightarrow$ Behauptung (i).

(ii) Behauptung:

$$\left|\det DP_n(r,\phi,\theta_1,\dots,\theta_{n-2})\right| = r^{n-1}\prod_{j=1}^{n-2}(\sin\theta_j)^j$$

Induktion nach $n$: I.A.: $n=2$. Nach Vorlesung $|\det DP_2(r,\phi)| = r$ ✓

I.S.: $(n-1) \to n$. Es ist:

- $$\det D\Psi_n(r,\phi,\theta_1,\dots,\theta_{n-2}) = \det \begin{pmatrix} \sin\theta_{n-2} & 0 & \cdots & 0 & r\cos\theta_{n-2} \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \\ \cos\theta_{n-2} & 0 & \cdots & 0 & -r\sin\theta_{n-2} \end{pmatrix}$$

$$\overset{\text{Laplace}}{=} \sin\theta_{n-2} \det\begin{pmatrix} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \\ 0 & \cdots & 0 & -r\sin\theta_{n-2} \end{pmatrix} + (-1)^{n+1} r\cos\theta_{n-2} \det\begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ \cos\theta_{n-2} & 0 & \cdots & 0 \end{pmatrix}$$

$$= -r\sin^2\theta_{n-2} + (-1)^{n+1}\cdot(-1)^n \cdot r\cos^2\theta_{n-2} = -r\sin^2\theta_{n-2} - r\cos^2\theta_{n-2}$$

$$= -r$$

$$\Rightarrow \quad |\det D\Psi_n(r,\phi,\theta_1,\dots,\theta_{n-2})| = r$$

- $$\det DZ_n(r,\phi,\theta_1,\dots,\theta_{n-3},x_n) = \det\begin{pmatrix} \boxed{DP_{n-1}(r,\phi,\theta_1,\dots,\theta_{n-3})} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0 \cdots 0 & 1 \end{pmatrix}$$

$$\Rightarrow |\det DZ_n(r,\phi,\theta_1,\dots,\theta_{n-3},x_n)| = |\det DP_{n-1}(r,\phi,\theta_1,\dots,\theta_{n-3})|$$

$$\overset{\text{I.V.}}{=} r^{n-2} \prod_{j=1}^{n-3} (\sin\theta_j)^j.$$

- Damit:

$$|\det DP_n(r,\phi,\theta_1,\dots,\theta_{n-2})| \overset{\text{Kettenregel}}{=} \left|\det\left(DZ_n(\Psi_n(r,\phi,\theta_1,\dots,\theta_{n-2}))\cdot D\Psi_n(r,\phi,\theta_1,\dots,\theta_{n-2})\right)\right|$$

$$= |\det(DZ_n(\Psi_n(r,\phi,\theta_1,\dots,\theta_{n-2})))| \cdot |\det D\Psi_n(r,\phi,\theta_1,\dots,\theta_{n-2})|$$

$$= |\det DZ_n(r\sin\theta_{n-2},\phi,\theta_1,\dots,\theta_{n-3},r\cos\theta_{n-2})| \cdot r$$

$$= (r\sin\theta_{n-2})^{n-2} \prod_{j=1}^{n-3}(\sin\theta_j)^j$$

$$= r^{n-2}\prod_{j=1}^{n-2}(\sin\theta_j)^j \qquad (>0)$$

$\Rightarrow$ Behauptung (ii) und insbesondere: $P_n$ ist $\mathcal{C}^1$-Diffeomorphismus

(iii) Dehne $P_n, Z_n, \Psi_n$ zu Abbildungen $\mathbb{R}^n \to \mathbb{R}^n$ aus, setze

$$D := [r,R] \times [-\pi,\pi] \times [0,\pi]^{n-2}.$$

Behauptung: $P_n(D) = B_{r,R}(0)$.

- Zunächst ist stets $\|P_n(s,\phi,\theta_1,\dots,\theta_{n-2})\| = |s|$.

