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Functors for Coalgebras

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Abstract. Functors preserving weak pullbacks provide the basis for a rich structure theory of coalgebras. We give an easy to use criterion to check whether a functor preserves weak pullbacks. We apply the characterization to the functor $\mathcal{F}$ which associates a set X with the set $\mathcal{F}(X)$ of all filters on X. It turns out that this functor preserves weak pullbacks, yet does not preserve weak generalized pullbacks. Since topological spaces can be considered as $\mathcal{F}$-coalgebras, in fact they constitute a covariety, we find that the intersection of subcoalgebras need not be a coalgebra, and 1-generated $\mathcal{F}$-coalgebras need not exist.

1. Introduction

Coalgebras have been introduced by Aczel and Mendler [AM89] to model various types of transition systems. Reichel [Rei95], and Jacobs [Jac96] show that coalgebras are well suited for modeling object oriented programming and for program verification. In [Rut96], J. J. M. M. Rutten develops the a fundamental theory of "universal coalgebra" along the lines of universal algebra ([Gra79, Coh81, Ihr93]). Various other authors have contributed further details to the theory, (e.g. [Bar93, GS98, RT94]).

The theory of coalgebras starts with a functor T from **Set**, the category of sets to itself. T provides the *type* for the coalgebras to be considered. A *coalgebra of type* T is then simply any map $\alpha : X \to T(X)$.

Of particular importance amongst all possible type functors are the identity functor $\mathcal{I}$, the powerset functor $\mathcal{P}(-)$, the finite-powerset functor $\mathcal{P}_{fin}(-)$ and functors of type $A \times (-)^B$ where A and B are fixed sets. The coalgebras belonging to these functors model different kinds of transition systems. They are *deterministic* in the first case, nondeterministic in the second, *image finite* in the case of the functor $\mathcal{P}_{fin}(-)$ and automata (with input alphabet B and output alphabet A) in the case of the functor $A \times (-)^B$.

An important observation at the outset of the development of universal coalgebra is that all functors which are of practical relevance as coalgebraic type functors preserve weak pullbacks, and that this property in turn is necessary (and largely sufficient) to obtain a rich structure theory along the lines of universal algebra.

In order to develop a satisfactory theory of cofree coalgebras, of covarieties and of coequational or coimplicational classes ([Gum98]), one additionally requires that T should

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be *bounded*, which amounts to saying that the cardinality of 1-generated T-coalgebras should not exceed a fixed cardinality. From the above functors, only the powerset functor $\mathcal{P}(-)$ fails to satisfy this additional requirement.

In this paper, we shall give a criterion to determine whether a functor T preserves weak pullbacks. We then apply the criterion to the *filter functor* $\mathcal{F}$ which associates with a set X the set $\mathcal{F}(X)$ of all filters on X. It turns out that $\mathcal{F}$ preserves weak pullbacks of two morphisms $\phi : A \to C$ and $\psi : B \to C$ but does not preserve *weak generalized pullbacks* (cf. [Rut96]).

Every topological space can be considered as an $\mathcal{F}$-coalgebra. Homomorphisms are precisely the continuous open maps and subcoalgebras are the open sets. We give an internal characterization of the class of all $\mathcal{F}$-coalgebras arising from some topological space and we show that this class is a covariety.

This example demonstrates, that an arbitrary intersection of $\mathcal{F}$-subcoalgebras need not result in a subcoalgebra and that 1-generated subcoalgebras need not exist.

As a consequence for the general theory, we conclude that preservation of weak pullbacks must be interpreted as preservation of *weak generalized pullbacks* if a satisfactory structure theory of universal coalgebra is to be achieved. We close with an appropriate criterion.

2. A criterion for preservation of weak pullbacks

2.1. *Retracts*

In any category $\mathcal{C}$ we can define an order relation amongst objects which will help us to describe the relations between pullbacks and weak pullbacks. If an object is a weak pullback of given morphisms, then so will be any other object that is larger in this order. The pullback of maps f and g, if it exists, will be the smallest element in this order of all weak pullbacks of f and g.

