Mathematik I, WiSe 2010/2011 – Lösung Blatt 12

12.2. Sei $W = \operatorname{span}\{(1,1,0),(1,2,3)\} \subset \mathbb{R}^3$. Bestimme $W^\perp$.

- Ist $v \in W \cap W^\perp$ dann: $\langle v,u \rangle = 0 \quad \forall u \in W$

$$\overset{v \in W}{\Longrightarrow} \quad \langle v,v \rangle = 0 \quad \overset{\langle,\rangle \text{ SP.}}{\Longrightarrow} \quad v = 0$$

Also ist $W \cap W^\perp = \{0\}$ d.h. $\dim W \cap W^\perp = 0$.

- $(1,1,0)$ und $(1,2,3)$ sind linear unabhängig

$\Rightarrow \dim W = 2$

Damit:

$$3 = \dim \mathbb{R}^3 \geq \dim W + W^\perp = \underbrace{\dim W}_{=2} + \dim W^\perp - \underbrace{\dim W \cap W^\perp}_{=0}$$

$\Rightarrow \dim W^\perp \leq 1$

- Betrachte $(3,-3,1) \in \mathbb{R}^3$.

Für $s,t \in \mathbb{R}$ beliebig:

$$\langle (3,-3,1),\ s(1,1,0) + t(1,2,3) \rangle$$

$$= s \langle (3,-3,1), (1,1,0) \rangle + t \langle (3,-3,1),(1,2,3) \rangle$$

$$= s \underbrace{(3\cdot 1 - 3\cdot 1 + 1\cdot 0)}_{=0} + t \underbrace{(3\cdot 1 - 3\cdot 2 + 3\cdot 1)}_{=0}$$

$$= 0$$

$\Rightarrow (3,-3,1) \in W^\perp$.

- Wegen $\dim W^\perp \leq 1$ folgt also: $W^\perp = \operatorname{span}\{(3,-3,1)\}$

$\blacksquare$

12.3. Bestimme den Winkel Θ zwischen den Vektoren

$$v = (1, 1, \sqrt{2}) \quad \text{und} \quad w = (1, 0, \sqrt{2})$$

des $\mathbb{R}^3$.

Nach VL gilt $\cos\Theta = \dfrac{\langle v, w\rangle}{\|v\| \cdot \|w\|}$. Es ist

$$\|v\| = (\langle v, v\rangle)^{\frac{1}{2}} = (1^2 + 1^2 + \sqrt{2}^2)^{\frac{1}{2}} = \sqrt{4} = 2$$

$$\|w\| = (\langle w, w\rangle)^{\frac{1}{2}} = (1^2 + 0^2 + \sqrt{2}^2)^{\frac{1}{2}} = \sqrt{3}$$

$$\langle v, w\rangle = 1 \cdot 1 + 1 \cdot 0 + \sqrt{2} \cdot \sqrt{2} = 3$$

$$\Rightarrow \quad \cos\Theta = \frac{3}{2\sqrt{3}} = \frac{3\sqrt{3}}{2\sqrt{3}\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \Theta = \frac{\pi}{6}$$

$\blacksquare$

<u>12.4.</u> Der Vektorraum $\mathbb{P}_2$ der reellen Polynome vom Grad ≤ 2 sei mit dem Skalarprodukt

$$(p\,|\,q) :=\int_0^1 p(t)\,q(t)\,dt$$

versehen. Konstruiere aus der Basis $\{1,t,t^2\}$ eine Orthonormalbasis von $\mathbb{P}_2$ und stelle $1+t+t^2 \in \mathbb{P}_2$ in dieser Basis dar.

Setze $w_1 := 1$, $w_2 := t$ und $w_3 := t^2$. Wende Gram-Schmidt an:

- $v_1 := \frac{w_1}{\|w_1\|}$, $\quad \|w_1\| = \left((w_1\,|\,w_1)\right)^{\frac{1}{2}} = \left(\int_0^1 1\cdot 1\,dt\right)^{\frac{1}{2}}$
$$= (1-0)^{\frac{1}{2}} = 1$$

$\Rightarrow v_1 = w_1 = 1.$

- $\tilde{v}_2 := w_2 - \sum_{j=1}^{2-1} (w_2\,|\,v_j)\cdot v_j = w_2 - (w_2\,|\,v_1)\cdot v_1$

$$(w_2\,|\,v_1) = \int_0^1 t\cdot 1\,dt = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}$$

$\Rightarrow \tilde{v}_2 = t - \frac{1}{2}$

- $v_2 := \frac{\tilde{v}_2}{\|\tilde{v}_2\|}$

$$\|\tilde{v}_2\| = \left((\tilde{v}_2\,|\,\tilde{v}_2)\right)^{\frac{1}{2}} = \left(\int_0^1 (t-\tfrac{1}{2})(t-\tfrac{1}{2})\,dt\right)^{\frac{1}{2}} = \left(\int_0^1 (t^2 - t + \tfrac{1}{4})\,dt\right)^{\frac{1}{2}}$$

$$= \left(\frac{1^3}{3} - \frac{1^2}{2} + \frac{1\cdot 1}{4} - \frac{0^3}{3} + \frac{0^2}{2} - \frac{1\cdot 0}{4}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{12}}$$

