Mathematik I WiSe 2010/2011 – Lösung Blatt 8

8.2. a) ges.: $\mathbb{L}_{A,b}$ mit

$$A = \begin{pmatrix} [1] & [2] & [0] & [1] & [7] \\ [1] & [0] & [0] & [3] & [5] \\ [3] & [2] & [0] & [7] & [17] \end{pmatrix} = \begin{pmatrix} [1] & [2] & [0] & [1] & [1] \\ [1] & [0] & [0] & [0] & [2] \\ [0] & [2] & [0] & [1] & [2] \end{pmatrix} \in \left(\mathbb{Z}/3\mathbb{Z}\right)^{3\times 5},$$

$$b = \begin{pmatrix} [3] \\ [1] \\ [5] \end{pmatrix} = \begin{pmatrix} [0] \\ [1] \\ [2] \end{pmatrix} \in \left(\mathbb{Z}/3\mathbb{Z}\right)^{3}.$$

Gauß-Algorithmus:

$$(A\,|\,b) = \left(\begin{array}{ccccc|c} [1] & [2] & [0] & [1] & [1] & [0] \\ [1] & [0] & [0] & [0] & [2] & [1] \\ [0] & [2] & [0] & [1] & [2] & [2] \end{array}\right) \begin{array}{l} \cdot(-[1]) \\ + \\ \end{array}$$

$$\rightarrow \left(\begin{array}{ccccc|c} [1] & [2] & [0] & [1] & [1] & [0] \\ [0] & [1] & [0] & [2] & [1] & [1] \\ [0] & [2] & [0] & [1] & [2] & [2] \end{array}\right) \begin{array}{l} \\ \cdot(-[2]) \\ + \end{array}$$

$$\rightarrow \left(\begin{array}{ccccc|c} [1] & [2] & [0] & [1] & [1] & [0] \\ [0] & [1] & [0] & [2] & [1] & [1] \\ [0] & [0] & [0] & [0] & [0] & [0] \end{array}\right) \begin{array}{l} + \\ \cdot(-[2]) \\ \end{array}$$

$$\rightarrow \left(\begin{array}{ccccc|c} [1] & [0] & [0] & [0] & [2] & [1] \\ [0] & [1] & [0] & [2] & [1] & [1] \\ [0] & [0] & [0] & [0] & [0] & [0] \end{array}\right)$$

$$\Rightarrow \mathbb{L}_{A,b} = \begin{pmatrix} [1] \\ [1] \\ [0] \\ [0] \\ [0] \end{pmatrix} + \left\langle \begin{pmatrix} [0] \\ [0] \\ [1] \\ [0] \\ [0] \end{pmatrix}, \begin{pmatrix} [0] \\ [1] \\ [0] \\ [1] \\ [0] \end{pmatrix}, \begin{pmatrix} [1] \\ [2] \\ [0] \\ [0] \\ [1] \end{pmatrix} \right\rangle.$$

b) <u>ges.:</u> $\mathbb{L}_{A,b}$ mit

$$A = \begin{pmatrix} [1] & [2] & [0] & [1] & [7] \\ [1] & [0] & [0] & [3] & [5] \\ [3] & [2] & [0] & [7] & [17] \end{pmatrix} = \begin{pmatrix} [1] & [2] & [0] & [1] & [2] \\ [1] & [0] & [0] & [3] & [0] \\ [3] & [2] & [0] & [2] & [2] \end{pmatrix} \in \left(\mathbb{Z}/5\mathbb{Z}\right)^{3\times 5},$$

$$b = \begin{pmatrix} [3] \\ [1] \\ [5] \end{pmatrix} = \begin{pmatrix} [3] \\ [1] \\ [0] \end{pmatrix} \in \left(\mathbb{Z}/5\mathbb{Z}\right)^{3}.$$