DEFINITION 2.1. Let A and B be objects, we say that A is a *retract* of B, or, equivalently, that B is a *coretraction* of A, and we write $A \preceq B$, if there are morphisms $\iota : A \to B$ and $\kappa : B \to A$ such that $\kappa \circ \iota = 1_A$. κ is also called a *split epi*, and we sometimes write $A \preceq_\kappa B$ to indicate the split epi witnessing $A \preceq B$.

We readily observe that $\preceq$ is a reflexive and transitive relation, i.e. a quasi-order, on the class of objects of $\mathcal{C}$, and that every functor $T : \mathcal{C} \to \mathcal{C}$ is order preserving.

2.2. *Weak Pullbacks*

Pullbacks are limits of two morphisms with a common codomain, *weak pullbacks* are weak limits of the same situation, specifically:

DEFINITION 2.2. Let $f : A \to C$ and $g : B \to C$ be morphisms. An object W with morphisms $\pi_1 : W \to A$ and $\pi_2 : W \to B$ is called a *weak pullback* of f and g, if $f \circ \pi_1 = g \circ \pi_2$ and for any other object W' with morphisms $\pi_1' : W' \to A$ and $\pi_2' : W' \to B$ satisfying $f \circ \pi_1' = g \circ \pi_2'$ there is a morphism $\epsilon : W' \to W$ such that $\pi_i' = \pi_i \circ \epsilon$ for $i = 1, 2$. If such an ϵ is always unique, then W is called a *pullback of f and g* and denoted by $pb(f, g)$.

$$\begin{tikzcd}
& A \arrow[r, "f"] & C \\
& W \arrow[u, "\pi_1"'] \arrow[r, "\pi_2"] & B \arrow[u, "g"'] \\
W' \arrow[uur, "\pi_1'"] \arrow[ur, dotted, "\epsilon"'] \arrow[urr, "\pi_2'"'] & &
\end{tikzcd}$$

Usually, π_1 and π_2 will be clear from the context. In that case we shall simply refer to the object W as a (weak) pullback of f and g. The following observation can be easily checked:

LEMMA 2.3. *If (W, π_1, π_2) is a weak pullback of f and g and $W \preceq_\kappa W'$, then $(W', \pi_1 \circ \kappa, \pi_2 \circ \kappa)$ is also a weak pullback of f and g.*

LEMMA 2.4. *If the pullback (P, π_1, π_2) of f and g exists, then (W, η_1, η_2) is a weak pullback of f and g, if and only if $P \preceq_\kappa W$, and $\eta_i = \pi_i \circ \kappa$.*

Thus, in a category where pullbacks exist, weak pullbacks are precisely the coretractions of pullbacks. Moreover, pullbacks are just the minimal elements, with respect to $\preceq$, amongst all weak pullbacks.

DEFINITION 2.5. Let $T : \mathcal{C} \to \mathcal{C}$ be a functor. We say that T *preserves (weak) pullbacks,* if T transforms every (weak) pullback (P, π_1, π_2) of $f : A \to C$ with $g : B \to C$ into a (weak) pullback $(T(P), T(\pi_1), T(\pi_2))$ of $T(f)$ with $T(g)$.

We say that T *weakly preserves pullbacks* if T transforms every pullback diagram into a weak pullback diagram.