  I.A.: $n=2$: $\quad \|P_2(s,\phi)\|^2 = s^2\cos^2\phi + s^2\sin^2\phi = s^2 \quad \checkmark$

  I.S.: $n-1 \to n$: $\quad \|P_n(s,\phi,\theta_1,\dots,\theta_{n-2})\|^2 = \|P_{n-1}(s\sin\theta_{n-2},\phi,\theta_1,\dots,\theta_{n-3})\|^2 + s^2\cos^2\theta_{n-2}$

  $$\overset{\text{I.V.}}{=} s^2\sin^2\theta_{n-2} + s^2\cos^2\theta_{n-2} = s^2 \quad \checkmark$$

- $P_n(D) \subset B_{r,R}(0)$. Sei $p = (s,\phi,\theta_1,\dots,\theta_{n-2}) \in D$.

  $\|P_n(p)\| = |s| = s \in [r,R]$, d.h. $P_n(p) \in B_{r,R}(0)$.

- Ist $p \in B_{r,R}(0) \cap \mathbb{R}^2 \setminus (-\infty,0] \times \mathbb{R}^{n-2}$, dann $\exists$ nach Vor.
  $q = (s,\phi,\theta_1,\dots,\theta_{n-2}) \in \mathbb{R}_{>0} \times (-\pi,\pi) \times (0,\pi)^{n-2}$ mit $P_n(q) = p$. Dann:

  $s = |s| \overset{\text{s.o.}}{=} \|P_n(q)\| = \|p\|$, d.h. $r \le s \le R$, also $q \in D$.

- Sei also schließlich $p \in B_{r,R}(0) \setminus (\mathbb{R}^2 \setminus (-\infty,0] \times \mathbb{R}^{n-2})$ d.h.
  $p = (x_1, 0, x_3, \dots, x_n)$ mit $x_1 \le 0$.

  Wir zeigen zunächst: $\forall\, z = (z_1, 0, z_3, \dots, z_n)$ mit $\|z\| \le R$, $z_1 \le 0$
  $\exists\, w = (s,\phi,\theta_1,\dots,\theta_{n-2}) \in [0,R] \times [-\pi,\pi] \times [0,\pi]^{n-2}$ mit $P_n(w) = z$

  I.A.: $n=2$: Setze $w := (|z_1|, \pi) \in [0,R] \times [-\pi,\pi]$.

  $$\Rightarrow P_2(w) = (|z_1|\cos\pi,\ |z_1|\sin\pi) = (-|z_1|, 0) \overset{z_1 \le 0}{=} (z_1, 0) = z \quad \checkmark$$

  I.S.: $n-1 \to n$: Setze $z' := (z_1, 0, z_3, \dots, z_{n-1}) \Rightarrow \|z'\|^2 = \|z\|^2 - z_n^2 \le \|z\|^2 \le R^2$

  $\overset{\text{I.V.}}{\Rightarrow} \exists\, w' = (s', \phi, \theta_1, \dots, \theta_{n-3}) \in [0,R] \times [-\pi,\pi] \times [0,\pi]^{n-3}$ mit $P_{n-1}(w') = z'$.

  $(s', z_n) \in [0,R] \times \mathbb{R} \Rightarrow \exists\, s \ge 0,\ \theta_{n-2} \in [0,\pi]$ mit $(s', z_n) = (s\sin\theta_{n-2},\ s\cos\theta_{n-2})$

  Setze $w := (s, \phi, \theta_1, \dots, \theta_{n-2})$. Dann:

  $$P_n(w) = Z_n \circ \Psi_n(w) = Z_n\big(\underbrace{s\sin\theta_{n-2}}_{=s'}, \phi, \theta_1, \dots, \theta_{n-3}, \underbrace{s\cos\theta_{n-2}}_{=z_n}\big)$$

  $$= \big(P_{n-1}(s', \phi, \theta_1, \dots, \theta_{n-3}),\ z_n\big) = (z_1, 0, z_3, \dots, z_n) = z$$

  und $s = |s| = \|P_n(w)\| = \|z\| \le R$. $\checkmark$

$\Rightarrow$ Zu unserem $p$ $\exists$ $q = (s,\phi,\theta_1,\dots,\theta_{n-2}) \in [0,R] \times [-\pi,\pi] \times [0,\pi]^{n-2}$ mit $P_n(q) = p$.