Gauß-Algorithmus:

$$(A|b) = \left(\begin{array}{ccccc|c} [1] & [2] & [0] & [1] & [2] & [3] \\ [1] & [0] & [0] & [3] & [0] & [1] \\ [3] & [2] & [0] & [2] & [2] & [0] \end{array}\right) \begin{array}{l} \cdot(-[1]) \quad \cdot(-[3]) \\ + \\ + \end{array}$$

$$\rightarrow \left(\begin{array}{ccccc|c} [1] & [2] & [0] & [1] & [2] & [3] \\ [0] & [3] & [0] & [2] & [3] & [0] \\ [0] & [1] & [0] & [4] & [1] & [1] \end{array}\right) \begin{array}{l} \\ |\cdot[2] \quad \cdot(-[1]) \\ + \end{array}$$

$$\rightarrow \left(\begin{array}{ccccc|c} [1] & [2] & [0] & [1] & [2] & [3] \\ [0] & [1] & [0] & [4] & [1] & [1] \\ [0] & [0] & [0] & [0] & [0] & [0] \end{array}\right) \begin{array}{l} + \\ \cdot(-[2]) \\ \end{array}$$

$$\rightarrow \left(\begin{array}{ccccc|c} [1] & [0] & [0] & [3] & [0] & [1] \\ [0] & [1] & [0] & [4] & [1] & [1] \\ [0] & [0] & [0] & [0] & [0] & [0] \end{array}\right)$$

$$\Rightarrow \mathbb{L}_{A,b} = \begin{pmatrix} [1] \\ [1] \\ [0] \\ [0] \\ [0] \end{pmatrix} + \left\langle \begin{pmatrix} [0] \\ [0] \\ [1] \\ [0] \\ [0] \end{pmatrix}, \begin{pmatrix} [2] \\ [1] \\ [0] \\ [1] \\ [0] \end{pmatrix}, \begin{pmatrix} [0] \\ [4] \\ [0] \\ [0] \\ [1] \end{pmatrix} \right\rangle.$$

8.3. $(a,b), (a',b') \in \mathbb{R}^2$,

$$\Delta := \{ \lambda(a,b) + \lambda'(a',b') \mid \lambda, \lambda' \geq 0, \ \lambda + \lambda' \leq 1 \}$$

Dreieck mit Ecken $(0,0)$, (a,b), (a',b').

<u>Beh.:</u> $\text{Fläche}(\Delta) = \pm \frac{1}{2} \det \begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$.

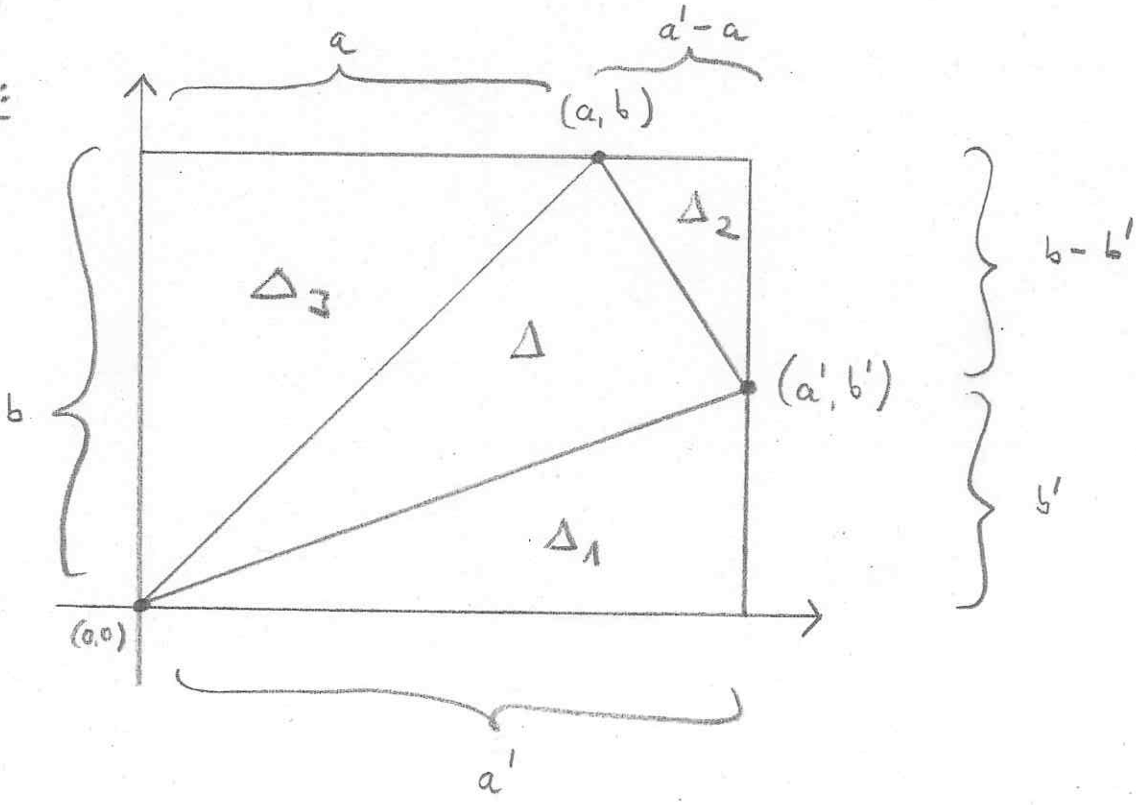
<u>Beweis:</u> Zunächst einige Vereinfachungen:

