Copower functors

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Abstract

We give a common generalization of two earlier constructions in [2], that yielded coalgebraic type functors for weighted, resp. fuzzy transition systems. Transition labels for these systems were drawn from a commutative monoid $\mathcal{M}$ or a complete semilattice $\mathcal{L}$, with the transition structure interacting with the algebraic structure on the labels. Here, we show that those earlier signature functors are in fact instances of a more general construction, provided by the so-called *copower functor*.

Exemplarily, we instantiate this functor in categories given by varieties $\mathfrak{V}$ of algebras. In particular, for the variety $\mathfrak{S}$ of all semigroups, or the variety $\mathfrak{M}$ of all (not necessarily commutative) monoids, and with $\mathcal{M}$ any monoid, we find that the resulting copower functors $\mathcal{M}_{\mathfrak{S}}[-]$ (resp $\mathcal{M}_{\mathfrak{M}}[-]$) weakly preserve pullbacks if and only if $\mathcal{M}$ is *equidivisible* (resp. *conical* and *equidivisible*).

Finally, we show that copower functors are universal in the sense that every $\mathcal{S}et$-functor can be seen as an instance of an appropreiate copower functor.

1 Introduction

Labeled transition systems and their many variations constitute fundamental examples of $\mathcal{S}et$-coalgebras and most notions and intuitions in abstract coalgebra are based on such examples. When the elements in the label set K are just considered to be names (of processes, actions, inputs), then the presence or absence of labels does not have much of an impact on the coalgebraic theory. A K-labeled transition system on a set A is just a family of Kripke structures $\mathcal{A}_k = (A, R_k)$, i.e. unlabeled transition systems with transition relations $R_k = \{(a, a') \in A^2 \mid a \xrightarrow{k} a'\}$, one for each $k \in K$. Definitions and theorems for Kripke structures will just have to be quantified over all labels.

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As an example, a map between K-labeled transition systems $\mathcal{A}$ and $\mathcal{B}$ is a homomorphism if and only if the following two conditions are satisfied for each $k \in K$ and $a, a' \in A$, $b, b' \in B$:

$$a \xrightarrow{k} a' \implies \varphi(a) \xrightarrow{k} \varphi(a') \tag{1}$$

$$\varphi(a) \xrightarrow{k} b' \implies \exists a' \in A. a \xrightarrow{k} a' \wedge \varphi(a') = b'. \tag{2}$$

When K carries some algebraic structure, more interesting behaviour can be modelled. If, for instance, $\mathcal{M} = (M, +, 0)$ is a commutative monoid, one can define a signature functor $\mathcal{M}_\omega^{(-)}$, so that coalgebras are image-finite transition structures with labels from $\mathcal{M}$, where homomorphisms obey the addition structure of $\mathcal{M}$, in the sense that a map φ between $\mathcal{M}_\omega$-coalgebras $\mathcal{A}$ and $\mathcal{B}$ is a coalgebra homomorphism if and only if

$$\varphi(a) \xrightarrow{m} b \iff m = \sum\{m' \mid a \xrightarrow{m'} b' \in \varphi^{-1}[b]\} \tag{3}$$

where $\sum$ denotes summation in $\mathcal{M}$, and $a \in A$, $b \in B$, see [2]. When the elements of $\mathcal{M}$ are interpreted as weights or strengths, then this means that the strength of a transition in the image is obtained by adding the strengths of corresponding transitions in the preimage. Observe that for the above formula to make sense, we need not only that the summation involves at most finitely many nonzero summands, but also that $\mathcal{M}$ is associative as well as commutative.

Choosing for $\mathcal{M}$ the additive monoid $\mathbb{R}^+$, the finite distribution functor $\mathcal{D}_\omega$ becomes a subfunctor of $\mathcal{M}_\omega$ and the above formula describes homomorphisms between Markov-chains, which are just coalgebras of signature $\mathcal{D}_\omega$, see, e.g. [7].

A different construction starts with the label set forming a complete semilattice $\mathcal{L} = (L, \bigvee)$. In that case we can define a signature functor $\mathcal{L}^{(-)}$ whose coalgebras are (not necessarily image-finite) $\mathcal{L}$-labeled systems where the homomorphisms condition turns out to be

$$\varphi(a) \xrightarrow{l} b \iff l = \bigvee\{l' \mid a \xrightarrow{l'} b' \in \varphi^{-1}[b]\}. \tag{4}$$

Despite its formal similarity with 3, observe, that the absence of image-finiteness now requires $\bigvee$ to be associative, commutative, and idempotent as well. Choosing $\mathcal{L} = \mathbf{2} = \{0, 1\}$ with the natural order, we can interpret $\mathcal{L}$-coalgebras as standard Kripke structures where the successors of a state s are all those s' with $s \rightarrow s'$. On the other hand, choosing for $\mathcal{L}$ the closed interval $[0, 1] \subseteq \mathbb{R}$ with its induced order, $\mathcal{L}$-coalgebras are just fuzzy transition systems.

Despite the obvious formal similarities in the above homomorphism formulae, it has not been clear, how to unify these different notions of labelled transition systems, in particular, since for monoid labelled transition systems image-finiteness seems to be an essential restriction, whereas semilattice-labelled transition systems seem to rely on idempotency in an essential manner. These observations seemed to stand in the way of a common generalization which at the same time ought to cover standard labelled transition systems, where no algebraic structure is available on the label set, even though we are able to equivalently rewrite the homomorphism conditions (1) and (2) in a manner reminiscent of the previous two formulas as

$$\varphi(a) \xrightarrow{k} b \iff k \in \{k' \mid a \xrightarrow{k'} b' \in \varphi^{-1}[b]\}. \tag{5}$$

In this note we shall show that indeed all mentioned types of labelled transition systems can be seen as instances of a single construction which yields a signature functor $\mathcal{A}_{\mathfrak{V}}[-]$ for an arbitrary variety of universal algebras $\mathfrak{V}$ and an arbitrary algebra $\mathcal{A} \in \mathfrak{V}$. Choosing for $\mathfrak{V}$ the class of all commutative monoids or the class of all complete semilattices, we obtain the above mentioned monoid-labeled, resp. semilattice-labeled systems. Specializing $\mathfrak{V}$ to the class of all algebras with empty signature (i.e. sets), we obtain the standard notion of labeled transition systems.

In order to demonstrate a novel application, we apply our construction to the case where the label set carries a semigroup or a not necessarily abelian monoid structure. We show that in both cases the signature functor preserves kernel pairs if and only if the semigroup (the monoid) is *equidivisible.* In the semigroup case, this is also equivalent to the functor preserving weak pullbacks. In the monoid case, it preserves weak pullbacks iff the monoid is *equidivisible* and *conical*, two important notions from semigroup theory, that will be discussed below.

2 Definitions and background

By a *signature*, we understand any endofunctor T on the category of sets. An *algebra* of signature (or *type*) T is simply any map $f : T(A) \to A$. Following tradition, we write an algebra as a pair $\mathcal{A} = (A, f^{\mathcal{A}})$, where A is a set, called the base set, and $f^{\mathcal{A}} : T(A) \to A$ is the so called *operation* of $\mathcal{A}$. An *algebra homomorphism* between algebras $\mathcal{A} = (A, f^{\mathcal{A}})$ and $\mathcal{B} = (B, f^{\mathcal{B}})$ is a map $\varphi : A \to B$ satisfying $\varphi \circ f^{\mathcal{A}} = f^{\mathcal{B}} \circ F(\varphi)$. Classical universal algebra is mostly concerned with the case where T is a polynomial functor $T(X) = \sum_{i \in I} X^{n_i}$. In this case $f : T(A) \to A$ is just a family of n_i-ary operations $(f_i : A^{n_i} \to A)_{i \in I}$.