$s = |s| = \|P_n(q)\| = \|p\| \overset{p \in B_{r,R}(0)}{\Longrightarrow} r \le s \le R$ d.h. $q \in D$.

Damit ist $P_n(D) = B_{r,R}(0)$ gezeigt.

(iv) Mit (i) und (ii) reduziert sich die behauptete Formel auf

$$\int_{B_{r,R}(0)} f(y)\,dy = \int_r^R \int_{-\pi}^{\pi} \int_{[0,\pi]^{n-2}} (f\circ P_n)(s,\phi,\theta_1,\dots,\theta_{n-2})\,|\det DP_n(s,\phi,\theta_1,\dots,\theta_{n-2})|\,ds\,d\phi\,d(\theta_1,\dots,\theta_{n-2}).$$

Da $P_n$ auf $\mathbb{R}^2 \supset D$ stetig diffbar und

$$\mathring{D} = (r,R)\times(-\pi,\pi)\times(0,\pi)^{n-2} \subset \mathbb{R}_{>0}\times(-\pi,\pi)\times(0,\pi)^{n-2},$$

d.h. $P_n|_{\mathring{D}}$ $\mathcal{C}^1$-Diffeomorphismus folgt die Formel wegen $P_n(D) = B_{r,R}(0)$ aus der Transformationsformel (2.4) der Vorlesung.

∎

**4.2.** Seien $n \in \mathbb{N}\setminus\{0\}$ und $R, \alpha > 0$. Berechne

$$\lim_{r\to 0} \int_{B_{r,R}(0)} \frac{1}{\|x\|_2^\alpha}\, dx \qquad \in \mathbb{R}\cup\{\infty\}.$$

Dazu:

- Vorab:

$$\mathrm{Vol}(B_1) = \int_{B_1} 1\, dx \overset{4.1.}{=} \int_0^1 \int_{-\pi}^{\pi} \int_{[0,\pi]^{n-2}} 1\cdot s^{n-1} \prod_{j=1}^{n-2} (\sin\theta_j)^j \, ds\, d\phi\, d(\theta_1,\dots,\theta_{n-2})$$

$$= \left(\int_0^1 s^{n-1}\, ds\right)\cdot \underbrace{\left(\int_{-\pi}^{\pi}\int_{[0,\pi]^{n-2}} \prod_{j=1}^{n-2}(\sin\theta_j)^j \, d\phi\, d(\theta_1,\dots,\theta_{n-2})\right)}_{=:\, K_n}$$

$$= \frac{K_n}{n}$$

$\Rightarrow K_n = n\cdot \mathrm{Vol}(B_1)$.

- Damit:

$$\int_{B_{r,R}(0)} \frac{1}{\|x\|_2^\alpha}\, dx \overset{4.1.}{=} \int_r^R \int_{-\pi}^{\pi} \int_{[0,\pi]^{n-2}} \underbrace{\|P_n(s,\phi,\theta_1,\dots,\theta_{n-2})\|_2^{-\alpha}}_{= s^{-\alpha} \text{ vgl. } 4.1}\, s^{n-1} \prod_{j=1}^{n-2} (\sin\theta_j)^j \, ds\, d\phi\, d(\theta_1,\dots,\theta_{n-2})$$

$$= \left(\int_r^R s^{n-1-\alpha}\, ds\right)\cdot K_n$$

Falls $\alpha = n$: $\lim_{r\to 0} \int_{B_{r,R}(0)} \frac{1}{\|x\|_2^\alpha}\, dx = \lim_{r\to 0} \int_r^R s^{-1} ds \cdot K_n = \lim_{r\to 0} \left(\ln(R) - \ln(r)\right)\cdot K_n = \infty$

$\alpha > n$: $\lim_{r\to 0} \int_{B_{r,R}(0)} \frac{1}{\|x\|_2^\alpha}\, dx = \lim_{r\to 0} K_n \left(\frac{R^{n-\alpha}}{n-\alpha} - \underbrace{\frac{r^{n-\alpha}}{n-\alpha}}_{\to -\infty}\right) = \infty$