From the above observations and from the fact that T is order preserving, one easily obtains:

LEMMA 2.6. *Let $\mathcal{C}$ be a category in which all pullbacks exist, and let $T : \mathcal{C} \to \mathcal{C}$ be a functor. Then the following are equivalent:*

(i) *T preserves weak pullbacks,*
(ii) *T weakly preserves pullbacks.*

COROLLARY 2.7. ([Rut96]). *If T preserves pullbacks, then T preserves weak pullbacks.*

2.3. **Set** *endofunctors*

We now turn to the category **Set**. Here pullbacks of two maps $f : A \to C$ with $g : B \to C$ always exist. They can be easily described as $(pb(f, g), \pi_1, \pi_2)$ where

$$pb(f, g) = \{(a, b) \mid f(a) = g(b)\},$$

and π_1, π_2 are just the natural projection to the components. Here we can formulate an easy criterion for weak pullback preservation:

THEOREM 2.8. *A functor T : **Set** $\to$ **Set** preserves weak pullbacks if and only if for all maps $f : A \to C$, $g : B \to C$ we have:*

For any pair $u \in T(A)$, $v \in T(B)$ with $T(f)(u) = T(g)(v)$ there is an element $w \in T(\{(x, y) \mid f(x) = g(y)\})$ with $T(\pi_1)(w) = u$ and $T(\pi_2)(w) = v$.

Proof. Let P with morphisms π_1 and π_2 be the pullback of f and g, and let Q with morphisms η_1 and η_2 be the pullback of $T(f)$ and $T(g)$. Clearly, $T(f) \circ T(\pi_1) = T(g) \circ T(\pi_2)$, so from the definition of pullback there is a unique map $\kappa : T(P) \to Q$ satisfying $\eta_i \circ \kappa = T(\pi_i)$ for $i = 1, 2$. If T preserves weak pullbacks, then $T(P)$ is a weak pullback, whence we also get a map $\iota : Q \to T(P)$ with $T(\pi_i) \circ \iota = \eta_i$. Now

$$\begin{aligned}
\eta_i \circ \kappa \circ \iota &= T(\pi_i) \circ \iota \\
&= \eta_i \\
&= \eta_i \circ id_Q,
\end{aligned}$$

hence $\kappa \circ \iota = id_Q$, as Q with morphisms η_1, η_2 is a pullback. In particular, κ is onto, so every element $(u, v) \in Q = pb(T(f), T(g))$ must have a κ-preimage $w \in T(P) = T(pb(f, g))$, which is the content of the criterion.

$$\begin{tikzcd}
& T(A) \arrow[r, "T(f)"] & T(C) \\
& Q \arrow[u, "\eta_1"'] \arrow[r, "\eta_2"] & T(B) \arrow[u, "T(g)"'] \\
T(P) \arrow[uur, "T(\pi_1)"] \arrow[ur, dotted, "\kappa"'] \arrow[urr, "T(\pi_2)"'] & &
\end{tikzcd}$$

Conversely, if the criterion is satisfied, then the map $\kappa : T(P) \to Q$, defined by

$$\kappa(w) := (T(\pi_1)(w), T(\pi_2)(w))$$

is surjective and $\eta_i \circ \kappa = T(\pi_i)$ for $i = 1, 2$. In **Set** every surjective map is a split epi, so the result follows from Lemma 2.4. $\square$

3. The filter functor

We apply the criterion to the functor $\mathcal{F}$ on **Set** which associates a set X with the set of all filters on X. For this we need a few definitions.

DEFINITION 3.1. Let A, C be sets and $\varphi : A \to C$ a map. A collection G of subsets of A is called *downward directed*, if for $U, V \in G$ there always exists a $W \in G$ with $W \subseteq U \cap V$. A nonempty downward directed collection G is called a *filter on* A, if $V \supseteq U \in G$ always implies $V \in G$.

Given a nonempty downward directed set $G \subseteq \mathcal{P}(A)$, then

$$\uparrow G = \{U \subseteq A \mid \exists U' \in G.\ U \supseteq U'\}$$

is the filter generated by G. With $\varphi(G)$ we denote the set of all φ-images of sets from G, i.e.

$$\varphi(G) = \{\varphi(U) \mid U \in G\}.$$

LEMMA 3.2. *Let G and H be collections of subsets of A.*

(i) *If G is downward directed, then so is $\varphi(G)$.*
(ii) *If G and H are filters then $G \subseteq H$ iff for all $U \in G$ there exists a $V \in H$ such that $V \subseteq U$.*
(iii) $\uparrow \varphi(G) = \uparrow \varphi(\uparrow G)$.