Dually, a *coalgebra* of signature T is a map $\alpha : A \to T(A)$ and we also write it as a pair $\mathcal{A} = (A, \alpha_{\mathcal{A}})$ where A is called the *state set* and $\alpha_{\mathcal{A}} : A \to F(A)$

the *structure map*. A map between between coalgebras $\mathcal{A} = (A, \alpha_{\mathcal{A}})$ and $\mathcal{B} = (B, \alpha_{\mathcal{B}})$ is called a homomorphism from $\mathcal{A}$ to $\mathcal{B}$ if

$$T\varphi \circ \alpha_{\mathcal{A}} = \alpha_{\mathcal{B}} \circ \varphi. \tag{6}$$

A subset $U \subseteq A$ is called a *subcoalgebra*, if there esists a (necessarily unique) structure map $\alpha_U : U \to T(U)$ so that the inclusion map $\subseteq_U^A$is a homomorphism.

2.1 Deterministic systems

Fixing an arbitrary set K, deterministic labelled transition systems with labels from K are coalgebras for the functor sending a set X to the set $X \times K$. A coalgebra $\mathcal{A} = (A, \alpha_{\mathcal{A}})$ is given by a map $\alpha_{\mathcal{A}} : A \to A \times K$. We write $a \overset{k}{\twoheadrightarrow} a'$ iff $\alpha(a) = (a', k)$. The homomorphism condition translates immediately into

$$a \overset{k}{\twoheadrightarrow} a' \implies \varphi(a) \overset{k}{\twoheadrightarrow} \varphi(a') \text{ for each } k \in K.$$

2.2 Nondeterministic systems

Consider, for instance, the powerset functor $\mathbb{P}$, then a coalgebra of signature $\mathbb{P}$ is just a map $\alpha : A \to \mathbb{P}(A)$. Such a map is equivalently described by a binary relation $\twoheadrightarrow_A$ setting $a \twoheadrightarrow a' \;:\iff\; a' \in \alpha(a)$. (We shall drop superscripts to $\twoheadrightarrow$ and to α when they are clear from the context.) Let $\mathbb{P}_\omega(X)$ denote the set of all finite subsets of X, then $\mathbb{P}_\omega$ is a subfunctor of $\mathbb{P}$ and its coalgebras are just image-finite transition systems.

2.3 Nondeterministic labeled transition systems (LTS)

For a $\mathbb{P}(K \times -)$-coalgebra $\mathcal{A} = (A, \alpha_{\mathcal{A}})$ and $a, a' \in A$, we write $a \overset{k}{\twoheadrightarrow} a'$ iff $(k, a') \in \alpha(a)$. The homomorphism equation (6) translates into the previously mentioned conditions (1) and (2), which are together equivalent to (5). The subclass of image-finite systems is made up of the coalgebras for the subfunctor $\mathbb{P}_\omega(K \times -)$.

2.4 Commutative monoid labeled system

Given a commutative monoid $\mathcal{M} = (M, +, 0)$ and a set X, we let $\mathcal{M}_\omega^{(X)}$ be the monoid of all maps $\sigma : X \to M$ with finite support, i.e. for which $\sigma(x) = 0$ for all but finitely many $x \in X$. Then $\mathcal{M}_\omega^{(-)}$ becomes a functor, when we define it on maps $f : X \to Y$ as

$$\mathcal{M}_\omega^f(\sigma)(y) := \sum\{\sigma(x) \mid x \in X,\ f(x) = y\},$$

see [2]. An $\mathcal{M}_\omega^{(-)}$-coalgebra $\mathcal{A} = (A, \alpha_\mathcal{A})$ can be considered an $\mathcal{M}$-labeled transition systems by writing

$$a \xrightarrow{m} a' \iff \alpha(a)(a') = m.$$

In [2] coalgebra homomorphisms between $\mathcal{M}_\omega^{(-)}$-coalgebras $\mathcal{A} = (A, \alpha_\mathcal{A})$and $\mathcal{B} = (B, \alpha_\mathcal{B})$ were characterized as maps $\varphi : A \to B$ satisfying condition (3). Thus an $\mathcal{M}$-labeled system $\mathcal{A} = (A, \alpha_\mathcal{A})$ is not just a labeled transition system whose label set happens to carry a monoid structure. For one, we have a different notion of homomorphism, but we also observe that there is always exactly one transition between a and a' from A and its weight is given as $\alpha(a)(a')$. Nevertheless, we may choose to interpret a transition of weight 0 as nonexistent, so we do have a natural interpretation of single systems as image finite LTS.

2.5 Semilattice labeled systems

In order to get rid of the image-finiteness inherent in the previous example, we now assume that $\mathcal{L}$ is a complete semilattice $\mathcal{L} = (L, \bigvee)$. The corresponding covariant signature functor $\mathcal{L}^{(-)}$ associates a set X with the set $\mathcal{L}^X$ of all maps $\sigma : X \to L$ and a map $f : X \to Y$ with a map $\mathcal{L}^f : L^X \to L^Y$defined as

$$\mathcal{L}^f(\sigma)(y) = \bigvee\{\sigma(x) \mid f(x) = y\}.$$

In [2] it was shown that this defines a functor and that a map $\varphi : A \to B$ is a homomorphism between $\mathcal{L}^{(-)}$-coalgebras $\mathcal{A} = (A, \alpha_\mathcal{A})$ and $\mathcal{B} = (B, \alpha_\mathcal{B})$ if and only if the following two conditions are satisfied for all $a, a' \in A$, all $b \in B$ and all $l \in \mathcal{L}$:

$$a \xrightarrow{l} a' \implies \varphi(a) \xrightarrow{l'} \varphi(a') \text{ for some } l' \geq l \tag{7}$$

$$\varphi(a) \xrightarrow{l} b' \implies l \leq \bigvee\{l' \mid \exists a' \in A.a \xrightarrow{l'} a', \varphi(a') = b'\}, \tag{8}$$

where again we use the notation $a \xrightarrow{l} a' \ := \alpha(a)(a') = l$. Observing that $a \xrightarrow{l} a' \wedge a \xrightarrow{l'} a' \implies l = l'$, we check that:

Lemma 1 *Conditions (7) and (8) are together equivalent to condition (4).*

PROOF. Given the above two conditions, and assuming $\varphi(a) \xrightarrow{l} b'$ we need to show $l = \bigvee\{l' \mid a \xrightarrow{l'} a' \in \varphi^{-1}[b']\}$. Condition (8) takes care of one inequality. For each l' with $a \xrightarrow{l'} a' \in \varphi^{-1}[b']$ we obtain $\varphi(a) \xrightarrow{l''} b'$ for some $l'' \geq l'$using (7). Therefore, $l = l'' \geq l'$, and the supremum of all these l' is below l.

Given $l = \bigvee\{l' \mid a \xrightarrow{l'} a' \in \varphi^{-1}[b']\}$, we also need to show $\varphi(a) \xrightarrow{l} b'$. Now, if $\varphi^{-1}[b'] = \emptyset$ the assumption yields $l = 0$, which by (8) entails $\varphi(a) \xrightarrow{0} b'$. Otherwise, condition (7) implies that $\varphi(a) \xrightarrow{l''} b'$ for some $l'' \geq l$ and condition (8) guarantees that $l'' \leq l$.

For the converse, we need to derive conditions (7) and (8) from (4). Assuming $a \xrightarrow{l} a'$ and using $a' \in \varphi^{-1}[\varphi(a')]$, the right-to-left direction of (4) implies $\varphi(a) \xrightarrow{l'} \varphi(a')$ for some $l' \geq l$. Finally, (8) follows trivially from (4).