$\alpha < n$: $\lim_{r\to 0} \int_{B_{r,R}(0)} \frac{1}{\|x\|_2^\alpha}\, dx = \lim_{r\to 0} K_n \left(\frac{R^{n-\alpha}}{n-\alpha} - \underbrace{\frac{r^{n-\alpha}}{n-\alpha}}_{\to 0}\right) = K_n \cdot \frac{R^{n-\alpha}}{n-\alpha} = \frac{n\cdot \mathrm{Vol}(B_1) R^{n-\alpha}}{n-\alpha}$

Zusammenfassung:

$$\lim_{r\to 0} \int_{B_{r,R}(0)} \frac{1}{\|x\|_2^\alpha}\, dx = \begin{cases} \infty & \text{falls } \alpha \geq n \\ \dfrac{n\cdot \mathrm{Vol}(B_1)\cdot R^{n-\alpha}}{n-\alpha} & \text{falls } 0 < \alpha < n \end{cases}$$

$\blacksquare$

4.3. Sei $A \in \mathbb{R}^{n\times n}$ symmetrisch und positiv definit.

a) Berechne $\lim\limits_{a\to\infty} \int\limits_{[-a,a]^n} e^{-\langle Ax|x\rangle}\,dx$.

b) Berechne das Volumen des Ellipsoids $E = \{x \in \mathbb{R}^n \mid \langle Ax|x\rangle \leq 1\}$.

Dazu:

a) • $A$ symmetrisch, pos. definit $\overset{\text{Spektral-}}{\underset{\text{Satz}}{\Longrightarrow}}$ $\exists$ positive Diagonalmatrix $D' \in \mathbb{R}^{n\times n}$ und invertierbares $C \in \mathbb{R}^{n\times n}$ mit $C^{-1} = C^t$, so dass $A = C^t D' C$.

$$D' = \begin{pmatrix} d_1' & & 0 \\ & \ddots & \\ 0 & & d_n' \end{pmatrix} \text{ mit } d_i' > 0. \text{ Setze } D := \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \text{ mit } d_i = \sqrt{d_i'}.$$

Dann: $A = C^t D' C = C^t D D C = C^t D^t D C = (DC)^t (DC) = B^t B$

mit $B := DC$. Es ist

$$(\det B)^2 = (\det B^t)(\det B) = \det B^t B = \det A \quad > 0$$

$$\Rightarrow \quad |\det B| = \sqrt{\det A}.$$

Außerdem: $\langle Ax|x\rangle = \langle B^t Bx|x\rangle = \langle Bx|Bx\rangle$.

• Mit der Transformationsformel folgt:

$$\int\limits_{[-a,a]^n} e^{-\langle Ax|x\rangle}\,dx = \int\limits_{[-a,a]^n} e^{-\langle Bx|Bx\rangle}\,dx = \frac{1}{|\det B|} \int\limits_{B([-a,a]^n)} e^{-\langle x|x\rangle}\,dx$$

• Es ist $\|B(x)\|_\infty \leq K' \|B(x)\|_2 \leq K' \|B\|\,\|x\|_2 \leq \underbrace{K'K''\|B\|}_{=:K} \|x\|_\infty$

$\leq Ka \quad \forall\, x \in [-a,a]^n$

d.h. $B([-a,a]^n) \subset [-Ka,Ka]^n$. Andererseits: $C^t = C^{-1}$ d.h. $C$ orthonormal.

$$\left.\begin{array}{lll} \Rightarrow \|Cx\|_2 = \|x\|_2 & \text{d.h. } C(B_a) \subset B_a & \\ \|C^{-1}x\|_2 = \|x\|_2 & \Rightarrow C^{-1}(B_a) \subset B_a & \text{d.h. } B_a \subset C(B_a) \end{array}\right\} \Rightarrow C(B_a) = B_a$$

Also $B_a = C(B_a) \subset C([-a,a]^n)$ (da $B_a \subset [-a,a]^n$).

