FROM T-COALGEBRAS TO FILTER STRUCTURES AND TRANSITION SYSTEMS

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ABSTRACT. For any set-endofunctor $T : \mathcal{S}et \to \mathcal{S}et$ there exists a largest sub-cartesian transformation μ to the filter functor $\mathbb{F} : \mathcal{S}et \to \mathcal{S}et$. Thus we can associate with every T-coalgebra A a certain filter-coalgebra $A_{\mathbb{F}}$.

Precisely, when T weakly preserves preimages, μ is natural, and when T weakly preserves intersections, μ factors through the covariant powerset functor $\mathbb{P}$, thus providing for every T-coalgebra A a Kripke structure $A_{\mathbb{P}}$.

The paper characterizes weak preservation of preimages, of intersections, and preservation of both preimages and intersections by a functor T via the existence of transformations from T to either $\mathbb{F}$ or $\mathbb{P}$.

Moreover, we define for arbitrary T-coalgebras $\mathcal{A}$ a next-time operator $\bigcirc_{\mathcal{A}}$ with associated modal operators $\Box$ and $\Diamond$ and relate their properties to weak limit preservation properties of T. In particular, for any T-coalgebra $\mathcal{A}$ there is a transition system $\mathcal{K}$ with $\bigcirc_A = \bigcirc_K$ if and only if T weakly preserves intersections.

1. INTRODUCTION

The importance of weak preservation properties of coalgebraic type functors has been clear since the seminal work of Rutten [Rut00]. Many of the results in the original 1996 preprint-version of his work assumed that the coalgebraic type functor weakly preserves pullbacks, or even arbitrary intersections.

In joint works with T. Schröder, we have subsequently shown that weak preservation of pullbacks decomposes into two more basic preservation properties, namely preservation of preimages and weak preservation of kernels. We have given numerous (co-)algebraic properties that depend, in a one-to-one fashion, to these preservation properties of the type functor.

The current paper studies a transformation μ between an arbitrary $\mathcal{S}et$-endofunctor T and the *filter functor* that associates with a set X the set $\mathbb{F}(X)$ of all filters on a set X.

The basic idea is to capture the notion of *successors* of a point a, which plays a central role in Kripke Structures, and make it available for coalgebras of arbitrary type T. Equivalently, taking a logical viewpoint, one may generalize the nexttime operator $\bigcirc$ of Kripke structures, which associates to a subset $S \subseteq A$ of a Kripke structure on A the set of all points whose successors are all contained in S.

It turns out that, unless T preserves intersections, one cannot speak of a single set of successors, but must consider a family of successor sets. Fortunately, however, the successor sets form a filter.

Therefore, one can construct a transformation μ between $T(X)$ and $\mathbb{F}(X)$, for arbitrary $\mathcal{S}et$-endofunctors T. Even though μ is not a natural transformation in general, it is enough to observe that it is *sub-natural*, a term defined below. The mentioned preservation properties of the functor T correspond to μ being natural, *sub-cartesian*, or cartesian.

For arbitrary T-coalgebras $\mathcal{A}$ this has the consequence that one always can define a filter-coalgebra on the same base set which has the same subcoalgebras as $\mathcal{A}$. Closer connections between $\mathcal{A}$ and its associated filter-coalgebra or its associated Kripke structures correspond to the mentioned preservation properties of T.

2. Categorical Notions

We need only basic category theoretic notions and facts, as found in the first few chapters of any textbook, such as e.g. [AHS90].

A functor $F : \mathcal{C} \to \mathcal{D}$ is said to *preserve monos*, if Ff is mono, whenever f was. When monos are left-invertible, as e.g. in the category of nonempty sets and mappings, they are, of course, automatically preserved.

Pullbacks are limits of two morphisms $f : A \to C$ and $g : B \to C$ with common codomain. Thus, the pullback of f and g is an object P with morphisms $p_1 : P \to A$ and $p_2 : P \to B$, so that $f \circ p_1 = g \circ p_2$, and for any other object Q with morphisms $p_1 : P \to A$ and $p_2 : Q \to B$ satisfying $f \circ q_1 = g \circ q_2$ there exists a unique "mediating" morphism $d : Q \to P$ with $p_i \circ d = q_i$, for $i = 1, 2$.

$$\begin{tikzcd}
 & A \arrow[r, "f"] & C \\
 & P \arrow[u, "p_1"'] \arrow[r, "p_2"] & B \arrow[u, "g"'] \\
Q \arrow[uur, dashed, "p_1"] \arrow[ur, dotted, "\exists! d"] \arrow[urr, dashed, "p_2"'] & &
\end{tikzcd}$$

Weak pullbacks are the corresponding weak limits, i.e. where the uniqueness requirement for the mediating morphism is dropped.

Preimages are pullbacks where g is mono. Observe, that in this case, p_1 will automatically be mono, too. This is not necessarily the case for *weak preimages.* However, a weak pullback, in which one of p_1, p_2 is mono, is already a pullback.

Intersections (*weak intersections*) are limits (weak limits) of families of monomorphisms $(f_i : A_i \rightarrowtail A)_{i \in I}$ with common codomain.

A functor $F : \mathcal{C} \to \mathcal{D}$ is said to *weakly preserve pullbacks*, if it transforms every pullback diagram in $\mathcal{C}$ into a weak pullback diagram in $\mathcal{D}$. If pullbacks always exist in $\mathcal{C}$, then it is easily seen that F weakly preserves pullbacks iff F *preserves weak pullbacks*, i.e. F transforms weak pullback diagrams into weak pullback diagrams.

Correspondingly, we say that F *weakly preserves preimages*, if it transforms each preimage diagram into a weak pullback diagram.

We say that F *weakly preserves intersections*, if F transforms every intersection diagram into a weak limit diagram.

If F preserves monos, as will often be the case, *weak preservation* of preimages (resp. intersections) is the same as *preservation* of preimages (resp. intersections).

3. Sub-natural and sub-cartesian transformations

Given categories $\mathcal{C}$, $\mathcal{D}$ and functors $F, G : \mathcal{C} \to \mathcal{D}$, a *transformation* $\nu : F \to G$ is just a family of $\mathcal{D}$-morphisms $\nu_A : FA \to GA$ for every object A in $\mathcal{C}$. It is called a *natural transformation*, if for every $\mathcal{C}$-morphism $f : A \to B$ the diagram

$$\begin{tikzcd}
FB \arrow[r, "\nu_B"] & GB \\
FA \arrow[u, "Ff"] \arrow[r, "\nu_A"] & GA \arrow[u, "Gf"']
\end{tikzcd}$$

commutes, and it is called *cartesian*, if the same diagram is a pullback.