1. O.B.d.A. seien $a, b, a', b' > 0$, die anderen Fälle behandelt man analog.
2. O.B.d.A. sei $a \neq a'$, $b \neq b'$ und $(a,b), (a',b')$ linear unabhängig, die anderen Fälle erhält man als „Grenzfälle" der folgenden Betrachtungen.
3. O.B.d.A. sei $a < a'$, sonst vertausche man die Bezeichnungen.

Es bleiben dann zwei „wesentliche" Fälle:

<u>1. Fall:</u> $b > b'$ („spitzwinkliges Dreieck")

<u>Skizze:</u>

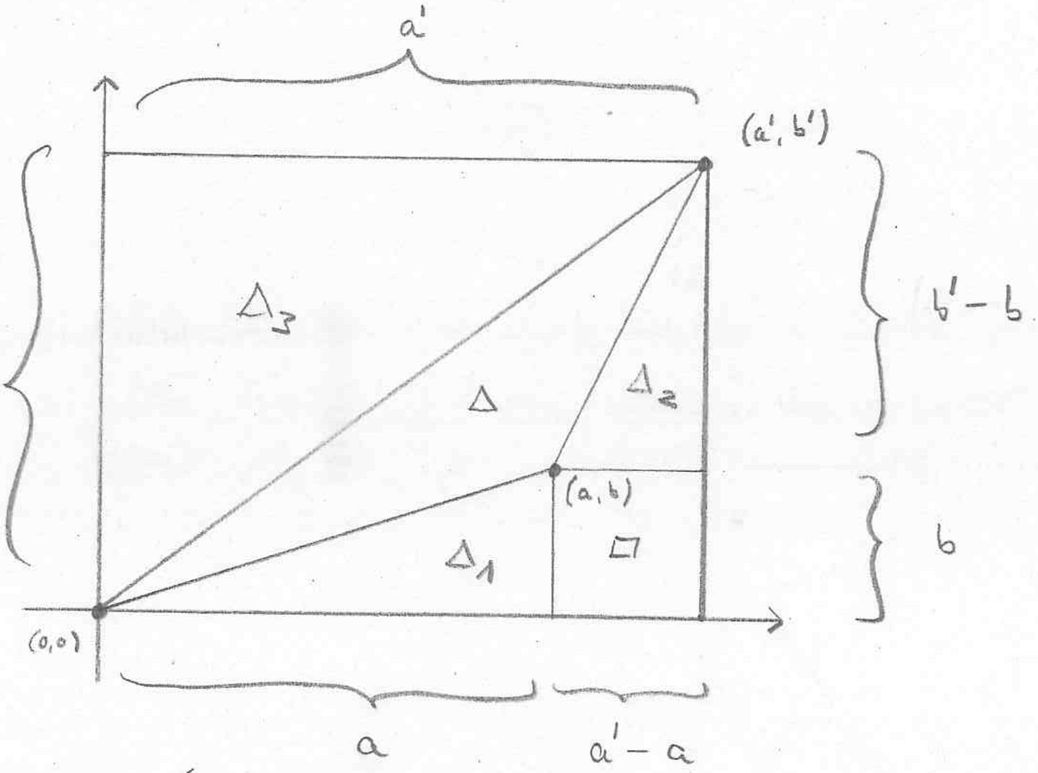


<u>Hier gilt:</u>

$$
\begin{aligned}
\text{Fläche}(\Delta) &= a'b - \text{Fläche}(\Delta_1) - \text{Fläche}(\Delta_2) - \text{Fläche}(\Delta_3)\\
&= a'b - \tfrac{1}{2}a'b' - \tfrac{1}{2}(a'-a)(b-b') - \tfrac{1}{2}ab\\
&= a'b - \cancel{\tfrac{1}{2}a'b'} - \tfrac{1}{2}a'b + \cancel{\tfrac{1}{2}a'b'} + \cancel{\tfrac{1}{2}ab} - \tfrac{1}{2}ab' - \cancel{\tfrac{1}{2}ab}\\
&= \tfrac{1}{2}(a'b - ab')\\
&= -\tfrac{1}{2}\det\begin{pmatrix} a & b\\ a' & b'\end{pmatrix}.
\end{aligned}
$$

<u>2. Fall:</u> $b < b'$ („stumpfwinkliges Dreieck")

<u>Skizze:</u>



<u>Hier gilt:</u> Fläche(Δ)

$$
\begin{aligned}
&= a'b' - \text{Fläche}(\Delta_1) - \text{Fläche}(\Delta_2) - \text{Fläche}(\Delta_3) - \text{Fläche}(\square)\\
&= a'b' - \tfrac{1}{2}ab - \tfrac{1}{2}(a'-a)(b'-b) - \tfrac{1}{2}a'b' - (a'-a)b\\