$\Rightarrow W_{\frac{1}{\sqrt{2}}a} \subset B_a \subset C([-a,a]^n)$ ($W_{\frac{1}{\sqrt{2}}a}$ sei Würfel entsprechender Seitenlänge)

Offenbar ist $D(W_{\frac{1}{\sqrt{2}}a}) = [-\frac{d_1}{\sqrt{2}}a, \frac{d_1}{\sqrt{2}}a] \times \dots \times [-\frac{d_n}{\sqrt{2}}a, \frac{d_n}{\sqrt{2}}a] =: Q_a$,

also $Q_a = D(W_{\frac{1}{\sqrt{2}}a}) \subset DC([-a,a]^n) = B([-a,a]^n)$.

Insgesamt: $Q_a \subset B([-a,a]^n) \subset [-Ka, Ka]^n$.

Also folgt für alle $a$:

$$\int_{Q_a} e^{-\langle x|x\rangle} dx \leq \int_{B([-a,a])} e^{-\langle x|x\rangle} dx \leq \int_{[-Ka,Ka]^n} e^{-\langle x|x\rangle} dx . \qquad (*)$$

- Wir berechnen

$$\int_{[-c_1a, c_1a]\times\dots\times[-c_na, c_na]} e^{-\langle x|x\rangle} dx = \int_{-c_1a}^{c_1a} \dots \int_{-c_na}^{c_na} e^{-(x_1^2+\dots+x_n^2)} dx_n \dots dx_1$$

$$= \left(\int_{-c_1a}^{c_1a} e^{-x_1^2} dx_1\right) \cdot \ldots \cdot \left(\int_{-c_na}^{c_na} e^{-x_n^2} dx_n\right)$$

$$\xrightarrow{a\to\infty} \sqrt{\pi} \cdot \ldots \cdot \sqrt{\pi} = \pi^{\frac{n}{2}} \qquad \text{(vgl. VL, Bsp. 2.5 (2))}$$

- Also: $\int_{Q_a} e^{-\langle x|x\rangle} dx \xrightarrow{a\to\infty} \pi^{\frac{n}{2}}$ und $\int_{[-Ka,Ka]} e^{-\langle x|x\rangle} dx \xrightarrow{a\to\infty} \pi^{\frac{n}{2}}$

$$\overset{(*)}{\Rightarrow} \lim_{a\to\infty} \int_{B([-a,a]^n)} e^{-\langle x|x\rangle} dx = \pi^{\frac{n}{2}}$$

$$\Rightarrow \lim_{a\to\infty} \int_{[-a,a]^n} e^{-\langle Ax|x\rangle} dx = \frac{1}{|\det B|} \lim_{a\to\infty} \int_{B([-a,a]^n)} e^{-\langle x|x\rangle} dx$$

$$= \frac{\pi^{\frac{n}{2}}}{\sqrt{\det A}} .$$

b) Wie in a) sei $A = B^t B$ mit $|\det B| = \sqrt{\det A}$, also $\langle Ax | x \rangle = \langle Bx | Bx \rangle$

Damit:

$$E = \{x \in \mathbb{R}^n \mid \langle Bx | Bx \rangle \leq 1\} = \{x \in \mathbb{R}^n \mid \|Bx\|_2^2 \leq 1\}$$

$$= \{x \in \mathbb{R}^n \mid \|B(x)\|_2 \leq 1\} = \{x \in \mathbb{R}^n \mid B(x) \in B_1\}$$

$$= B^{-1}(B_1).$$

Also:

$$\operatorname{Vol}(E) = \int_E 1 \, dx = \int_{B^{-1}(B_1)} 1 \, dx \overset{\text{Trafo}}{=} \int_{B_1} |\det B^{-1}| \, dx$$

$$= \frac{1}{|\det B|} \int_{B_1} 1 \, dx = \frac{\operatorname{Vol}(B_1)}{\sqrt{\det A}}$$

$$= \frac{1}{\sqrt{\det A}} \cdot \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}.$$