$\Rightarrow v_2 = \sqrt{12}\,t - \frac{\sqrt{12}}{2} = 2\sqrt{3}\,t - \sqrt{3}.$

- $\tilde{v}_3 := w_3 - \sum_{j=1}^{3-1} (w_3 | v_j)\, v_j = w_3 - (w_3 | v_1) v_1 - (w_3 | v_2) v_2$

$$(w_3 | v_1) = \int_0^1 t^2 \cdot 1 \, dt = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

$$(w_3 | v_2) = \int_0^1 t^2 (2\sqrt{3}\, t - \sqrt{3})\, dt = \sqrt{3} \int_0^1 (2t^3 - t^2)\, dt$$

$$= \sqrt{3} \left(\frac{1^4}{2} - \frac{1^3}{3} - \frac{0^4}{2} + \frac{0^3}{3} \right) = \frac{\sqrt{3}}{6}$$

$$\Rightarrow \tilde{v}_3 = t^2 - \frac{1}{3} \cdot 1 - \frac{\sqrt{3}}{6} (2\sqrt{3}\, t - \sqrt{3}) = t^2 - t + \frac{1}{6}$$

- $v_3 := \frac{\tilde{v}_3}{\|\tilde{v}_3\|}$

$$\|\tilde{v}_3\| = \left((\tilde{v}_3 | \tilde{v}_3) \right)^{\frac{1}{2}} = \left(\int_0^1 (t^2 - t + \tfrac{1}{6})(t^2 - t + \tfrac{1}{6})\, dt \right)^{\frac{1}{2}}$$

$$= \left(\int_0^1 (t^4 - 2t^3 + \tfrac{4}{3} t^2 - \tfrac{1}{3} t + \tfrac{1}{36})\, dt \right)^{\frac{1}{2}}$$

$$= \left(\frac{1^5}{5} - 2 \cdot \frac{1^4}{4} + \frac{4}{3} \cdot \frac{1^3}{3} - \frac{1}{3} \cdot \frac{1^2}{2} + \frac{1 \cdot 1}{36} - 0 \right)^{\frac{1}{2}} = \left(\frac{1}{180} \right)^{\frac{1}{2}}$$

$$= \frac{1}{6\sqrt{5}}$$

$$\Rightarrow v_3 = 6\sqrt{5} \left(t^2 - t + \frac{1}{6} \right) = 6\sqrt{5}\, t^2 - 6\sqrt{5}\, t + \sqrt{5}$$

Also ist die gesuchte Orthonormalbasis von $\mathbb{P}_2$

$$\{ 1,\ 2\sqrt{3}\, t - \sqrt{3},\ 6\sqrt{5}\, t^2 - 6\sqrt{5}\, t + \sqrt{5} \}$$

- $u := 1 + t + t^2 \in \mathbb{P}_2$. Da $\{v_1, v_2, v_3\}$ eine Orthonormalbasis ist, folgt:

$$u = (u|v_1)v_1 + (u|v_2)v_2 + (u|v_3)v_3$$

$$(u|v_1) = \int_0^1 (1+t+t^2)\cdot 1\, dt = \frac{1^3}{3} + \frac{1^2}{2} + 1 - 0 = \frac{11}{6}$$

$$(u|v_2) = \int_0^1 (1+t+t^2)(2\sqrt{3}t - \sqrt{3})\, dt$$

$$= \int_0^1 (2\sqrt{3}t^3 + \sqrt{3}t^2 + \sqrt{3}t - \sqrt{3})\, dt$$

$$= \sqrt{3}\left(2\cdot\frac{1^4}{4} + \frac{1^3}{3} + \frac{1^2}{2} - 1 - 0\right) = \frac{\sqrt{3}}{3}$$

$$(u|v_3) = \int_0^1 (1+t+t^2)(6\sqrt{5}t^2 - 6\sqrt{5}t + \sqrt{5})\, dt$$

$$= \int_0^1 (6\sqrt{5}t^4 + \sqrt{5}t^2 - 5\sqrt{5}t + \sqrt{5})\, dt$$

$$= \sqrt{5}\left(6\cdot\frac{1^5}{5} + \frac{1^3}{3} - 5\cdot\frac{1^2}{2} + 1 - 0\right) = \frac{\sqrt{5}}{30}$$

Also folgt für die Darstellung von u:

$$u = \frac{11}{6}\cdot 1 + \frac{\sqrt{3}}{3}\cdot(2\sqrt{3}t - \sqrt{3}) + \frac{\sqrt{5}}{30}\cdot(6\sqrt{5}t^2 - 6\sqrt{5}t + \sqrt{5})$$

$\blacksquare$