The functor $\mathcal{L}^{(-)}$ generalizes the covariant powerset functor in that choosing the two-element ordered set $\mathbf{2} = \{0,1\}$ with $0 \leq 1$, we obtain $\mathbf{2}^{(-)} \cong \mathbb{P}(-)$. It also generalizes the signature functor of nondeterministic labeled transition systems, as $(\mathbf{2}^K)^{(-)} \cong \mathbb{P}(K \times -)$. More interestingly we can choose for $\mathcal{L}$ the real unit-interval $[0,1] \subseteq \mathbb{R}$ and interpret $[0,1]$-coalgebras as fuzzy transition systems, with $a \xrightarrow{r} a'$ indicating a transition with certainty r. The homomorphism conditions have a very natural interpretion in this context.

3 Copower functors

We now show that the functors $\mathcal{L}^{(-)}$ and $\mathcal{M}_\omega^{(-)}$ studied previously, are instances of a more general construction. To this end, assume that $\mathfrak{C}$ is a category with fixed object $\mathcal{M} \in \mathfrak{C}$ so that for every set X the coproduct $\coprod_X \mathcal{M}$ exists in $\mathfrak{C}$. To ease notation, we write $\mathcal{M} \cdot X$ for $\coprod_X \mathcal{M}$.

Let $U : \mathcal{C} \to \mathcal{S}et$ be any functor. We claim that $\mathcal{M}_{\mathfrak{C}}[X] := U(\mathcal{M} \cdot X)$ is the object map of a $\mathcal{S}et$ endofunctor.

For every $x \in X$, let $e_x : \mathcal{M} \to \mathcal{M} \cdot X$ be the canonical sum injection. For a set map $f : X \to Y$, the source $(e_{f(x)} : \mathcal{M} \to \mathcal{M} \cdot Y)_{x \in X}$ is a competitor to the sum $(e_x : \mathcal{M} \to \mathcal{M} \cdot X)_{x \in X}$. This provides a unique homomorphism $\mathcal{M} \cdot f : \mathcal{M} \cdot X \to \mathcal{M} \cdot Y$ with $\mathcal{M} \cdot f \circ e_x = e_{f(x)}$ for all $x \in X$.

$$\begin{tikzcd} \mathcal{M} \cdot X \arrow[rr, dashed, "\mathcal{M} \cdot f"] & & \mathcal{M} \cdot Y \\ & \mathcal{M} \arrow[ul, "e_x"] \arrow[ur, "e_{f(x)}"'] & \end{tikzcd} \tag{9}$$

Theorem 2 *Given a category $\mathfrak{C}$ with object $\mathcal{M} \in \mathfrak{C}$ so that all copowers $\mathcal{M} \cdot X$ exist in $\mathfrak{C}$ and a functor $U : \mathfrak{C} \to \mathcal{S}et$, then*

$$\begin{aligned} \mathcal{M}_{\mathfrak{C}}[X] &:= U(\mathcal{M} \cdot X) \quad \textit{and} \\ \mathcal{M}_{\mathfrak{C}}[f] &:= U(\mathcal{M} \cdot f) \end{aligned}$$

for any sets X and Y and any map $f : X \to Y$, defines a $\mathcal{S}et$-endofunctor.

PROOF. Obviously, $\mathcal{M}_{\mathfrak{C}}[id_X] = id_{\mathcal{M}_{\mathfrak{C}}[X]}$, so let $f : X \to Y$ and $g : Y \to Z$ be maps. We claim that $\mathcal{M}_{\mathfrak{C}}[g \circ f] = \mathcal{M}_{\mathfrak{C}}[g] \circ \mathcal{M}_{\mathfrak{C}}[f]$, i.e. $U(\mathcal{M} \cdot (g \circ f)) = U(\mathcal{M} \cdot g) \circ U(\mathcal{M} \cdot f)$. But for each $x \in X$ we get by (9):

$$\mathcal{M} \cdot (g \circ f) \circ e_x = e_{(g \circ f)(x)} = \mathcal{M} \cdot g \circ e_{f(x)} = \mathcal{M} \cdot g \circ \mathcal{M} \cdot f \circ e_x$$

whence $\mathcal{M} \cdot (g \circ f) = \mathcal{M} \cdot g \circ \mathcal{M} \cdot f$.

In most cases, U will be an underlying-set functor, so we shall often suppress it, since it will always be clear from the context, whether we are looking at some $\mathcal{A}$ in $\mathfrak{C}$ or at $U(\mathcal{A}) \in \mathcal{S}et$. We now show that both our functors $\mathcal{M}_\omega^{(-)}$ and $\mathcal{L}^{(-)}$ are instances of the above construction.

3.1 Commutative Monoids

Proposition 3 *Let $\mathfrak{Mc}$ be the category of commutative monoids and $\mathcal{M} \in \mathfrak{Mc}$. Then $\mathcal{M}_{\mathfrak{Mc}}[-] = \mathcal{M}_\omega^{(-)}$*

PROOF. It is well known that in the category of commutative monoids the copower $\coprod_X \mathcal{M}$ is given by $\{\sigma : X \to M \mid \sigma(x) =_{a.e.} 0\}$, i.e the set of all maps $\sigma : X \to M$ that are almost everywhere zero, with addition defined pointwise, so $\mathcal{M}_{\mathfrak{Mc}}[X] = \mathcal{M}_\omega^{(X)}$. The injections $e_x : \mathcal{M} \to \mathcal{M}_\omega^X$ are defined by

$$e_x(m)(x') := \begin{cases} m & \text{if } x = x' \\ 0 & \text{otherwise.} \end{cases}$$

Each $\sigma \in \mathcal{M}_\omega^X$ can be written as a sum $\sigma = \sum_{x\in X} e_x(\sigma(x))$, so let $f : X \to Y$ be any set map, then

$$\begin{aligned}
\mathcal{M}_{\mathfrak{Mc}}[f](\sigma)(y) &= (\mathcal{M}_{\mathfrak{Mc}}[f])(\sum_{x\in X} e_x(\sigma(x)))(y) \\
&= (\sum_{x\in X} \mathcal{M}_{\mathfrak{Mc}}[f](e_x(\sigma(x))))(y) \\
&= (\sum_{x\in X} e_{f(x)}(\sigma(x)))(y) \\
&= \sum_{x\in X} (e_{f(x)}(\sigma(x))(y)) \\
&= \sum_{x\in X} \{\sigma(x) \mid f(x) = y\} \\
&= \mathcal{M}_\omega^f(\sigma)(y).
\end{aligned}$$

Hence $\mathcal{M}_{\mathfrak{Mc}}[f] = \mathcal{M}_\omega^f$.

3.2 $\bigvee$-Semilattices

Next. we show, that for a complete $\bigvee$-semilattice $\mathcal{L}$, the semilattice functor $\mathcal{L}^{(-)}$ is a copower-functor, as well.