All these properties are easy to check.

We now denote by $\mathcal{F}(X)$ the set of all filters on X. $\mathcal{F}$ can be made into a functor $\mathcal{F} : \mathbf{Set} \to \mathbf{Set}$ by defining it on maps $\varphi : A \to B$ as

$$\mathcal{F}(\varphi)(G) = \uparrow \varphi(G)$$

where G stands for an arbitrary filter on A.

LEMMA 3.3. *$\mathcal{F}$ is a covariant functor.*

Proof.

$$\begin{aligned}\mathcal{F}(id)(G) &= \uparrow id(G)\\ &= \uparrow G\\ &= G.\end{aligned}$$

$$\begin{aligned}\mathcal{F}(\varphi\circ\psi)(G) &= \uparrow(\varphi\circ\psi)(G)\\ &= \uparrow\varphi(\psi(G))\\ &= \uparrow\varphi(\uparrow\psi(G))\\ &= \mathcal{F}(\varphi)(\mathcal{F}(\psi)(G))\\ &= (\mathcal{F}(\varphi)\circ\mathcal{F}(\psi))(G).\end{aligned}$$

□

The next lemma will turn out to be crucial for the proof of the main theorem:

LEMMA 3.4. *Let $\varphi : A \to C$ and $\psi : B \to C$ be maps, G a filter on A and H a filter on B. If $\mathcal{F}(\varphi)(G) = \mathcal{F}(\psi)(H)$ then for any $U \in G$ and $V \in H$ we can find $\hat{U} \in G$, $\hat{V} \in H$ with $\hat{U} \subseteq U$, $\hat{V} \subseteq V$ and $\varphi(\hat{U}) = \psi(\hat{V})$.*

Proof. Since $U \in G$ and $\mathcal{F}(\varphi)(G) \subseteq \mathcal{F}(\psi)(H)$ there exists $V' \in H$ with $\psi(V') \subseteq \varphi(U)$, hence for $V'' = V \cap V'$ we also have $\psi(V'') \subseteq \varphi(U) \cap \psi(V)$, so $V'' \subseteq \psi^{-1}(\varphi(U) \cap \psi(V))$. Put

$$\hat{V} = V \cap \psi^{-1}(\varphi(U) \cap \psi(V)),$$

then $V'' \subseteq \hat{V} \subseteq V$ and $\hat{V} \in H$. Clearly, $\psi(\hat{V}) \subseteq \varphi(U) \cap \psi(V)$, but the reverse inclusion holds too, since any $c \in \varphi(U) \cap \psi(V)$ can be written as $\psi(y)$ for some $y \in V \cap \psi^{-1}(\varphi(U) \cap \psi(V))$, yielding $c \in \psi(\hat{V})$. Thus $\psi(\hat{V}) = \varphi(U) \cap \psi(V)$ which, by symmetry, is equal to $\varphi(\hat{U})$. □

Now we have everything in place to state and prove the main result of this section:

THEOREM 3.5. *$\mathcal{F}$ preserves weak pullbacks.*

Proof. Let $\varphi : A \to C$ and $\psi : B \to C$ be maps. Let us abbreviate their pullback by

$$K_{\varphi,\psi} = \{(x, y) \in A \times B \mid \varphi(x) = \psi(y)\}.$$

Given $G \in \mathcal{F}(A)$ and $H \in \mathcal{F}(B)$ with $\mathcal{F}(\varphi)(G) = \mathcal{F}(\psi)(H)$, Theorem 2.8 requires us to find a filter R on $K_{\varphi,\psi}$ with $\mathcal{F}(\pi_1)(R) = G$ and $\mathcal{F}(\pi_2)(R) = H$. Put

$$R = \uparrow\{(U \times V) \cap K_{\varphi,\psi} \mid U \in G,\ V \in H,\ \varphi(U) = \psi(V)\}.$$