We shall need to work with transformations, which are neither natural nor cartesian, but satisfy a weaker property:

Definition 3.1. *A transformation $\nu : F \to G$ will be called* sub-natural *if the above diagram commutes for every monomorphism f. It is called* sub-cartesian*, if the diagram is a weak pullback for every monomorphism f.*

The following observation will become important in later sections:

Theorem 3.2. *Assume that F preserves monos, and let $\nu : F \to G$ be a sub-cartesian transformation. Then*

(i) *if G weakly preserves intersections then so does F.*
(ii) *if ν is natural and G weakly preserves preimages, then F preserves preimages.*

Proof. To show (i), start with a family of monomorphisms $(e_i : A_i \hookrightarrow A)_{i \in I}$ and its limit M with morphisms $f_i : M \hookrightarrow A_i$, satisfying $e_i \circ f_i = e_k \circ f_k$ for all $i, k \in I$. Applying F and G and inserting the transformation morphisms, we obtain the following diagram, where the top row is a weak limit by the assumption on G. Since ν is subcartesian and the e_i and f_i are monos, the squares commute.

$$\begin{tikzcd}
& GM \arrow[r, "Gf_i"] & GA_i \arrow[r, "Ge_i"] & GA \\
& FM \arrow[u, "\nu_A"'] \arrow[r, hook, "Ff_i"] & FA_i \arrow[u, "\nu_{A_i}"'] \arrow[r, hook, "Fe_i"] & FA \arrow[u, "\nu_A"'] \\
Q \arrow[uur, dashed, bend left, "e"] \arrow[ur, dotted, "d_i"'] \arrow[urr, dashed, bend right, "q_i"'] & & &
\end{tikzcd}$$

To show that the bottom row is a weak limit, too, let Q be a competitor with morphisms $q_i : Q \to FA_i$ satisfying $Fe_i \circ q_i = Fe_k \circ q_k$ for all $i, k \in I$. Then Q with morphisms $\nu_{A_i} \circ q_i$ becomes a competitor to the weak limit GM, yielding a morphism $e : Q \to GM$ with $\nu_{A_i} \circ q_i = Gf_i \circ e$ for all $i \in I$. Since ν is sub-cartesian, we obtain morphisms $d_i : Q \to FM$ for each $i \in I$ with $Ff_i \circ d_i = q_i$. We need to show that all the d_i are equal to a single morphism d. A diagram chase, utilizing $Fe_i \circ q_i = Fe_k \circ q_k$ and $e_k \circ f_k = e_i \circ f_i$, yields $Fe_i \circ Ff_i \circ d_i = Fe_i \circ Ff_i \circ d_k$. Since F preserves monos, we can cancel $Fe_i \circ Ff_i$ and obtain $d_i = d_k$.

To show (ii), let P with morphisms $p_1 : P \to A$ and $p_2 : P \to B$ be the preimage of $f : A \to C$ and monomorphism $g : B \hookrightarrow C$. It follows that p_1 is mono.

Applying F and G to this diagram and filling in the transformation ν, which we now assume to be natural, we obtain the commutative cube of the following figure:

$$\begin{tikzcd}
 & & GA \arrow[rr, "Gf"] & & GC \\
 & FA \arrow[ur, "\nu_A"] \arrow[rr, "Ff"] & & FC \arrow[ur, "\nu_C"] & \\
 & & GP \arrow[uu, "Gp_1"'] \arrow[rr, "Gp_2"] & & GB \arrow[uu, "Gg"'] \\
 & FP \arrow[uu, hook, "Fp_1"] \arrow[ur, "\nu_P"] \arrow[rr, "Fp_2"] & & FB \arrow[uu, hook, "Fg"'] \arrow[ur, "\nu_B"'] & \\
Q \arrow[uuur, dashed, "q_1"] \arrow[ur, dotted, "e"] \arrow[uurr, dashed, "d"'] \arrow[urrr, dashed, "q_2"'] & & & &
\end{tikzcd}$$

Since G weakly preserves preimages, the back face is a weak pullback diagram. Since F preserves monos, Fp_1 and Fg will be mono. We need to show that the front face is a weak pullback, too.

Given a competitor, i.e. an object Q with morphisms $q_1 : Q \to FA$ and $q_2 : Q \to FB$ satisfying $Ff \circ q_1 = Fg \circ q_2$, we extend q_1 and q_2 with the transformation morphisms ν_A and ν_B, to make Q into a competitor to the weak pullback of the back face. This yields a mediating morphism $d : Q \to GP$ with $\nu_A \circ q_1 = Gp_1 \circ d$. Since ν is sub-cartesian, the left face is a pullback, so we obtain a morphism $e : Q \to FP$ with $Fp_1 \circ e = q_1$. It follows that

$$\begin{aligned} Fg \circ Fp_2 \circ e &= Ff \circ Fp_1 \circ e \\ &= Ff \circ q_1 \\ &= Fg \circ q_2, \end{aligned}$$

so canceling the monomorphism Fg yields $Fp_2 \circ e = q_2$. □

4. Functors on the category $\mathcal{S}et$

Given a set X with subset $U \subseteq X$, we denote the canonical injection by $\subseteq_U^X : U \hookrightarrow X$. We sometimes drop the sub- and superscripts of $\subseteq$ when they are clear from the context.

Given a map $f : X \to Y$ with $U \subseteq X$ and $V \subseteq Y$, we denote by $f[U]$ the image of U and by $f^{-}V$ the preimage of V under f. Occasionally, we write $[f]$ for the image $f[X]$ of f.

4.1. $\mathcal{S}et$-**Functors.** For a $\mathcal{S}et$-functor $T : \mathcal{S}et \to \mathcal{S}et$, and for $U \subseteq X$, we write T_U^X for the application of T to the inclusion map $\subseteq_U^X$, and $[T_U^X]$ for the image of TU under the said map T_U^X, i.e.

$$[T_U^X] := T_U^X[TU].$$

We note two simple lemmas:

Lemma 4.1. *If $U \subseteq V \subseteq X$, then $[T_U^X] \subseteq [T_V^X]$.*

Proof. $[T_U^X] = T_U^X[TU] = (T_V^X \circ T_U^V)[TU] = T_V^X[T_U^V] \subseteq T_V^X[TV] = [T_V^X]$. □

Lemma 4.2. *If $f : X \to Y$ and $U \subseteq X$, then $(Tf)[T_U^X] = [T_{f[U]}^Y]$.*

Proof. Let $f_|$ be the domain-codomain-restriction of f to U, then $f \circ \subseteq_U^X = \subseteq_{f[U]}^Y \circ f_|$, hence $Tf \circ T_U^X = T_{f[U]}^Y \circ Tf_|$. Applying the left side to TU yields $(Tf)[T_U^X]$. Since $f_|$ is surjective, it has a right inverse, hence $Tf_| : TU \to T(f[U])$ must be surjective, too. Therefore, $(T_{f[U]}^Y \circ Tf_|)[TU] = T_{f[U]}^Y[T(f[U])] = [T_{f[U]}^Y]$. □

4.2. ***Set*-functors preserve finite nonempty intersections.** For a $\mathcal{S}et$-functor $T : \mathcal{S}et \to \mathcal{S}et$ we may assume that $T(X) \neq \emptyset$, unless $X = \emptyset$, for otherwise T would have to be the trivial functor with $T(Y) = \emptyset$ for every set Y.

For nonempty sets X and Y, any injective $f : X \to Y$ has a left inverse. Hence Tf is invertible, too. As a consequence, every functor T on $\mathcal{S}et$ preserves monos with nonempty domain.

Rather surprisingly, every $\mathcal{S}et$-endofunctor T also preserves nonempty intersections. To be precise:

Lemma 4.3. [Trn69] *Whenever $U \cap V \neq \emptyset$, then $[T_U^W] \cap [T_V^W] = [T_{U\cap V}^W]$.*

A proof of this result can be found in [GS02]. A corresponding theorem for infinite intersections is not valid in general.