Lemma 4 *For a $\bigvee$-semilattice $\mathcal{L}$ and a set X, the X-fold coproduct is $\mathcal{L}^X$ with injections $e_x : \mathcal{L} \to \mathcal{L}^X$ given by $e_x(l)(x) = l$, and $e_x(l)(x') = 0$ for $x \neq x'$. In particular, products and coproducts are the same in the category of $\bigvee$-semilattices.*

PROOF. Let $(h_x : \mathcal{L} \to \mathcal{S})_{x\in X}$ be an X-indexed family of $\bigvee$-homomorphisms. We must show that there is a unique $\bigvee$-homomorphism $\varphi : \mathcal{L}^X \to \mathcal{S}$ with $\varphi \circ e_x = h_x$ for all $x \in X$. Since every element $\sigma \in \mathcal{L}^X$ can be written as $\bigvee_{x\in X} \mathrm{e}_x(\sigma(x))$, such a φ must necessarily satisfy:

$$\begin{aligned}
\varphi(\sigma) &= \varphi(\bigvee_{x\in X} e_x(\sigma(x))) \\
&= \bigvee_{x\in X} \varphi(e_x(\sigma(x))) \\
&= \bigvee_{x\in X} h_x(\sigma(x)).
\end{aligned}$$

Taking the last line as a definition, we check that φ is indeed a $\bigvee$-homomorphism:

$$\begin{aligned}\varphi(\bigvee_{i\in I}\sigma_i) &= \bigvee_{x\in X} h_x(\bigvee_{i\in I}\sigma_i(x))\\ &= \bigvee_{x\in X}\bigvee_{i\in I} h_x(\sigma_i(x))\\ &= \bigvee_{i\in I}\bigvee_{x\in X} h_x(\sigma_i(x))\\ &= \bigvee_{i\in I}\varphi(\sigma_i).\end{aligned}$$

To check that $\varphi\circ e_x = h_x$, we calculate for arbitrary $l\in\mathcal{L}$:

$$(\varphi\circ e_x)(l) = \varphi(e_x(l)) = \bigvee_{x'\in\mathcal{L}} h_{x'}(e_x(l)(x')) = h_x(l),$$

where in the last step we have used the fact that $e_x(l)(x') = l$ for $x = x'$ and 0 otherwise.

Theorem 5 *Let $\mathfrak{SL}$ be the category of complete $\bigvee$-semilattices and $\mathcal{L}\in\mathfrak{SL}$, then $\mathcal{L}^{(-)} = \mathcal{L}_{\mathfrak{SL}}[-]$.*

PROOF. By the previous lemma, we have $\mathcal{L}_{\mathfrak{SL}}[X] = \mathcal{L}^X$ and $\mathcal{L}_{\mathfrak{SL}}[f](\sigma) = \bigvee_{x\in X} e_{f(x)}(\sigma(x))$ for any function $f: X\to Y$. We calculate:

$$\begin{aligned}\mathcal{L}_{\mathfrak{SL}}[f](\sigma)(y) &= (\bigvee_{x\in X} e_{f(x)}(\sigma(x)))(y)\\ &= \bigvee_{x\in X}(e_{f(x)}(\sigma(x)))(y)\\ &= \bigvee\{\sigma(x)\mid x\in X, y = f(x)\}\\ &= \mathcal{L}^f(\sigma)(y).\end{aligned}$$

3.3 Algebras in Varieties

It is well known that varieties of universal algebras are complete categories, in which all coproducts exist. They are known as *free products*, see [6]. Let $\mathfrak{V}$ be a variety of universal algebras of signature T and let $\mathcal{A}$ be an algebra in $\mathfrak{V}$. We are only interested in copowers, so given any set X, the X-fold sum of $\mathcal{A} = (A, f^{\mathcal{A}})$ is constructed in $\mathfrak{V}$ as follows:

We start with the set $A\times X$, and maps $e_x: A\to A\times X$ representing the $X-$fold sum of A in $\mathcal{S}et$. Then we form $F_{\mathfrak{V}}(A\times X)$, the free algebra in $\mathfrak{V}$ with variables from $A\times X$. If the signature of $\mathfrak{V}$ is finitary, then this is just the set of all $\mathfrak{V}$-terms with variables from $A\times X$. These variables are naturally embedded (assuming $\mathfrak{V}$ is nontrivial) by $\iota: A\times X\to F_{\mathfrak{V}}(A\times X)$. In general, this will not yet be the free product, simply, because the compositions $\iota \circ e_x : \mathcal{A} \to F_{\mathfrak{V}}(A \times X)$ need not be algebra homomorphisms. Thus we let Θ be the smallest congruence relation on $\mathcal{F} := F_{\mathfrak{V}}(A \times X)$ containing all pairs

$$\Big((\iota \circ e_x \circ f^{\mathcal{A}})(u)\ ,\ (f^{\mathcal{F}} \circ T(\iota \circ e_x))(u)\Big) \tag{10}$$

where $u \in T(A)$ and $f^{\mathcal{F}}$ is the T-operation on $F_{\mathfrak{V}}(A \times X)$. If T is a finitary polynomial functor, then this reads as

$$\Big((f^{\mathcal{A}}(a_1, \ldots, a_n), x), f^{\mathcal{F}}((a_1, x), \ldots, (a_n, x))\Big)$$

for each n-ary operation symbol f and elements $a_1, \ldots, a_n \in A$.

Let π_Θ be the canonical projection $\pi_\Theta : F_{\mathfrak{V}}(A \times X) \to F_{\mathfrak{V}}(A \times X)/\Theta$, then we claim:

Lemma 6 *$\mathcal{A}_{\mathfrak{V}}[X] := F_{\mathfrak{V}}(A \times X)/\Theta$ with embeddings $\epsilon_x := \pi_\Theta \circ \iota \circ e_x : \mathcal{A} \to \mathcal{A}_{\mathfrak{V}}[X]$ is the X-fold sum of $\mathcal{A}$ in $\mathfrak{V}$.*

PROOF. The proof consists of verifying that

- $\pi_\Theta \circ \iota \circ e_x : \mathcal{A} \to F_{\mathfrak{V}}(A \times X)/\Theta$ is a homomorphism, and
- every homomorphism $\phi : F_{\mathfrak{V}}(A \times X) \to \mathcal{B}$ for which $\phi \circ \iota \circ e_x$ is a homomorphism, factors uniquely through π_Θ.

$$\begin{tikzcd}
F_{\mathfrak{V}}(A \times X) \arrow[r, "\pi_\Theta"] \arrow[ddr, "\phi"] & F_{\mathfrak{V}}(A \times X)/\Theta \arrow[dd, dashed, "\varphi"] \\
A \times X \arrow[u, dotted, "\iota"] \arrow[dr, dotted, "f"] & \\
\mathcal{A} \arrow[u, dotted, "e_x"] \arrow[r, "\varphi_x"'] & \mathcal{B}
\end{tikzcd}$$

Given then a family $(\varphi_x : \mathcal{A} \to \mathcal{B})_{x \in X}$ of homomorphism, let $f : A \times X \to B$ be the sum map in $\mathcal{S}et$ and $\phi : F_{\mathfrak{V}}(A \times X) \to \mathcal{B}$ the unique homomorphic extension of f. By the above, ϕ factors uniquely through π_Θ, providing the sum homomorphism $\varphi : F_{\mathfrak{V}}(A \times X)/\Theta \to \mathcal{B}$.

Observe that it may well happen, that variables (a, x) and (a, y) with $x \neq y$ are being identified by Θ. In particular, this occurs whenever a is the result of a constant operation in $\mathcal{A}$. In classical universal algebras the signature functor T is a polynomial functor $T(X) = \sum_{i \in I} D_i \times X^{n_i}$, so that each element $d \in D_i$ represents an n_i-ary operation symbol. When $d \in D_j$ with $n_j = 0$, the equality (10) implies $(d^{\mathcal{A}}, x)\, \Theta\, d^{\mathcal{F}}$ for all $x \in X$ hence $(d^{\mathcal{A}}, x) = (d^{\mathcal{A}}, y)$ in $F_{\mathfrak{V}}(A \times X)/\Theta$ for all $x, y \in X$.

The elements of $F_{\mathfrak{V}}(A\times X)$ are $\mathfrak{V}$-terms, which we can represent as equivalence classes of finite trees, where each node is an element of some D_i, having n_i subtrees. The leaves of these trees are variables $(a,x)\in A\times X$. By the above, this includes 0-ary nodes. Two trees p and q are identified in $F_{\mathfrak{V}}(A\times X)/\Theta$, iff there exist a sequence $p_0, p_1, \ldots, p_n$ of trees with $p=p_0$, $p_n=q$, and for each $i<n$ either $p_i=p_{i+1}$ is a $\mathfrak{V}$-equation or $p_i=t((a_1,x),\ldots,(a_n,x))$ and $p_{i+1}=(t^{\mathcal{A}}(a_1,\ldots,a_n),x)$, or conversely. It is customary to write ax instead of (a,x) so that formally, $\mathcal{A}_{\mathfrak{V}}[X]$ is the set of $\mathfrak{V}$-terms in the ax subject to the requirement that "right multiplication" with x becomes a homomorphism.