Here Lemma 3.4 will be needed to verify that R is downward directed, hence a filter on $K_{\varphi,\psi}$. Indeed, given $U_1, U_2 \in G$ and $V_1, V_2 \in H$ with $\varphi(U_i) = \psi(V_i)$, then Lemma 3.4 provides us with $\hat{U} \subseteq U_1 \cap U_2$ and $\hat{V} \subseteq V_1 \cap V_2$, satisfying $\hat{U} \in G$, $\hat{V} \in H$, and $\varphi(\hat{U}) = \psi(\hat{V})$, thus $(\hat{U} \times \hat{V}) \cap K_{\varphi,\psi} \subseteq ((U_1 \times V_1) \cap K_{\varphi,\psi}) \cap ((U_2 \times V_2) \cap K_{\varphi,\psi})$.

Whenever $U \in G$ and $V \in H$ with $\varphi(U) = \psi(V)$, we obtain $\pi_1((U \times V) \cap K_{\varphi,\psi}) = U$, hence $\pi_1(R) \subseteq G$. It follows that $\mathcal{F}(\pi_1)(R) = \uparrow\{\pi_1(W) \mid W \in R\} \subseteq G$. Conversely, for any $U \in G$ choose an arbitrary $V \in H$. Again Lemma 3.4 provides $\hat{U}, \hat{V}$ with $\hat{U} \subseteq U$ and $(\hat{U} \times \hat{V}) \cap K_{\varphi,\psi} \in R$. Hence $U \in \uparrow\{\pi_1(W) \mid W \in R\} = \mathcal{F}(\pi_1)(R)$. Thus $G = \mathcal{F}(\pi_1)(R)$ and, symmetrically, $H = \mathcal{F}(\pi_2)(R)$, as was required to be shown in the application of Theorem 2.8. □

4. $\mathcal{F}$-coalgebras

We now study coalgebras of the functor $\mathcal{F}$. Let $T : \mathbf{Set} \to \mathbf{Set}$ be any functor, A a set and $\alpha : A \to T(A)$ a mapping. The pair (A, α) is called a *T-coalgebra*. When α is clear from the context, we shall simply call A a T-coalgebra.

In the case of the filter functor $\mathcal{F}$, $\alpha(a)$ is a set, so borrowing notation from transition systems, we sometimes write $a \overset{\alpha}{\longrightarrow} U$ instead of $U \in \alpha(a)$. Again, if α is understood, we simply write $a \longrightarrow U$. Thus the fact that $\alpha(a)$ is a filter translates into the conditions

(i) $a \to U,\ a \longrightarrow V \Longrightarrow a \longrightarrow U \cap V$, and
(ii) $a \longrightarrow U,\ U \subseteq V \Longrightarrow a \longrightarrow V$.

4.1. *Homomorphisms*

Let (A, α_A) and (B, α_B) be two T-coalgebras, then a map $\varphi : A \to B$ is called a *homomorphism* if the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\alpha_A"'] & B \arrow[d, "\alpha_B"] \\ T(A) \arrow[r, "T(\varphi)"] & T(B) \end{tikzcd}$$

In the case of $\mathcal{F}$-coalgebras we obtain the following conditions.

PROPOSITION 4.1. *A map φ between $\mathcal{F}$-coalgebras A and B is a homomorphism iff the following two conditions are satisfied for any $a \in A$, $U \subseteq A$ and $V \subseteq B$:*

(i) $a \longrightarrow U \Longrightarrow \varphi(a) \longrightarrow \varphi(U)$
(ii) $\varphi(a) \longrightarrow V \Longrightarrow a \longrightarrow \varphi^{-1}(V)$.