4.3. **Discharging empty sets and mappings.** The proviso about the empty set can be discarded by modifying the functor T on the empty set $\emptyset$ and on the empty mappings $\emptyset_A : \emptyset \to A$. To this end, consider the two-element set $\mathbf{2} = \{\mathbf{0}, \mathbf{1}\}$ with canonical injections $e_0, e_1 : \mathbf{1} \to \mathbf{2}$. Let $e : P \to T\mathbf{1}$ be the equalizer of Te_0 and Te_1.

$$\begin{tikzcd} P \arrow[r, "e"] & T\mathbf{1} \arrow[r, shift left, "Te_0"] \arrow[r, shift right, "Te_1"'] & T\mathbf{2} \end{tikzcd}$$

Then define a functor T^+ on objects X by

$$T^+(X) = \begin{cases} P, & \text{if } X = \emptyset, \\ T(X), & \text{otherwise,} \end{cases}$$

Identifying any $y \in Y$ with the map $\mathbf{1} \to Y$ with value y, we have $Ty : T\mathbf{1} \to TY$, and we can define for any $f : X \to Y$:

$$T^+f := \begin{cases} Ty \circ e, & \text{if } X = \emptyset, y \in Y \\ T(f), & \textit{otherwise.} \end{cases}$$

Due to the construction of e as an equalizer, one easily checks that the definition of T^+f does not depend on the choice of $y \in Y$. Then the following lemma can be verified:

Lemma 4.4. *(*[Trn69]*) T^+ is a $\mathcal{S}et$-functor, preserving all monos and all finite intersections. T^+ agrees with T on all nonempty sets and on all mappings with nonempty domain.*

The above description of T^+ is from Barr([Bar93]) and it differs slightly from the original construction of Trnková, who defined $T^+(\emptyset)$ as the set of all natural transformations from $\hat{\mathbf{1}}$ to T, where $\hat{\mathbf{1}}$ is the functor with $\hat{\mathbf{1}}(\emptyset) = \emptyset$ and $\hat{\mathbf{1}}(X) = \mathbf{1}$ for $X \neq \emptyset$. Barr's description has the advantage that equalizers (in the category $\mathcal{S}et$) are usually easier to calculate than natural transformations.

The following corollary will be needed later:

Corollary 4.5. *T^+ preserves preimages of injective maps.*

Since we are interested in coalgebras, nothing changes, when we replace T by T^+, so we will assume from now on that all $\mathcal{S}et$-functors under consideration satisfy the property of T^+ in the previous lemma. In particular, they preserve monos, finite intersections and preimages of injective maps.

4.4. **Preservation properties of $\mathcal{S}et$-functors.** Checking, whether a diagram is a pullback (weak pullback) is especially easy in the category $\mathcal{S}et$. Essentially, this is due to the fact that each set is a sum of one-element sets, so the pullback condition can be formulated elementwise. Thus, in order to check whether a $\mathcal{S}et$-functor preserves weak pullbacks, we can use the following criterion:

Lemma 4.6. *A set functor $T : \mathcal{S}et \to \mathcal{S}et$ weakly preserves the pullback (P, p_1, p_2) of the maps $f : X \to Z$ and $g : Y \to Z$, iff for any $u \in TX$ and $v \in TY$ with $(Tf)(u) = (Tg)(v)$ there is a $w \in TP$ with $(Tp_1)(w) = u$ and $(Tp_2)(w) = v$.*

Lemma 4.7. *A functor $T : \mathcal{S}et \to \mathcal{S}et$ preserves intersections iff for each family $(U_i \subseteq X)_{i\in I}$*

$$\bigcap_{i\in I} [T^X_{U_i}] = [T^X_{\bigcap_{i\in I} U_i}].$$

Proof. By lemma 4.1, the inclusion "$\supseteq$" is always true.

Assuming that T preserves intersections, for each $u \in \bigcap_{i\in I} [T^X_{U_i}]$ and each $i \in I$ there exists $u_i \in TU_i$ with $T^X_{U_i}(u_i) = u$. Abbreviate $W := \bigcap_{i\in I} U_i$, then TW with the maps $T^{U_i}_W$ is a limit of the sink $(T^X_{U_i})_{i\in I}$, so there exists some $w \in TW$ with $T^{U_i}_W(w) = u_i$, hence $T^X_W(w) = (T^X_{U_i} \circ T^{U_i}_W)(w) = T^X_{U_i}(u_i) = u$, so $u \in [T^X_W]$.

Conversely, assume that the formula is true and let P with maps $f_i : P \to X_i$ be the limit of the monomorphisms $e_i : X_i \rightarrowtail X$. Each e_i factors as $e_i = \subseteq \circ\, g_i$ with $g_i : X_i \to U_i \subseteq X$ bijective. It follows that there is a bijective map $g : P \to \bigcap_{i\in I} U_i$ with $g_i \circ f_i = \subseteq \circ\, g$. Applying T, we have the following commutative diagram:

$$\begin{tikzcd}
& TP \arrow[r, tail, "Tf_i"] \arrow[d, tail, "Tg"'] & TX_i \arrow[r, tail, "Te_i"] \arrow[d, tail, "Tg_i"] & TX \\
T\bigcap_{i\in I} U_i \arrow[r, phantom, "="] & TW \arrow[r, hook, "T^{U_i}_W"] & TU_i \arrow[ur, hook, "T^X_{U_i}"'] &
\end{tikzcd}$$

To see that TP with the maps Tf_i is the limit of the Te_i, assume a family of elements $u_i \in TX_i$ be given with $(Te_j)(u_j) = (Te_k)(u_k) =: u$ for all $j, k \in I$. We need to find an element $p \in TP$ with $Tf_i(p) = u_i$ for all $i \in I$.

Now, $T^X_{U_i}(Tg_i(u_i)) = u$ for all $i \in I$, so $u \in \bigcap_{i\in I} [T^X_{U_i}]$, hence by the assumption $u \in [T^X_{\bigcap_{i\in I} U_i}]$. Abbreviating $W := \bigcap_{i\in I} U_i$, then $u = T^X_W(v)$ for some $v \in TW$. We claim that $p := Tg^{-1}(v)$ is the sought element in TP. Indeed, $(Te_i \circ Tf_i)(p) = (T^X_{U_i} \circ T^{U_i}_W \circ Tg)(Tg^{-1}(v)) = T^X_W(v) = u = Te_i(u_i)$ for each $i \in I$. Since the Te_i are monos, it follows $Tf_i(p) = u_i$ for all $i \in I$. $\square$

The following is an easy but relevant corollary:

Corollary 4.8. *T preserves infinite intersections iff for any $u \in TX$ there is a smallest $U \subseteq X$ with $u \in [T^X_U]$.*

Proof. The smallest U with $u \in [T_U^X]$ must be $U := \bigcap_{i\in I}\{V \subseteq X \mid u \in [T_V^X]\}$. To check whether indeed $u \in [T_U^X]$, we apply the formula of the previous lemma and obtain the triviality $u \in \bigcap\{\ [T_V^X] \mid u \in [T_V^X]\ \}$.

For the other direction, choose any $x \in \bigcap_{i\in I}[T_{U_i}^X]$ and let W be the smallest $W \subseteq X$ with $x \in [T_W^X]$. Then $W \subseteq \bigcap_{i\in I} U_i$, so $x \in [T^X_{\bigcap_{i\in I} U_i}]$ by lemma 4.1. □

5. The filter functor

A *filter* $\mathcal{G}$ on a set X is a collection of subsets of X that is closed under finite intersections and supersets. In other words, $\mathcal{G} \subseteq \mathbb{P}(X)$ and

- $G_1, G_2 \in \mathcal{G} \implies G_1 \cap \mathcal{G}_2 \in \mathcal{G}$, and
- $G \in \mathcal{G}$, and $G \subseteq H \subseteq X \implies H \in \mathcal{G}$.