4 Semigroups and monoids

Recall that a semigroup $\mathcal{S}=(S,\cdot)$ is just a set with a binary associative operation. A monoid $\mathcal{M}=(M,\cdot,1)$ is a semigroup with a two-sided unit. Whe shall write $\mathcal{M}=(M,+,0)$ when the operation is commutative. Let $\mathfrak{S}$, (resp. $\mathfrak{M}$) be the variety of all semigroups (resp. monoids) and $\mathfrak{Sc}$ (resp. $\mathfrak{Mc}$) be the varieties of all commutative semigroups, (resp. commutative monoids).

4.1 Conical, refinable and equidivisible semigroups

The following notions from semigroup theory will be needed in the sequel.

Definition 7 *A monoid $\mathcal{M}$ is called* conical*, if no element $m\neq 1$ is invertible, i.e. if $m_1\cdot m_2=1$ implies $m_1=1=m_2$.*

A semigroup $\mathcal{S}$ can always be embedded into a monoid by adjoining a fresh element $1\notin S$. The resulting monoid $\mathcal{S}^1$ is clearly conical and conversely, every conical monoid $\mathcal{M}$ arises this way from a semigroup.

Definition 8 *[1] A semigroup $\mathcal{S}$ is called* refinable*, if $a_1\cdot a_2=b_1\cdot b_2$ implies the existence of $s_{11}, s_{12}, s_{21}, s_{22}$ so that $s_{i1}\cdot s_{i2}=a_i$, and $s_{1i}\cdot s_{2i}=b_i$ for $i=1,2$.*

In other words, given $a_1\cdot a_2=b_1\cdot b_2$, there is a matrix, whose i-th row multiplies to a_i and whose i-th column to b_i.

$$\begin{array}{cc|c} s_{11} & s_{12} & a_1 \\ s_{21} & s_{22} & a_2 \\ \hline b_1 & b_2 & \end{array}$$

Refinability can be considered as a generalized distributive law. Indeed, as was shown in [2], a lattice $\mathcal{L}$ considered as a semilattice is refinable if and only if $\mathcal{L}$ is distributive.

By an easy induction, one checks that refinability implies a matrix decomposition of equal products of arbitrarily many factors, in other words, if $a_1, \ldots, a_m \in S$, and $b_1, \ldots, b_n \in S$ with $a_1 \cdot \ldots \cdot a_m = b_1 \cdot \ldots \cdot b_n$ then there exists an $m \times n$-matrix $(s_{i,j})$ of elements from S, whose i-th row multiplies to a_i and whose j-th colum to b_j for each $i \leq m$, and $j \leq n$.

$$\begin{array}{ccc|c} s_{11} & \cdots & s_{1n} & a_1 \\ \vdots & & \vdots & \vdots \\ s_{m1} & \cdots & s_{mn} & a_m \\ \hline b_1 & \cdots & b_n & \end{array}$$

The final and related notion we need is equidivisibility:

Definition 9 *[4] A semigroup S is called* equidivisible*, if $a_1 \cdot a_2 = b_1 \cdot b_2$ implies that there exists some h so that either $a_1 = b_1 \cdot h$ and $h \cdot a_2 = b_2$ or $b_1 = a_1 \cdot h$ and $h \cdot b_2 = a_2$.*

Equidivisible monoids are obviously refinable, but the converse does not hold, as the following example demonstrates.

Example 10 *A lattice, considered as $\vee$-semilattice (or as $\wedge$-semilattice) is refinable, iff it is distributive and it is equidivisible iff it is a chain.*

To see that any equidivisible semigroup $\mathcal{S} = (S, \cdot)$ is refinable, too, let us assume $a_1 \cdot a_2 = b_1 \cdot b_2$. Equidivisibility yields an element h such that w.l.o.g. $a_1 \cdot h = b_1$ and $h \cdot b_2 = a_2$. Thus we can partially fill the required matrix as

$$\begin{array}{cc|c} a_1 & & a_1 \\ h & b_2 & a_2 \\ \hline b_1 & b_2 & \end{array}$$

In a monoid, we may just fill the as yet empty space with the unit element, but in a semigroup, such a unit need not be available. Fortunately, however, equidivisibilty forces the existence, for any pair $a, b \in S$, of an element $e_{a,b}$ which is at the same time a right unit for a and a left unit for b. To see this, apply equidivisibility to the equality $a \cdot b = a \cdot b$. Now filling e_{a_1,b_2} into the above matrix, we have shown:

Proposition 11 *Every equidivisible semigroup is refinable.*

It is equidivisibility, rather than refinability, which allows us to obtain a common refinement for equal products of elements. To see this, observe first that we can visualize equidivisibility as follows:

$$a_1 \cdot a_2 = b_1 \cdot b_2 \qquad \Longrightarrow \qquad \overbrace{a_1 \cdot h}^{b_1} \underbrace{\!\!\cdot b_2}_{a_2} \quad \text{or} \quad \overbrace{b_1 \cdot h}^{a_1} \underbrace{\!\!\cdot a_2}_{b_2}\,.$$

This idea can be extended to products with arbitrarily many factors. Given $a_1 \cdot \ldots \cdot a_m = b_1 \cdot \ldots \cdot b_n$, we can find a common refinement as a product of smaller building blocks $h_1 \cdot \ldots \cdot h_{m+n-1}$ so that all the factors a_i and b_j are products of adjacent groups of the h_r:

$$h_1 \cdot \ldots \cdot \overbrace{h_{i_{r-1}+1} \cdot \ldots \cdot h_{j_{s-1}+1} \cdot \ldots \cdot h_{i_r}}^{a_i} \cdot \ldots \cdot h_{j_s} \cdot \ldots \cdot h_{m+n-1}\,.$$

$$\text{with } b_j = h_{j_{s-1}+1} \cdot \ldots \cdot h_{i_r} \cdot \ldots \cdot h_{j_s}$$

A precise formulation is given in the following result, which will be needed later:

Lemma 12 *$\mathcal{M}$ is equidivisible iff given $a_1 \cdot \ldots \cdot a_m = b_1 \cdot \ldots \cdot b_n$ there exists some $k < m+n$, elements $h_1, h_2, \ldots, h_k$ and partitions $0 = i_0 < i_1 < \ldots < i_m = k$ as well as $0 = j_0 < j_1 < \ldots < j_m = k$ so that $a_r = h_{i_{r-1}+1} \cdot \ldots \cdot h_{i_r}$ for each $r \le m$ and likewise $b_s = h_{j_{s-1}+1} \cdot \ldots \cdot h_{j_s}$ for each $s \le n$.*

PROOF. We use induction on $m+n$. For $m = 1$ or $n = 1$ the statement is trivial, for $m = n = 2$ it is just the definition of equidivisibility. Thus, assume $a_1 \cdot \ldots \cdot a_{m+1} = b_1 \cdot \ldots \cdot b_n$ with $m, n \ge 2$. Put $a := a_1 \cdot \ldots \cdot a_m$ and $b := b_1 \cdot \ldots \cdot b_{n-1}$, then $a \cdot a_{m+1} = b \cdot b_n$. Equidivisibility yields h with either $a \cdot h = b$ and $h \cdot b_n = a_{m+1}$ or $b \cdot h = a$ and $h \cdot a_{m+1} = b_n$. In the first case, the inductive hypothesis yields a common refinement $h_1 \cdot \ldots \cdot h_{m+n-1}$ of $a_1 \cdot \ldots \cdot a_m \cdot h$ and $b_1 \cdot \ldots \cdot b_{n-1}$. With $h_{m+n} := b_n$we extend it to a common refinement of $a_1 \cdot \ldots \cdot a_{m+1}$ and $b_1 \cdot \ldots \cdot b_n$. The second case is handled similarly.