Proof. The homomorphism diagram requires that $\mathcal{F}(\varphi)(\alpha_A(a)) = \alpha_B(\varphi(a))$. The inclusion "$\subseteq$" translates directly into the first condition, whereas the reverse inclusion translates into

$$\varphi(a) \longrightarrow V \Longrightarrow \exists U.\, a \longrightarrow U, \varphi(U) \subseteq V.$$

The conclusion of this condition implies $U \subseteq \varphi^{-1}(V)$, hence $a \longrightarrow \varphi^{-1}(V)$. □

Note that using (i) we can infer $a \longrightarrow \varphi^{-1}(V) \Longrightarrow \varphi(a) \longrightarrow \varphi\varphi^{-1}(V) \subseteq V \Longrightarrow \varphi(a) \longrightarrow V$, which is just the converse of (ii). Similarly, from the converse of (ii) we can infer (i), so we get:

COROLLARY 4.2. *$\varphi : A \to B$ is a homomorphism iff for all $a \in A$ and all $V \subseteq B$*

$$\varphi(a) \longrightarrow V \iff a \longrightarrow \varphi^{-1}(V).$$

4.2. *Subcoalgebras*

If $A \subseteq B$ and (A, α_A), (B, α_B) are coalgebras, then A is called a *subcoalgebra* of B, if the natural inclusion map $\iota : A \to B$ is a homomorphism. The above corollary, with $\varphi = \iota$, requires that for any $a \in A$ and $V \subseteq B$:

$$a \xrightarrow{\alpha_B} V \iff a \xrightarrow{\alpha_A} A \cap V.$$

In particular, for any $U \subseteq A$ this means

$$a \xrightarrow{\alpha_B} U \iff a \xrightarrow{\alpha_A} U.$$

The formula shows how the coalgebra structure on the subcoalgebra A is uniquely determined by the coalgebra structure on B. Therefore, we can speak of a subset A of B as being a subcoalgebra and we write $A \leq B$ in this case.

PROPOSITION 4.3. *A subset $S \subseteq A$ is an $\mathcal{F}$-subcoalgebra of (A, α) iff $s \xrightarrow{\alpha} S$ for each $s \in S$.*

Proof. If S is to be a coalgebra, then for each $s \in S$ we must have $s \xrightarrow{\alpha_S} S$, hence $s \xrightarrow{\alpha} S$. Conversely, we define a transition structure on S by

$$\alpha_S(s) = \{S \cap V \mid V \in \alpha(s)\}.$$

The condition $s \xrightarrow{\alpha} S$, i.e. $S \in \alpha(s)$ guarantees that $\alpha_S(s)$ is a filter on S, so (S, α_S) becomes an $\mathcal{F}$-coalgebra. For every $V \subseteq S$, obviously $s \xrightarrow{\alpha} V \iff s \xrightarrow{\alpha_S} S \cap V$, so (S, α_S) is indeed a subcoalgebra of (A, α). $\square$

From the general theory of coalgebras ([Rut96]) or directly from this proposition one can infer that arbitrary unions and – as a consequence of $\mathcal{F}$ preserving pullbacks – also finite intersections of subcoalgebras are again subcoalgebras. Thus the collection of all subcoalgebras of a given $\mathcal{F}$-coalgebra defines a topological space.

4.3. *Sums*

Sums of coalgebras always exist. Given a family $(A_i, \alpha_i)_{i \in I}$ then the disjoint union $A = (\bigcup_{i \in I} A_i)$ may canonically be endowed with a coalgebra structure α so that each (A_i, α_i) is a subcoalgebra of (A, α). In the case of the functor $\mathcal{F}$, we have $a \xrightarrow{\alpha} U$ in (A, α) if and only if $a \xrightarrow{\alpha_i} A_i \cap U$ in (A_i, α_i) where A_i is the component of A containing a.