On any set X, let $\mathbb{F}(X)$ be the set of all filters on X. $\mathbb{F}$ can be made into an endo-functor on $\mathcal{S}et$ by defining $\mathbb{F}f$ for any map $f : X \to Y$ as

$$(\mathbb{F}f)(\mathcal{G})\ := \uparrow \{f[G] \mid G \in \mathcal{G}\}.$$

where $\mathcal{G}$ is an arbitrary filter on X. Here $\uparrow \mathcal{H}$, for any system of subsets $\mathcal{H} \subseteq \mathbb{P}(X)$, denotes the set of all supersets of sets in $\mathcal{H}$, i.e. $\uparrow \mathcal{H} = \{W \subseteq X \mid \exists H \in \mathcal{H}.H \subseteq W\}$.

It was shown in [Gum01] that $\mathbb{F}$ is a functor and that the following theorem holds:

Proposition 5.1. $\mathbb{F}$ *weakly preserves pullbacks, but does not preserve infinite intersections.*

Here we shall work with the following equivalent definition for $\mathbb{F}f$:

Lemma 5.2. *For any map $f : X \to Y$, and any filter $\mathcal{G}$ on X, we have*

$$(\mathbb{F}f)(\mathcal{G})\ =\ \{V \subseteq Y \mid f^- V \in \mathcal{G}\}.$$

Proof. If $f^- V \in \mathcal{G}$ then $V \supseteq f[f^- V]$. Conversely, if $V \supseteq f[G]$ for some $G \in \mathcal{G}$, then $f^- V \supseteq f^- f[G] \supseteq G$, so $f^- V \in \mathcal{G}$. □

Clearly, the covariant powerset functor $\mathbb{P}$ is a subfunctor of the filter functor $\mathbb{F}$. The natural embedding $\varepsilon : \mathbb{P} \to \mathbb{F}$ associates a set $U \subseteq X$ with the filter of all supersets of U in X. There is also an obvious transformation

$$\bigcap : \mathbb{F} \to \mathbb{P}$$

in the other direction, given by intersection. We have $\bigcap \circ \epsilon = id_F$, but $\bigcap$ is *not* a natural transformation. Instead we find:

Lemma 5.3. $\bigcap : \mathbb{F} \to \mathbb{P}$ *is sub-natural, but not sub-cartesian.*

Proof. For sub-naturality, it suffices to check that for every injective map $f : X \to Y$ and any $\mathcal{G} \in \mathbb{F}X$:

$$(\mathbb{P}f \circ \bigcap)(\mathcal{G}) = f[\bigcap \mathcal{G}] = \bigcap\{f[G] \mid G \in \mathcal{G}\} = \bigcap \uparrow \{f[G] \mid G \in \mathcal{G}\} = (\bigcap \circ \mathbb{F}f)(\mathcal{G}).$$

$\mathbb{P}$ preserves intersections, and $\mathbb{F}$ preserves monos. Therefore, we can invoke 3.2 to argue that if $\bigcap$ was sub-cartesian, then by (i), $\mathbb{F}$ would have to preserve intersections too, which is not the case, see e.g. [Gum01]. □

6. A sub-cartesian transformation to the filter functor

For an arbitrary functor $T : \mathcal{S}et \to \mathcal{S}et$, we now define a transformation $\mu : T \to \mathbb{F}$. When T preserves preimages, μ will be natural, otherwise it will be natural on injective maps only. Nevertheless, this property will suffice to prove our coalgebraic results of the following section.

Definition 6.1. *For any set X and any functor $T : \mathcal{S}et \to \mathcal{S}et$ define a map $\mu_X : T(X) \to \mathbb{F}(X)$ by*

$$\mu_X(u) := \{U \subseteq X \mid u \in [T_U^X]\}.$$

To see that μ_X is indeed a filter, we need to invoke lemmas 4.1 and 4.3 in combination with 4.4. In general, μ is not a natural transformation, but we always have:

Lemma 6.2. *μ is a sub-cartesian transformation.*

Proof. For an injective map $f : X \to Y$, we first need to show commutativity of the following square:

$$\begin{tikzcd} TY \arrow[r, "\mu_Y"] & \mathbb{F}Y \\ TX \arrow[u, "Tf"] \arrow[r, "\mu_X"'] & \mathbb{F}X \arrow[u, "\mathbb{F}f"'] \end{tikzcd}$$

Given $u \in TX$, we have for any $V \subseteq Y$:

$$\begin{aligned} V \in \mu_Y((Tf)(u)) &\iff (Tf)(u) \in [T_V^Y] \\ &\iff \exists v \in TV.(Tf)(u) = T_V^Y(v) \\ &\stackrel{!}{\iff} \exists w \in T(f^- V).T_{f^- V}^X(w) = u \\ &\iff u \in [T_{f^- V}^X] \\ &\iff f^- V \in \mu_X(u) \\ &\iff V \in (\mathbb{F}f)(\mu_X(u)). \end{aligned}$$

The third (marked) equivalence is due to the fact that T preserves preimages with respect to injective maps, see corollary 4.5.

To check that the diagram is a weak pullback, let $v \in TY$ and $\mathcal{G} \in \mathbb{F}X$ be given with $\mu_Y(v) = (\mathbb{F}f)(\mathcal{G})$. This implies that for every $V \subseteq Y$ we have:

$$v \in [T_V^Y] \iff f^- V \in \mathcal{G}.$$

Choosing $V := f[X]$, we have $f^- f[X] = X \in \mathcal{G}$, so we obtain $v \in [T_{f[X]}^Y]$. Hence, there exists $w \in T(f[X])$ with $T_{f[X]}^Y(w) = v$. Since $f = \subseteq_{f[X]}^Y \circ f'$ with f' bijective, we have that $Tf = T_{f[X]}^Y \circ Tf'$ with Tf' bijective. This yields an element $u \in TX$ with $(Tf')(u) = w$, i.e. $(Tf)(u) = v$. The condition $\mu_X(u) = \mathcal{G}$ follows from the commutativity of the diagram together with the fact that $\mathbb{F}f$ is mono. □

Note that we did not claim that μ should be natural. In fact, we shall soon describe when this is the case. In the meantime, we can characterize μ amongst all sub-cartesian transformations:

Theorem 6.3. *μ is the largest sub-cartesian transformation from T to $\mathbb{F}$.*

Proof. Suppose $\nu : T \to \mathbb{F}$ is any sub-cartesian transformation. We need to prove $\nu_X(q) \subseteq \mu_X(q)$ for every set X and every $q \in TX$.

Put $\mathcal{G} := \nu_X(q)$ and assume $U \in \mathcal{G}$. For $\mathcal{G}_U := \{U \cap G \mid G \in \mathcal{G}\}$ we have $\mathcal{G} = \uparrow \mathcal{G}_U = (\mathbb{F}_U^X)\mathcal{G}_U$. We obtain the following situation:

$$\begin{tikzcd}
q & \in & TX \arrow[r, "\nu_X"] & \mathbb{F}X & \ni & \mathcal{G} & \ni & U \\
 & & TU \arrow[u, hook, "T_U^X"'] \arrow[r, "\nu_U"] & \mathbb{F}U \arrow[u, hook, "\mathbb{F}_U^X"'] & \ni & \mathcal{G}_U \arrow[u, mapsto] & &
\end{tikzcd}$$

Since ν is sub-cartesian, there exists some $w \in TU$ with $(T_U^X)(w) = q$. Hence $q \in (T_U^X)[TU]$ which means $U \in \mu_X(q)$. $\square$

We can now formulate our first characterization theorem:

Theorem 6.4. *For a set functor $T : \mathcal{S}et \to \mathcal{S}et$ the following are equivalent:*