4.2 Copowers of nonabelian monoids

For the rest of this section, we are concerned mostly with the functor $\mathcal{M}_{\mathfrak{M}}[-]$ where $\mathfrak{M}$ is the variety of all monoids and $\mathcal{M} = (M, \cdot, 1) \in \mathfrak{M}$. Given a set X, the elements of $\mathcal{F} := F_{\mathfrak{M}}(M \times X)$ are the monoid terms in variables from $M \times X$. We use $\star$ as multiplikation symbol in $\mathcal{F}$ in order to avoid ambiguities. Writing mx instead of (m, x), we can represent the elements of $\mathcal{F}$ as formal polynomials

$$m_1 x_1 \star \ldots \star m_n x_n$$

where $m_i \in S$ and $x_i \in X$. The elements of the X-fold copower $\mathcal{M}_{\mathfrak{M}}[X]$ are obtained by reducing with the equations corresponding to (10):

$$m_1 x \star m_2 x = (m_1 \cdot m_2)x \tag{11}$$

$$1x = \varepsilon. \tag{12}$$

Writing $[m_1 x_1 \star \ldots \star m_n x_n]_{\mathfrak{M}}$ for the Θ-class of $m_1 x_1 \star \ldots \star m_n x_n$, it follows that

$$\mathcal{M}_{\mathfrak{M}}[X] = \{[m_1 x_1 \star \ldots \star m_n x_n]_{\mathfrak{M}} \mid n \in \mathbb{N},\, m_i \in M,\, x_i \in X\}.$$

The rules (11) and (12) reduce a formal polynomial $m_1 x_1 \star \ldots \star m_n x_n$ to a normal form with respect to Θ, so that we may equivalently write

$$\mathcal{M}_{\mathfrak{M}}[X] = \{m_1 x_1 \star \ldots \star m_n x_n \mid n \in \mathbb{N},\;\; m_i \in M - \{1\},\, x_i \in X,\, x_i \neq x_{i+1}\}$$

where the case $n = 0$ accounts for the empty product, representing the unit 1. From now on, when we write $m_1 x_1 \star \ldots \star m_n x_n$ by itself, we assume that it is already in normal form, otherwise we write $[m_1 x_1 \star \ldots \star m_n x_n]_{\mathfrak{M}}$. Consequently, $[m_1 x_1 \star \ldots \star m_n x_n]_{\mathfrak{M}}$ can be read as "the normal form of $m_1 x_1 \star \ldots \star m_n x_n$".

4.3 Monoid labeled coalgebras

Coalgebras for the functor $\mathcal{M}_{\mathfrak{M}}[-]$ are transition systems where each state has a list of successor states with the transitions to these states labeled by elements of M. States appearing twice at adjacent positions in the list are combined and their labels multiplied according to $\mathcal{S}$ and transitions with label 1 are dropped.

The coalgebraic theory of the class of coalgebras for a given functor F depends on the properties of the functor F. A prominent concern is weak preservation of certain limits. The early coalgebraic literature (see [5]) in fact, was primarily concerned with functors that weakly preserved pullbacks. I was later shown in [3], that weak preservation of pullbacks can be separated into (weak) preservation of preimages and weak preservation of kernel pairs. Structure theoretically, the first property is equivalent to homomorphic preimages of subcoalgebras being subcoalgebras, and the latter preservation property guarantees that bisimilarity agrees with observational equivalence.

For these reasons we shall be concerned with the question, under which conditions the functor $\mathcal{M}_{\mathfrak{M}}[-]$ weakly preserves preimages, kernels, or both, i.e. weak pullbacks. For the functor $\mathcal{M}_\omega^{(-)}$ which due to proposition 3 agrees with $\mathcal{M}_{\mathfrak{M}_c}[-]$, the following result has been proved in [2] :

Theorem 13 *Let* $\mathfrak{Mc}$ *be the class of all commutative monoids and* $\mathcal{M} \in \mathfrak{M}_{\mathfrak{c}}$*, then*

- $\mathcal{M}_{\mathfrak{Mc}}[-]$ *(weakly) preserves preimages iff* $\mathcal{M}$ *is conical,*
- $\mathcal{M}_{\mathfrak{Mc}}[-]$ *weakly preserves kernel pairs iff* $\mathcal{M}$ *is refinable.*

As an immediate consequence one obtains:

Corollary 14 $\mathcal{M}_{\mathfrak{Mc}}[-]$ *weakly preserves pullbacks iff* $\mathcal{M}$ *is conical and refinable.*

We are now trying to obtain similar characterizations when we replace the class $\mathfrak{Mc}$ by the class of all monoids $\mathfrak{M}$ or by the class of all semigroups $\mathfrak{S}$. We begin with a characterization of $\mathcal{M}_{\mathfrak{M}}$-coalgebras and some useful lemmata:

Lemma 15 *Let* $\mathcal{A} = (A, \alpha)$ *be an* $\mathcal{M}_{\mathfrak{M}}[-]$*-coalgebra, then* $U \subseteq A$ *is a subcoalgebra iff for each* $u \in U$ *with* $\alpha(u) = m_1 a_1 \star \ldots \star m_n a_n$ *we have* $a_i \in U$ *for all* $i \leq n$*.*

PROOF. Note that the functor $\mathcal{M}_{\mathfrak{M}}[-]$ preserves inclusions. Hence U is a subcoalgebra iff $\alpha_{\mathcal{A}}(u) \in \mathcal{M}_{\mathfrak{M}}[U]$ for each $u \in U$, so if $\alpha(u) = m_1 a_1 \star \ldots \star m_n a_n$ is in normal form, we must have $a_i \in U$ for all $i \leq n$.

Lemma 16 *If* $\mathcal{M}$ *is conical, then* $U \subseteq A$ *is a subcoalgebra iff for all* $u \in U$ *with* $\alpha(u) = [m_1 a_1 \star \ldots \star m_n a_n]_{\mathfrak{M}}$ *we have* $a_i \in U$ *for all* $i \leq n$ *with* $m_i \neq 1$*.*

PROOF. If $\mathcal{M}$ is conical, then reducing $m_1 a_1 \star \ldots \star m_n a_n$ to normal form, can only make those a_i disappear whose coefficients m_i are equal to 1.

After these preparations, we first show:

Proposition 17 $\mathcal{M}_{\mathfrak{M}}[-]$ *(weakly) preserves preimages if and only if* $\mathcal{M}$ *is conical.*

PROOF. According to [3], a $\mathcal{S}et$-functor T (weakly) preserves preimages if and only if homomorphic preimages of T-subcoalgebras are subcoalgebras, to be precise, if

$$V \leq \mathcal{B} \implies \varphi^{-}[V] \leq \mathcal{A}$$

for every homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ among T-coalgebras.

Assuming that $\mathcal{M}$ is conical, let $\mathcal{A} = (A, \alpha)$ and $\mathcal{B} = (B, \beta)$ be arbitrary $\mathcal{M}_{\mathfrak{M}}[-]$-coalgebras, $V \leq \mathcal{B}$ and $\varphi : \mathcal{A} \to \mathcal{B}$ any homomorphism. Given $u \in$

$\varphi^-[V]$ and $\alpha(u) = m_1a_1\star\ldots\star m_na_n$, lemma 15 requires us to show $a_i \in \varphi^-[V]$, that is $\varphi(a_i) \in V$. Note that each $m_i \neq 1$. Since φ is a homomorphism and V a subcoalgebra, $\beta(\varphi(u)) = (\mathcal{M}_{\mathfrak{M}}[\varphi]\circ\alpha)(u) = [m_1\varphi(a_1)\star\ldots\star m_n\varphi(a_n)]_{\mathfrak{M}} \in V$. By lemma 16, we conclude that each $\varphi(a_i) \in V$, so each $a_i \in \varphi^-[V]$, as required.