$\blacksquare$

4.4. Gib mit Hilfe des Integrals $\int_{B_R} e^{-(x_1^2+\dots+x_n^2)}dx$ einen Beweis der Formel

$$\Omega_n(R) = \frac{R^n \pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$$

für das Volumen $\Omega_n(R)$ der Kugel $B_R \subset \mathbb{R}^n$ und zeige, dass

$$\lim_{n\to\infty} \Omega_n(R) = 0.$$

Beweis:

- $$\mathrm{Vol}(B_R) = \int_{B_R} 1\,dx = \int_0^R \int_{-\pi}^{\pi} \int_{[0,\pi]^{n-2}} s^{n-1} \prod_{j=1}^{n-2} (\sin\theta_j)^j \, ds\, d\phi\, d(\theta_1,\dots,\theta_{n-2})$$

$$= \left(\int_0^R s^{n-1}ds\right) \underbrace{\left(\int_{-\pi}^{\pi} \int_{[0,\pi]^{n-2}} \prod_{j=1}^{n-2} (\sin\theta_j)^j \, d\phi\, d(\theta_1,\dots,\theta_{n-2})\right)}_{=:K_n} = \frac{R^n \cdot K_n}{n}$$

- Mit 4.3.a, oder einer einfachen Rechnung: $\int_{[-r,r]^n} e^{-(x_1^2+\dots+x_n^2)}dx \xrightarrow{r\to\infty} \pi^{\frac{n}{2}}$, d.h. wegen $[-\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}] \subset B_r \subset [-r,r]$: $\int_{B_r} e^{-(x_1^2+\dots+x_n^2)}dx \xrightarrow{r\to\infty} \pi^{\frac{n}{2}}$

Andererseits:

$$\int_{B_r} e^{-(x_1^2+\dots+x_n^2)}dx = \int_0^r \int_{-\pi}^{\pi} \int_{[0,\pi]^{n-2}} e^{-s^2} s^{n-1} \prod_{j=1}^{n-2} (\sin\theta_j)^j \, ds\, d\phi\, d(\theta_1,\dots,\theta_{n-2})$$

$$= K_n \cdot \int_0^r s^{n-1} e^{-s^2} ds \overset{\text{Subst.}}{\underset{t=s^2}{=}} \frac{K_n}{2} \int_0^{r^2} t^{\frac{n-2}{2}} e^{-t} dt$$

$$\xrightarrow{r\to\infty} \frac{K_n}{2} \int_0^{\infty} t^{(\frac{n}{2}-1)} e^{-t} dt = \frac{K_n}{2}\Gamma\left(\frac{n}{2}\right)$$

$$\Rightarrow \pi^{\frac{n}{2}} = \frac{K_n}{2}\Gamma\left(\frac{n}{2}\right) \quad \text{d.h.} \quad K_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$$

$$\Rightarrow \mathrm{Vol}(B_R) = \frac{R^n \cdot K_n}{n} = \frac{R^n \cdot \pi^{\frac{n}{2}}}{\frac{n}{2}\Gamma(\frac{n}{2})} = \frac{R^n \cdot \pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}.$$

- $$a_n := \Omega_n(R) = \frac{R^n \pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} = \frac{(R^2\pi)^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}. \quad \text{Damit:}$$

$$(a_{2n})_n = \left(\frac{(R^2\pi)^n}{\Gamma(n+1)}\right)_n = \left(\frac{(R^2\pi)^n}{n!}\right)_n \longrightarrow 0 \quad \text{da} \quad \exp(R^2\pi) = \sum_{k=0}^{\infty} \frac{(R^2\pi)^k}{k!}$$

$$(a_{2n+1})_n = \left(\frac{(R^2\pi)^{n+\frac{1}{2}}}{\Gamma(n+\frac{1}{2}+1)}\right)_n \underset{\substack{\uparrow \\ n \text{ groß genug}}}{\leq} \sqrt{R^2\pi}\left(\frac{(R^2\pi)^n}{\Gamma(n+1)}\right)_n = \sqrt{R^2\pi}\left(\frac{(R^2\pi)^n}{n!}\right)_n \longrightarrow 0.$$

Damit folgt: $\lim_{n\to\infty} \Omega_n(R) = \lim_{n\to\infty} a_n = 0.$ ∎