(i) *T (weakly) preserves preimages*
(ii) *$\mu : T \to \mathbb{F}$ is a natural transformation*
(iii) *There exists a natural transformation $\nu : T \to \mathbb{F}$ which is sub-cartesian.*

Proof. (i) $\Longrightarrow$ (ii): Let T weakly preserve preimages and let $f : X \to Y$ be any map. We need to show that the following diagram commutes:

$$\begin{tikzcd}
TY \arrow[r, "\mu_Y"] & \mathbb{F}Y \\
TX \arrow[u, "Tf"] \arrow[r, "\mu_X"'] & \mathbb{F}X \arrow[u, "\mathbb{F}f"']
\end{tikzcd}$$

Given any $u \in TX$ we calculate

$$\begin{aligned}
(\mathbb{F}f)(\mu_X(u)) &= \{V \subseteq Y \mid u \in [T_{f^-V}^X]\}, \\
\mu_Y((Tf)(u)) &= \{V \subseteq Y \mid (Tf)(u) \in [T_V^Y]\}.
\end{aligned}$$

The inclusion "$\subseteq$" between the above sets always holds, for given $u \in [T_{f^-V}^X]$, there is some $w \in T(f^-V)$ with $T_{f^-V}^X(w) = u$. Applying Tf, we find

$$\begin{aligned}
(Tf)(u) &= (Tf)(T_{f^-V}^X(w)) \\
&= T_V^Y((Tf_|)(w))
\end{aligned}$$

due to the commutativity of the following diagram which arises from applying T to the diagram describing the preimage of V under f:

$$\begin{tikzcd}
TX \arrow[r, "Tf"] & TY \\
T(f^-V) \arrow[u, hook, "T_{f^-V}^X"'] \arrow[r, "Tf_|"'] & TV \arrow[u, hook, "T_V^Y"']
\end{tikzcd}$$

Hence $(Tf)(u) \in [T_V^Y]$.

For the other inclusion "$\supseteq$", we need to assume that T preserves preimages, which is to say that the above square is in fact a (weak) pullback. Given some $V \in \mu_Y((Tf)(u))$, i.e. $(Tf)(u) \in [T_V^Y]$, we have $v \in TV$ with $(Tf)(u) = T_V^Y(v)$. By the weak pullback property, there exists an element $w \in T(f^- V)$ with $T^X_{f^- V}(w) = u$, hence $f^- V \in \mu_X(u)$, i.e. $V \in (\mathbb{F}f)(\mu_X(u))$.

(ii) $\Longrightarrow$ (iii) follows from lemma 6.2.

For (iii) $\Longrightarrow$ (i), we recall from [Gum01] that the filter functor weakly preserves pullbacks, in particular, preimages. Thus we are in a position to apply theorem 3.2 to obtain the desired result. $\square$

6.1. **Examples.** It is instructive to calculate $\mu : T \to \mathbb{F}$ for some familiar functors on $\mathcal{S}et$. In the following table, the first column lists functors $T : \mathcal{S}et \to \mathcal{S}et$ and the second column gives, for an arbitrary $q \in TX$ the value of $\mu^T_X(q) \subseteq X$.

Functor T	$\mu_X(q)$
Id	$\uparrow \{q\}$
$\mathbb{P}(-)$	$\uparrow q$
$\mathbb{P}\mathbb{P}(-)$	$\uparrow \bigcup q$
$\mathbb{F}(-)$	q
$A \times -$	$\uparrow \{\pi_2(q)\}$
$(-)^I$	$\uparrow q[I]$
$\bar{\mathbb{P}}\bar{\mathbb{P}}(-)$	$\{U \subseteq X \mid \forall\ V \in q.\forall\ W \subseteq X.V \cap U = W \cap U \implies W \in q\}$

The last mentioned functor is the composition of the contravariant powerset functor $\bar{\mathbb{P}}$ with itself. Exemplary, we calculate $\mu_X(q)$ for this case:

On objects, $\bar{\mathbb{P}}(X) = \mathbb{P}(X)$, but for a map $f : X \to Y$, one has $\bar{\mathbb{P}}(f) = f^- : \mathbb{P}(Y) \to \mathbb{P}(X)$, hence $\bar{\mathbb{P}}(\subseteq^Y_X)(V) = X \cap V$ for $V \subseteq Y$. Next, for $\mathcal{S} \in \bar{\mathbb{P}}\bar{\mathbb{P}}(U)$ one gets $(\bar{\mathbb{P}}\bar{\mathbb{P}} \subseteq^X_U)(\mathcal{S}) = \{V \subseteq X \mid V \cap U \in \mathcal{S}\}$, hence

$$U \in \mu_X(q) \iff \exists\ \mathcal{S} \in \bar{\mathbb{P}}\bar{\mathbb{P}}(U).\ q = \{V \subseteq X \mid V \cap U \in \mathcal{S}\}.$$

The following formulas allow us to construct μ for functors that are combinations of simpler ones. We use the name of the functor T as an upper index for μ, since we have to deal with several functors at the same time, each one having its own sub-cartesian $\mu^T : T \to \mathbb{F}$. So, given T, T_1, and T_2 with associated sub-cartesian transformations μ^T, μ^{T_1}, and μ^{T_2} to $\mathbb{F}$, we get the following transformation for their sums, products and powers:

Functor	$\mu^T_X(q)$
$T_1 + T_2$	$\texttt{if}\ q \in T_i(X)\ \texttt{then}\ \mu^{T_i}_X(q)$
$T_1 \times T_2$	$\mu^{T_1}_X(\pi_1(q)) \cap \mu^{T_2}_X(\pi_2(q))$
T^I	$\{U \subseteq X \mid q[I] \subseteq [T^X_U]\}$

A system with state set X that takes an input from a set I and either produces an error $e \in E$ or moves to a new state, while producing an output $o \in O$, can be modeled by a coalgebra of type $T(X) = (E + O \times X)^I$. The above table tells us how to calculate μ^T. Using the fact that $[F^X_U] = FU$ for the standard functor $F(-) = E + O \times (-)$, we obtain: $\mu^T_X(q) = \uparrow q[I]$.

7. A sub-natural transformation to the powerset functor

The covariant powerset functor $\mathbb{P}$ which associates with a map $f : X \to Y$ the map $\mathbb{P}f : \mathbb{P}X \to \mathbb{P}Y$ with $(\mathbb{P}f)(U) := f[U]$ is obviously a subfunctor of the filter functor $\mathbb{F}$. The natural embedding is given by $\epsilon_X(U) := \uparrow \{U\}$ for any $U \in \mathbb{P}(X)$. When does the transformation $\mu : T \to \mathbb{F}$ factor through this embedding?

$$\begin{tikzcd}
\mathbb{F} \arrow[rr, shift left, "\bigcap"] & & \mathbb{P} \arrow[ll, shift left, "\epsilon"] \\
& T \arrow[ul, "\mu"] \arrow[ur, "\tau"'] &
\end{tikzcd}$$

From $\bigcap \circ \epsilon = id_{\mathbb{P}}$, we obtain immediately:

Lemma 7.1. *The only transformation $\tau : T \to \mathbb{P}$ with $\epsilon \circ \tau = \mu$ is given by $\tau := \bigcap \circ \mu$.*

In other words, for an arbitrary functor $T : \mathcal{S}et \to \mathcal{S}et$ the following definition yields a transformation, which is sub-natural due to lemmas 5.3 and 6.2:

Definition 7.2. *For an arbitrary set X, put*

$$\tau_X(u) := \bigcap \mu_X(u).$$

The just defined transformation τ is special amongst all sub-natural transformations $T \to \mathbb{P}$, for in analogy with theorem 6.3, we obtain:

Theorem 7.3. *τ is the largest sub-natural transformation $T \to \mathbb{P}$.*

Proof. For any sub-natural $\nu : T \to \mathbb{P}$ and for any $q \in TX$, we need to show

$$\nu_X(q) \subseteq \bigcap \{U \subseteq X \mid q \in [T_U^X]\}.$$

Thus, for any $U \subseteq X$ with $q \in [T_U^X]$, we need to show $\nu_X(q) \subseteq U$.