For the converse, consider $\mathcal{M}_{\mathfrak{M}}[-]$-coalgebras $\mathcal{A} = (\{x, x_1, x_2\}, \alpha)$ and $\mathcal{B} = (\{y, y_1\}, \beta)$ where $\alpha(x) = m_1x_1 \star m_2x_2$, $\beta(y) = (m_1 \cdot m_2)y_1$ and $\alpha(x_1) = \alpha(x_2) = \beta(y_1) = \varepsilon$. Clearly, the map φ, with $\varphi(x) = y$ and $\varphi(x_1) = \varphi(x_2) = y_1$ is a homomorphism. Assuming $(m_1 \cdot m_2) = 1$ makes $\{y\}$ a subcoalgebra of $\mathcal{B}$, but $\varphi^-[\{y\}] = \{x\}$ is not a subcoalgebra of $\mathcal{A}$, unless $m_1 = m_2 = 1$. Thus, weak preservation of preimages forces $\mathcal{M}$ to be conical.

For the proof of the main theorem of this section, we need a further technical lemma which is easily proved by induction:

Lemma 18 *Suppose $[m_1u_1 \star \ldots \star m_nu_n]_{\mathfrak{M}} = c_1x_1 \star c_2x_2 \in \mathcal{M}_{\mathfrak{M}}[\{x_1, x_2\}]$ with $x_1 \neq x_2$, then $m_1 \cdot \ldots \cdot m_k = c_1$ and $m_{k+1} \cdot \ldots \cdot m_n = c_2$ for some $k < n$. If all m_i are different from 1 and $\mathcal{M}$ is conical then additionally $u_1 = \ldots u_k = x_1$ and $u_{k+1} = \ldots u_n = x_2$.*

PROOF. The proof is by induction on n. For the premise to hold, n must be at least 2, so the base step is trivial. Assume, the claim holds for some n, then for $n + 1$ we start with $[m_1u_1 \star \ldots \star m_nx_n \star m_{n+1}x_{n+1}]_{\mathfrak{M}} = c_1x_1 \star c_2x_2$. The left hand side of this equation must be reducible to $c_1x_1 \star c_2x_2$ with the above rules (11) or (12). After the first reduction step, the formal polynomial becomes shorter and the induction hypothesis can be applied. If, for instance, rule (i) was used, then we get w.l.o.g. $c_1 = m_1 \cdot \ldots (m_i \cdot m_{i+1}) \ldots m_k$ and $c_2 = m_{k+1} \cdot \ldots \cdot m_n$. The decomposition of the original formal polynomial is obvious. The same holds, if rule (ii) was applied. The second statement of the lemma is obvious, since under the stated hypothesis the second rule (12) can never be applied.

The main result of this section is then:

Theorem 19 *$\mathcal{M}_{\mathfrak{M}}[-]$ weakly preserves pullbacks if and only if $\mathcal{M}$ is conical and equidivisible.*

PROOF. If $\mathcal{M}_{\mathfrak{M}}[-]$ weakly preserves pullbacks, it weakly preserves preimages and kernel pairs. By proposition 17, $\mathcal{M}$ is conical. Also, $\mathcal{M}_{\mathfrak{M}}[-]$ weakly preserves kernel pairs, so let $c_1, c_2, d_1, d_2 \in \mathcal{M}$ be given with $c_1 \cdot c_2 = m = d_1 \cdot d_2$. Consider sets $X := \{x_1, x_2\}$ and $Z := \{z\}$ and the unique map $f : X \to Z$. Then the elements $c_1x_1 \star c_2x_2$ and $d_1x_1 \star d_2x_2$ are in the kernel of $\mathcal{M}_{\mathfrak{M}}[f]$. With

$X \times X$ being the kernel of this particular f in the base category $\mathcal{S}et$, weak kernel preservation guarantees the existence of some element $w \in \mathcal{M}_{\mathfrak{M}}[X \times X]$ with $\mathcal{M}[\pi_1](w) = c_1x_1 \star c_2x_2$ and $\mathcal{M}[\pi_2](w) = d_1x_1 \star d_2x_2$, where the $\pi_i : X^2 \to X$ are the canonical projections.

Any such w can be written as $m_1(u_1, v_1) \star \ldots \star m_n(u_n, v_n)$ where $n \in \mathbb{N}$, $u_i, v_i \in X$ and all $m_i \neq 1$, hence

$$[m_1u_1 \star \ldots \star m_nu_n]_{\mathfrak{M}} = c_1x_1 \star c_2x_2$$

and

$$[m_1v_1 \star \ldots \star m_nv_n]_{\mathfrak{M}} = d_1x_1 \star d_2x_2.$$

Now lemma 18 yields $c_1 = m_1 \cdot \ldots \cdot m_k$ and $c_2 = m_{k+1} \cdot \ldots \cdot m_n$ and likewise $d_1 = m_1 \cdot \ldots m_l$ and $d_2 = m_{l+1} \cdot \ldots \cdot m_n$. If $k = l$, we choose $h = 1$, otherwise w.l.o.g $k < l$ and with $h = m_{k+1} \cdot \ldots \cdot m_l$ we have $c_1 \cdot h = d_1$ and $h \cdot d_2 = c_2$, so $\mathcal{M}$ is equidivisible.

Conversely, assuming that $\mathcal{M}$ is equidivisible, let $f : X \to Z$ be any set map and let $Ker\, f := \{(x, y) \in X^2 \mid f(x) = f(y)\}$ be its kernel with $\pi_1$and π_2 its projection maps. Given $u, v \in \mathcal{M}_{\mathfrak{M}}[X]$ with $\mathcal{M}_{\mathfrak{M}}[f](u) = \mathcal{M}_{\mathfrak{M}}[f](v)$ we must find an element $w \in \mathcal{M}_{\mathfrak{M}}[Ker\, f]$ such that $\mathcal{M}_{\mathfrak{M}}[\pi_1](w) = u$ and $\mathcal{M}_{\mathfrak{M}}[\pi_2](w) = v$.

Now, we have $u = a_1x_1 \star \ldots \star a_mx_m$ and $v = b_1y_1 \star \ldots \star b_ny_n$ for appropriate $m, n \in \mathbb{N}$ and a_i,$b_j \in \mathcal{M}$ and x_i,$y_j \in X$. Note that this notation implies that all a_i and all b_j are $\neq$1, which we can safely assume, so

$$\mathcal{M}_{\mathfrak{M}}[f](u) = [a_1f(x_1)\star\ldots\star a_mf(x_a)]_{\mathfrak{M}} = [b_1f(y_1)\star\ldots\star b_nf(y_n)]_{\mathfrak{M}} = \mathcal{M}_{\mathfrak{M}}[f](v)\,.$$

It follows that both terms have a common normal form $NF = t_1z_1 \star \ldots \star t_rz_r$. To reduce to this normal form, only rule 11 is available, since none of the coefficients a_i, b_j is equal to 1 and since 1 cannot be created in the reduction process, due to the assumption that $\mathcal{M}$ is conical.