&= \cancel{\tfrac{1}{2}a'b'} - \cancel{\tfrac{1}{2}ab} - \cancel{\tfrac{1}{2}a'b'} + \tfrac{1}{2}a'b + \tfrac{1}{2}ab' - \cancel{\tfrac{1}{2}ab} - a'b + \cancel{ab}\\
&= \tfrac{1}{2}(ab' - a'b) = \tfrac{1}{2}\det\begin{pmatrix} a & b\\ a' & b'\end{pmatrix}.
\end{aligned}
$$

8.4.

a) $$\begin{vmatrix} 3 & 0 & 2 & 0 \\ 1 & 4 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 4 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 3 & 0 & 2 & 0 \\ 0 & 4 & 0 & 1 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 2 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & -6 & -1 & -3 \\ 0 & 4 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\underset{\substack{\uparrow \\ \text{obere} \\ \Delta\text{-Matrix}}}{=} -1 \cdot 2 \cdot (-1) \cdot 1 = \underline{\underline{2}}.$$

b) $$\begin{vmatrix} 3 & 0 & 1 & 2 \\ 0 & 2 & 2 & 2 \\ 3 & 2 & 3 & 2 \\ 0 & 4 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 3 & 0 & 1 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 4 & 4 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 0 & 1 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{vmatrix} \underset{\substack{\uparrow \\ \text{obere} \\ \Delta\text{-Matrix}}}{=} 3 \cdot 2 \cdot 0 \cdot 1 = \underline{\underline{0}}.$$