By assumption, there is some $w \in TU$ with $q = (T_U^X)(w)$. This implies:

$$\begin{array}{rcl}
\nu_X(q) & = & (\nu_X \circ T_U^X)(w) \\
& = & (\mathbb{P}_U^X \circ \nu_U)(w) \\
& = & \nu_U(w) \\
& \in & \mathbb{P}(U).
\end{array}$$

□

Further properties of τ will require conditions on the functor T. In particular, we are interested in preservation of intersections. Our aim is to characterize preservation of intersection by the existence of transformations to the powerset functor, in analogy to theorem 6.4.

Theorem 7.4. *For a functor $T : \mathcal{S}et \to \mathcal{S}et$ the following are equivalent:*

(i) *T (weakly) preserves intersections.*
(ii) $\mu = \varepsilon \circ \tau$
(iii) *$\tau : T \to \mathbb{P}$ is sub-cartesian.*
(iv) *There exists a sub-cartesian transformation $\nu : T \to \mathbb{P}$*

Proof. (i) $\Longrightarrow$ (ii): By corollary 4.8, for every $u \in TX$, there is a smallest $U \subseteq X$ with $u \in [T_U^X]$. It follows that $U = \bigcap\{V \subseteq X \mid a \in [T_V^X]\}$ and $\mu_X(u) = \uparrow U = \varepsilon \circ \tau(u)$.

(ii) $\Longrightarrow$ (iii): Since we know that $\mu = \varepsilon \circ \tau$ is sub-cartesian, the outer square of the following figure is a weak pullback. As ε_X is mono, one easily checks that the left square is a weak pullback, too, hence τ is sub-cartesian.

$$\begin{tikzcd}
TY \arrow[r, "\tau_Y"] & \mathbb{P}Y \arrow[r, tail, "\varepsilon_Y"] & \mathbb{F}Y \\
TX \arrow[u, hook, "Tf"] \arrow[r, "\tau_X"] & \mathbb{P}X \arrow[u, hook, "\mathbb{P}f"] \arrow[r, tail, "\varepsilon_X"] & \mathbb{F}X \arrow[u, hook, "\mathbb{F}f"]
\end{tikzcd}$$

(iii) $\Longrightarrow$ (i): This is part (i) of theorem 3.2, since $\mathbb{P}$ obviously preserves arbitrary intersections. $\square$

8. A natural transformation to $\mathbb{P}$

We now would like to characterize when τ is a natural transformation. This property is brought about jointly by the preservation of intersections and of preimages.

Theorem 8.1. *For any functor $T : \mathcal{S}et \to \mathcal{S}et$ the following are equivalent:*

(i) *T preserves preimages and infinite intersections.*
(ii) *$\tau : T \to \mathbb{P}$ is natural and sub-cartesian.*
(iii) *There exists a natural transformation $\nu : T \to \mathbb{P}$ which is sub-cartesian.*

Proof. (i) $\Longrightarrow$ (ii): By theorem 7.4, τ is sub-cartesian and $\mu = \varepsilon \circ \tau$, and by theorem 6.4, μ is a natural transformation. So in the naturality diagram for an arbitrary map $f : X \to Y$,

$$\begin{tikzcd}
TY \arrow[r, "\tau_X"] & \mathbb{P}Y \arrow[r, tail, "\varepsilon_Y"] & \mathbb{F}Y \\
TX \arrow[u, "Tf"] \arrow[r, "\tau_Y"] & \mathbb{P}Y \arrow[u, "\mathbb{P}f"] \arrow[r, tail, "\varepsilon_X"] & \mathbb{F}X \arrow[u, "\mathbb{F}f"]
\end{tikzcd}$$

we have the outer and the right square commuting. Since ε_Y is mono, the left square also commutes.

(iii) $\Longrightarrow$ (i): Since $\mathbb{P}$ preserves preimages and intersections, this is once more a consequence of theorem 3.2. $\square$

9. Modal Operators on coalgebras.

Coalgebras of the filter functor $\mathbb{F}$ have been described in [Gum01]. Given an $\mathbb{F}$-coalgebra $\mathcal{A} = (A, \alpha)$, i.e. a (structure)map $\alpha : A \to \mathbb{F}A$, one defines a relation $\to$ between A and $\mathbb{P}A$ by

$$a \to U :\Longleftrightarrow U \in \alpha(a).$$

Then one has

(i) $a \to U$ and $a \to V \implies a \to U \cap V$,
(ii) $a \to U \subseteq V \implies a \to V$.

and conversely, a relation $\rightarrow$ between A and $\mathbb{P}A$ satisfying (i) and (ii) arises from a filter coalgebra on A.

Our sub-cartesian transformation μ can be used to associate to a coalgebra $\mathcal{A} = (A, \alpha)$ of arbitrary type T a filter coalgebra $\mathcal{A}_{\mathbb{F}} = (A, \alpha_{\mathbb{F}})$ on the same base set. The fact that μ is sub-cartesian has as consequence that the subcoalgebra structure is preserved and reflected:

Theorem 9.1. *To every coalgebra $\mathcal{A}_T = (A, \alpha)$, one can construct a filter-coalgebra $\mathcal{A}_{\mathbb{F}}$ on the same underlying set, so that $\mathcal{A}_T$ and $\mathcal{A}_{\mathbb{F}}$ have the same subcoalgebras.*

Proof. Define $\mathcal{A}_{\mathbb{F}} = (A, \mu \circ \alpha)$. From a T-coalgebra structure $\alpha : A \to TA$, we obtain the $\mathbb{F}$-coalgebra structure $\alpha_{\mathbb{F}} := \mu_A \circ \ \alpha$. Since μ is sub-natural, every subcoalgebra of $\mathcal{A}$ becomes a subcoalgebra of $\mathcal{A}_{\mathbb{F}}$, to. Since μ is sub-cartesian, every subcoalgebra of U of $\mathcal{A}_{\mathbb{F}}$ arises from a subcoalgebra of $\mathcal{A}$ on the same set. The required T-structure map on U arises as the mediating map for the weak pullback square in in the following figure:

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[rr, bend left, "\alpha_{\mathbb{F}}^A"] & TA \arrow[r, "\mu_A"] & \mathbb{F}A \\
U \arrow[u, hook, "\subseteq"'] \arrow[r, dashed] \arrow[rr, bend right, "\alpha_{\mathbb{F}}^U"'] & TU \arrow[u, hook, "T_U^A"'] \arrow[r, "\mu_U"'] & \mathbb{F}U \arrow[u, hook, "\mathbb{F}_U^A"']
\end{tikzcd}$$

$\square$

It would be nice, if in the previous theorem, we could replace the filter functor by the powerset functor. However, coalgebras of the powerset functors, i.e. Kripke-structures, have the property that the system of subcoalgebras is closed under arbitrary intersection. Therefore, we can hope for a similar theorem only when T preserves infinite intersections.