Let us first consider the special case $r = 1$. Then $f(x_1) = \ldots = f(x_m) = z_1 = f(y_1) = \ldots = f(y_n)$ and $a_1 \cdot \ldots \cdot a_m = t_1 = b_1 \cdot \ldots \cdot b_n$. Using the notation of lemma 12, we find $h_1, h_2, \ldots, h_k$ refininig the products so that each h_i is part of the product decomposition of a unique $a_{l(i)}$ and of a unique $b_{r(i)}$. We now consider the formal polynomial

$$w = h_1(x_{l(1)}, y_{l(1)}) \star \ldots \star h_k(x_{l(k)}, y_{r(k)})$$

then $w \in \mathcal{M}_{\mathfrak{M}}[Ker\, f]$ and

$$\mathcal{M}_{\mathfrak{M}}[\pi_1](w) = h_1x_{l(1)} \star \ldots \star h_kx_{l(k)} = a_1x_1 \star \ldots \star a_mx_m = u$$

likewise

$$\mathcal{M}_{\mathfrak{M}}[\pi_2](w) = h_1 y_{r(1)} \star \ldots \star h_k y_{r(k)} = b_1 y_1 \star \ldots \star b_n y_n = v$$

as requested.

In the general case where the normal form $NF = t_1 z_1 \star \ldots \star t_r z_r$ has $r > 1$, we will find partitions of the indices $0 = i_0 < i_1 < \ldots < i_r = p$ and $0 = j_0 < j_1 < \ldots < j_r = q$ so that for each k the products agree:

$$m_{i_k+1} \cdot \ldots \cdot m_{i_{k+1}} = s_{j_k+1} \cdot \ldots \cdot s_{j_{k+1}}$$

and also

$$f(x_{i_k+1}) = \ldots = f(x_{i_{k+1}}) = z_{k+1} = f(y_{j_k+1}) = \ldots = f(y_{j_{k+1}}).$$

For each such class of the partition we proceed as in the special case above, and find a formal polynomial $w_k \in \mathcal{M}_{\mathfrak{M}}[Ker\, f]$ and define $w = w_1 * \ldots \star w_r$.

4.4 The semilattice functor $\mathcal{M}_{\mathfrak{S}}[-]$

If $\mathfrak{S}$ is the class of all semilattices, only rule (12) is to be used in computing the congruence Θ. In this context, even if $\mathcal{M}$ should happen to be a monoid, the label 1 does not play any special role. It is therefore easy to see that for any semigroup $\mathcal{M}$, the functor $\mathcal{M}_{\mathfrak{S}}[-]$ preserves preimages. The case of kernel preservation is dealt with as before, but it is not necessary to consider any condition replacing "conical". We therefore obtain:

Theorem 20 *For any semigroup $\mathcal{M}$, the functor $\mathcal{M}_{\mathfrak{S}}[-]$ weakly preserves preimages. $\mathcal{M}_{\mathfrak{S}}[-]$ weakly preserves kernel pairs iff it weakly preserves pullbacks iff the semigroup $\mathcal{M}$ is equidivisible.*

5 Copower functors are universal

5.1 What is special about sum functors

Generalizing earlier constructions, we have studied set functors arising from powers of a fixed object $\mathcal{M}$ in some category $\mathfrak{C}$ as $F(X) = U(\coprod_{x \in X} \mathcal{M}) = U(\mathcal{M} \cdot X)$, where U is some (forgetful) functor to $\mathcal{S}et$. We have instantiated this construction in algebraic categories of semigroups, monoids and commutative monoids. An obvious question arises: What is special about such functors ? The somehow surprising answer is: *Nothing.* To be precise:

Theorem 21 *For every $\mathcal{S}et$-endofunctor T, there is a concrete category $\mathfrak{C}$ and an object $\mathcal{M} \in \mathfrak{C}$ such that $T(-) \cong U \circ \mathcal{M} \cdot (-)$, where U is the forgetful functor.*

PROOF. Let $\mathfrak{C}$ be the subcategory of $\mathcal{S}et$ consisting of all all sets $T(X)$ together with all maps Tf where X is a set and f a set map. Since each set X is the sum in $\mathcal{S}et$ of $|X|$ copies of 1, it is easy to check that $T(X)$ is indeed the sum in $\mathfrak{C}$ of $|X|$ copies of the $\mathfrak{C}$-object $T(1)$. It follows that $T(X) = U(\coprod_{x \in X} T(1))$ where $U : \mathfrak{C} \to \mathcal{S}et$ is the inclusion functor.

Thus copower functors represent not only an interesting but also a universal construction principle for $\mathcal{S}et$-functors.

5.2 Semigroups as categories

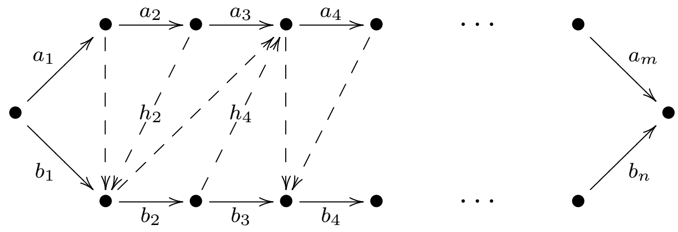
We have have considered monoids and classes of semigroups to construct $\mathcal{S}et$-functors with specially designed preservation properties. In a sense, we employed semigroup and monoid notions and methods to study category theoretic properties. One might ask, whether, conversely, category theoretic notions might benefit the study of monoids. A monoid $\mathcal{M}$, after all, is a category $\bullet_{\mathcal{M}}$, albeit a really simple looking one. It has just one single object $\bullet = \{0\}$, and it has an arrow $a : \bullet \to \bullet$ for each $a \in \mathcal{M}$. Composition of arrows is defined as monoid multiplication. A commutative square, in this category is a just collection of elements $a_1, a_2, b_1, b_2 \in \mathcal{M}$ satisfying $a_1 \cdot a_2 = b_1 \cdot b_2$. Obviously, then

$\mathcal{M}$ is equidivisible $\iff$ Each square in $\bullet_{\mathcal{M}}$ has a diagonal:

$$\text{either} \quad \begin{tikzcd} \bullet \arrow[r, "a_1"] \arrow[d, "b_1"'] & \bullet \arrow[d, "a_2"] \arrow[dl, dashed] \\ \bullet \arrow[r, "b_2"'] & \bullet \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} \bullet \arrow[r, "a_1"] \arrow[d, "b_1"'] & \bullet \arrow[d, "a_2"] \\ \bullet \arrow[r, "b_2"'] \arrow[ur, dashed] & \bullet \end{tikzcd}$$

Squares having diagonals is a very important aspect in category theory. It is, for instance, fundamental in the study of standard factorizations and factorization systems for morphisms. Here, we shall show, how this category theoretic view might be used to obtain a convincing graphical argument for the proof of the refinement lemma 12.

The given equality $a_1 \cdot \ldots \cdot a_m = b_1 \cdot \ldots \cdot b_n$, translates into two paths in the category $\bullet_{\mathcal{M}}$ with common starting and ending point. The above diagonal property allows us to insert arrows into this pair of paths. After the first arrow is inserted a new pair of paths arises, and so on. We obtain a figure such as e.g.:



It is required to find a path from the left to the right end that collects all a_i as well as all b_j. Note that each node always has at most two outgoing arrows, and that these are connected by another (necessarily dashed) arrow given by the diagonal property. The algorithm commences at the common starting point of the two paths and proceeds to its successor. In case where there are two successors, it chooses the one from which the connecting diagonal arrow arrow starts. It is easy to verify that this results in a path with the required properties. In the above example, with the diagonals labelled h_1,h_2, etc. from left to right, it would yield the path $a_1, a_2, h_2, b_2, h_4, a_4, \ldots$ providing the common decomposition:

$$\underbrace{a_1 \cdot a_2 \cdot \rlap{$\overbrace{}^{a_3}$}h_2}_{b_1} \cdot b_2 \cdot \underbrace{h_4 \cdot a_4 \cdot \ldots}_{b_3} \cdots$$

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