Nevertheless, we can define an abstract next-time-operator $\bigcirc$, which recovers the vital properties of the next-time operator on Kripke-Structures. Indeed, if T preserves intersections, $\bigcirc P$ is the set of all states $s \in S$ whose immediate successors are all in P. The following lemmas, in particular 9.5, allow one to define $\bigcirc P$ for coalgebras in arbitrary categories. We begin with the following concrete definition:

Definition 9.2. *Let $\mathcal{A} = (A, \alpha)$ be a T-coalgebra on A. For any subset $P \subseteq A$ let*

$$\bigcirc_{\mathcal{A}} P := \{a \in A \mid \alpha(a) \in [T_P^A]\}.$$

We shall drop the subscript $_{\mathcal{A}}$, whenever the coalgebra structure is clear from context. When $\mathcal{A}$ is a Kripke structure, i.e. a $\mathbb{P}$-coalgebra, then $\bigcirc P$ is just the set of all states $a \in A$ such that all successors are in P. Guided by this intuition, we check the following easy properties:

Lemma 9.3. $\bigcirc : \mathbb{P}A \to \mathbb{P}A$ *is monotone.*

The algebraic relevance of the next-time operator is given by the following observation:

Lemma 9.4. *$U \subseteq A$ is a subcoalgebra iff $\bigcirc U \overset{\supseteq}{\cancel{\subseteq}} U$.*

In fact, this will follow from the following categorical characterization of $\bigcirc P$:

Lemma 9.5. *$\bigcirc P$ is the preimage of $TP \subseteq TA$ with respect to $\alpha : A \to TA$.*

Proof. By definition, $\alpha[\bigcirc P] \subseteq [T_P^A]$, and since T_P^A is injective, there is (precisely) one map $\varphi : P \to TP$ making the square in the following figure commutative.

$$\begin{tikzcd}
\bigcirc P \arrow[r, hook, "\subseteq"] \arrow[d, dashed, "\varphi"'] & A \arrow[d, "\alpha"] \\
TP \arrow[r, hook, "T_P^A"'] & TA
\end{tikzcd}$$

The square is in fact a preimage square, for given $a \in A$ and $q \in TP$ with $\alpha(a) = T_P^A(q)$, then $a \in \bigcirc P$ by definition and $(T_P^A \circ \varphi)(a) = (\alpha \circ \subseteq)(a) = \alpha(a) = T_P^A(q)$. Since T_P^A is injective, this means that $\varphi(a) = q$. $\square$

The fact that any coalgebra $\mathcal{A} = (A, \alpha)$ determines a monotone operator $\bigcirc_{\mathcal{A}}$ on $\mathbb{P}A$, allows us define new operators $\Box_{\mathcal{A}}$ and $\Diamond_{\mathcal{A}}$ as largest fixed points:

$$\Box_{\mathcal{A}} S := \nu X.S \cap \bigcirc_{\mathcal{A}} X,$$

and as smallest fixed point:

$$\Diamond_{\mathcal{A}} S := \mu X.S \cup \bigcirc_{\mathcal{A}} X.$$

As usual, we drop the subscript ${}_{\mathcal{A}}$, when there is no confusion.

Lemma 9.6. *Given a coalgebra $\mathcal{A} = (A, \alpha)$ and any subset $S \subseteq A$, then $\Box S$ is the largest subcoalgebra of $\mathcal{A}$ that is contained in S.*

Proof. We have $\Box S = S \cap \bigcirc \Box S$, so $\Box S \subseteq S$ and $\Box S$ is a subcoalgebra of $\mathcal{A}$ by lemma 9.4. According to Tarski's description of the largest fixed point,

$$\Box S = \bigcup \{X \subseteq A \mid X \subseteq S \cap \bigcirc X\},$$

and using lemma 9.4 again, $\Box S$ is the union of all subcoalgebras of S that are contained in S. $\square$

Lemma 9.7. *Let $\mathcal{A} = (A, \alpha)$ and $\mathcal{B} = (B, \beta)$ be coalgebras, $\varphi : \mathcal{A} \to \mathcal{B}$ a homomorphism. Then for any subset $Q \subseteq B$ we have:*

$$\bigcirc \varphi^- Q \subseteq \varphi^- \bigcirc Q.$$

Proof. In the following diagram, the bottom face arises from the application of functor T to a preimage square. The front face commutes, since φ is a homomorphism, and the left and right faces are in fact pullbacks, due to lemma 9.5.

$$\begin{tikzcd}
 & \bigcirc \varphi^- Q \arrow[rr, dotted] \arrow[dl, hook] \arrow[dd] & & \bigcirc Q \arrow[dl, hook] \arrow[dd] \\
A \arrow[rr, "\varphi"] \arrow[dd, "\alpha"] & & B \arrow[dd, "\beta"] & \\
 & T\varphi^- Q \arrow[rr] \arrow[dl, hook] & & TQ \arrow[dl, hook] \\
TA \arrow[rr, "T\varphi"] & & TB &
\end{tikzcd}$$

$\bigcirc \varphi^- Q$ with the obvious maps becomes a competitor to the pullback $\bigcirc Q$. This yields the dotted map, making the top face commutative. In particular, $\varphi[\bigcirc \varphi^- Q] \subseteq \bigcirc Q$, hence $\bigcirc \varphi^- Q \subseteq \varphi^- \bigcirc Q$. $\square$

Applying this lemma to a subset $Q = \varphi P$, we obtain the corollary:

Corollary 9.8. *For any homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ and any $P \subseteq A$ we have:*

$$\varphi[\bigcirc P] \subseteq \bigcirc \varphi[P].$$

Theorem 9.9. *T weakly preserves preimages if and only if for all homomorphisms $\varphi : \mathcal{A} \to \mathcal{B}$ we have:*

$$\varphi^- \bigcirc Q = \bigcirc \varphi^- Q.$$

Proof. We can reuse the figure from the previous proof. If T preserves preimages, then the bottom face is a weak pullback. We will show that the top face is a pullback, i.e. a preimage, too.

Given some $u \in \varphi^- \bigcirc Q$, i.e. $\varphi(u) \in \bigcirc Q$. Then

$$(T\varphi)(\alpha(u)) = \beta\varphi(u) \in [T_Q^B].$$

Thus there is a $w \in TQ$ with $T_Q^B(w) = (T\varphi)(\alpha(u))$. Thus the pullback $T\varphi^- Q$ contains an element $v \in T\varphi^- Q$ with $T^A_{\varphi^- Q}(v) = \alpha(u)$, i.e. $u \in \bigcirc \varphi^- Q$.

For the "only-if"-direction, it is enough, according to [GS], to show that for any subcoalgebra $V \leq \mathcal{B}$ we have that $\varphi^- V \leq \mathcal{A}$.

Given $V \leq \mathcal{B}$, we have $V \subseteq \bigcirc V$, i.e. $\varphi^- V \subseteq \varphi^- \bigcirc V$. The hypothesis therefore yields $\varphi^- V \subseteq \bigcirc \varphi^- V$, meaning that $\varphi^- V$ is a subcoalgebra of $\mathcal{A}$. $\square$

Theorem 9.10. *T preserves arbitrary intersections if and only if for every coalgebra $\mathcal{A}$ and each family $P_i \subseteq A$, $i \in I$, we have*

$$\bigcirc \bigcap_{i \in I} P_i = \bigcap_{i \in I} \bigcirc P_i.$$

Proof. Let T preserve intersections. In the following diagram the right squares are pullbacks by lemma 9.5 and the bottom row is a weak intersection. This yields a map $f : \bigcap_{i \in I} \bigcirc P_i \to T \bigcap_{i \in I} P_i$, completing the outer square.

$$\begin{tikzcd}
\bigcap_{i\in I} \bigcirc P_i \arrow[r, hook] \arrow[d, dotted, "f"'] & \bigcirc P_j \arrow[r, hook] \arrow[d, "f_j"'] & A \arrow[d, "\alpha"] \\
T \bigcap_{i\in I} P_i \arrow[r, hook] & TP_j \arrow[r, hook] & TA
\end{tikzcd}$$

If we can show that this square is indeed a weak pullback square, then the result follows again from lemma 9.5.

Any competitor Q for $\bigcap_{i \in I} P_i$ becomes a competitor to each of the $\bigcirc P_j$. The arising maps $\delta_j : Q \to \bigcirc P_j$ make Q into a copmpetitor to te intersection $\bigcap_{i \in I} \bigcirc P_i$. This yields the mediating map $\varepsilon : Q \to \bigcap_{i \in I} \bigcirc P_i$.

For the other direction, assume $\bigcap_{i \in I} \bigcirc_\alpha P_i \subseteq \bigcirc_\alpha \bigcap_{i \in I} P_i$ for every coalgebra $\mathcal{A} = (A, \alpha)$. This means

$$(\forall i \in I.\ \alpha(x) \in [T^A_{P_i}]) \implies \alpha(x) \in [T^A_{\bigcap_{i \in I} P_i}].$$

Given any element $u \in \bigcap [T^A_{P_i}]$, we consider the coalgebra on A with constant structure map $\alpha(a) = u$, in order to conclude that $u \in [T^A_{\bigcap_{i \in I} P_i}]$. Consequently,

$$\bigcap [T^A_{P_i}] \subseteq [T^A_{\bigcap_{i \in I} P_i}].$$

Lemma 4.1 provides the reverse inclusion, so by 4.7, T preserves intersections. $\square$

Now we can formulate the main theorem of this section:

Theorem 9.11. *For every T-coalgebra $\mathcal{A} = (A, \alpha)$ there is a Kripke structure $\mathcal{K} = (A, R)$ on the same set A with $\bigcirc_{\mathcal{A}} = \bigcirc_{\mathcal{K}}$ if and only if T weakly preserves intersections.*

The proof follows from the following lemma in combination with theorem 9.10.

Lemma 9.12. *Let $\mathcal{A} = (A, \alpha)$ be a coalgebra. There exists a Kripke-Structure $\mathcal{K} = (A, \to)$ on the base set A so that $\bigcirc_{\mathcal{A}} = \bigcirc_{K}$ if and only if*

$$\bigcap_{i \in I} \bigcirc_{\mathcal{A}} P_i = \bigcirc_{\mathcal{A}} \bigcap_{i \in I} P_i =$$

for every family of subsets $P_i \subseteq A$, $i \in I$.

Proof. As Kripke-structures satisfy the displayed formula, the necessity is clear. Conversely, we assume the validity of the formula and define a Kripke structure $\mathcal{K} = (A, \to)$ by

$$a \to b \iff b \in \bigcap \{P \subseteq A \mid a \in \bigcirc_{\mathcal{A}} P\}$$

Given $a \in \bigcirc_{A} Q$, then $a \to b$ implies that $b \in Q$ for every b, hence $a \in \bigcirc_{\mathcal{K}} Q$. For the other direction, compute:

$$\begin{aligned}
a \in \bigcirc_{\mathcal{K}} Q \quad &\iff \quad \forall b \in A.\ a \to b \implies b \in Q \\
&\iff \quad \bigcap \{P \subseteq A \mid a \in \bigcirc_{\mathcal{A}} P\} \subseteq Q \\
&\implies \quad \bigcirc_{\mathcal{A}} \bigcap \{P \subseteq A \mid a \in \bigcirc_{\mathcal{A}} P\} \subseteq \bigcirc_{\mathcal{A}} Q \\
&\implies \quad \bigcap \{\bigcirc_{\mathcal{A}} P \subseteq A \mid a \in \bigcirc_{\mathcal{A}} P\} \subseteq \bigcirc_{\mathcal{A}} Q \\
&\implies \quad a \in \bigcirc_{\mathcal{A}} Q
\end{aligned}$$

10. Discussion and further work

Starting with Rutten's seminal work [Rut00], weak preservation of pullbacks, resp. of intersections by the coalgebraic type functor has played an important role in the universal theory of coalgebra. Weak pullback preservation splits into weak preservation of kernels and (weak) preservation preimages [GS].

Here we have seen that (weak) preimage preservation yields a natural and sub-cartesian transformation of T to the filter functor $\mathbb{F}$, thus establishing an intimate relationship between T-coalgebras and filter coalgebras. Similarly, preservation of intersections makes for a sub-cartesian transformation from T to the powerset functor $\mathbb{P}$, thus relating every T-coalgebra with a Kripke-Structure.

In the work of E.G. Manes [Man98] we can find the definition of sub-natural, resp. sub-cartesian, transformations under the names "mono-transformation", resp. "taut transformation". Manes proves that finitary collection monads are precisely the taut quotients of polynomial functors.

For coalgebras of polynomial type, B. Jacobs had previously introduced a nexttime operator in [Jac02]. Since polynomial functors preserve weak pullbacks and intersections, they are rather special. Our definition of the nexttime operator does not rely on any assumption regarding the type functor, in fact, lemma 9.5 shows how a nexttime operator could even be defined for coalgebras over base categories other than $\mathcal{S}et$.

We have seen that preservation of preimages, resp. of intersections, are reflected in very natural preservation properties of the $\bigcirc$ operator. It can now serve as a starting point for a CTL-like logic, for arbitray coalgebras. As an example, coalgebras of the 3-2-functor $F(X) = \{(x_1, x_2, x_3) \mid card(\{x_1, x_2, x_3\}) \leq 2\}$ (see [AM89]) are characterized in this logic by the following formulae:

(1) $\bigcirc$ true,
(2) $\bigcirc(\varphi \vee \psi \vee \theta) \implies \bigcirc(\phi \vee \psi) \vee \bigcirc(\phi \vee \theta) \vee \bigcirc(\psi \vee \theta)$.

□

References

[AHS90] J. Adámek, H. Herrlich, and G.E. Strecker, *Abstract and concrete categories*, John Wiley & Sons, 1990.

[AM89] P. Aczel and N. Mendler, *A final coalgebra theorem*, Proceedings category theory and computer science (D.H. Pitt et al., eds.), Lecture Notes in Computer Science, Springer, 1989, pp. 357–365.

[Bar93] M. Barr, *Terminal coalgebras in well-founded set theory*, Theoretical Computer Science (1993), no. 144(2), 299–315.

[GS] H.P. Gumm and T. Schröder, *Types and coalgebraic structure*, Algebra Universalis, to appear.

[GS02] ———, *Coalgebras of bounded type*, Math. Struct. in Comp. Science (2002), no. 12, 565–578.

[Gum01] H.P. Gumm, *Functors for coalgebras*, Algebra Universalis (2001), no. 45 (2-3), 135–147.

[Jac02] B. Jacobs, *The temporal logic of coalgebras via Galois algebras*, Math. Struct. in Comp. Science (2002), no. 12, 875–903.

[Man98] E.G. Manes, *Implementing collection classes with monads*, Math. Struct. in Comp. Science **8** (1998), 231–276.

[Rut00] J.J.M.M. Rutten, *Universal coalgebra: a theory of systems*, Theoretical Computer Science (2000), no. 249, 3–80.

[Trn69] V. Trnková, *Some properties of set functors*, Comm. Math. Univ. Carolinae (1969), no. 10,2, 323–352.

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