5. Topological spaces, examples and counterexamples

5.1. *Topological coalgebras*

From topological spaces we can obtain concrete examples of $\mathcal{F}$-coalgebras. Given a topological space (A, τ), we turn it into an $\mathcal{F}$-coalgebra using the co-operation that associates with every point $a \in A$ the filter $\mathcal{U}_\tau(a)$ of all τ-neighbourhoods, i.e.:

$$V \in \mathcal{U}_\tau(a) \iff \exists\, \mathcal{O} \in \tau.\, a \in \mathcal{O} \subseteq V.$$

DEFINITION 5.1. An $\mathcal{F}$-coalgebra (A, α) is called *topological* if there exists a topology τ on A so that for all $a \in A$

$$\alpha(a) = \mathcal{U}_\tau(a).$$

With our criteria for $\mathcal{F}$-homomorphisms and $\mathcal{F}$-subcoalgebras (Propositions 4.1 and 4.3), the following proposition is easy to check:

PROPOSITION 5.2. *Let A and B be topological spaces. A map $\varphi : A \to B$ is a coalgebra homomorphism iff it is continuous and open. A subset S of a topological space A is a subcoalgebra iff it is open.*

Topological spaces - as coalgebras - may be characterized within the class of all $\mathcal{F}$-coalgebras. From the above we readily obtain an internal characterization of topological $\mathcal{F}$-coalgebras:

PROPOSITION 5.3. *An $\mathcal{F}$-coalgebra A is topological if and only if for every $a \in A$ and $U \subseteq A$ we have*

$$a \longrightarrow U \Longrightarrow \exists S \leq A.\, a \in S \subseteq U.$$

Essentially, this criterion tells us that A is topological precisely when it is topological with respect to the topology given by its collection of subcoalgebras. Note that the converse of the above implication is true in any $\mathcal{F}$-coalgebra.

5.2. *The covariety of topological coalgebras*

In analogy to notions in universal algebra, a class of coalgebras is called a *covariety* if it is closed under the formation of *homomorphic images*, *subcoalgebras* and *sums*. Using the above internal characterization of topological coalgebras we get:

THEOREM 5.4. *The class of all topological coalgebras is a covariety.*

Proof. Consider first a surjective homomorphism $\varphi : A \to B$ and assume that A is topological. For an arbitrary $b \in B$ choose an $a \in A$ with $\varphi(a) = b$. For any $V \subseteq B$ we calculate

$$\begin{aligned} b \longrightarrow V &\Longrightarrow \varphi(a) \longrightarrow V \\ &\Longrightarrow a \longrightarrow \varphi^{-1}(V) \\ &\Longrightarrow \exists S \leq A.\, a \in S \subseteq \varphi^{-1}(V) \\ &\Longrightarrow \exists S \leq A.\, b \in \varphi(S) \subseteq \varphi(\varphi^{-1}(V)) \\ &\Longrightarrow \exists T \leq B.\, b \in T \subseteq V. \end{aligned}$$

For the last implication we can use $T = \varphi(S)$ and the fact that the homomorphic image of a subcoalgebra is again a subcoalgebra of a homomorphic image.

Next, let (A, α) be a subcoalgebra of (B, β) and assume that the latter is topological, then for $a \in A$ and $U \subseteq A$ we conclude

$$\begin{aligned} a \xrightarrow{\alpha} U &\Longrightarrow a \xrightarrow{\beta} U \\ &\Longrightarrow \exists S \leq B.\, a \in S \subseteq U \\ &\Longrightarrow \exists S \leq A.\, a \in S \subseteq U, \end{aligned}$$

since $S \subseteq U \subseteq A$.

Finally, let (A, α) be the sum of the (A_i, α_i) and $a \in A$, then there exists an i with $a \in A_i$, thus for any $U \subseteq A$

$$\begin{aligned} a \xrightarrow{\alpha} U &\Longrightarrow \exists U_i \subseteq A_i.\, a \xrightarrow{\alpha_i} U_i \subseteq U \\ &\Longrightarrow \exists S \leq A_i.\, a \in S \subseteq U_i \subseteq U \\ &\Longrightarrow \exists S \leq A.\, a \in S \subseteq U. \end{aligned}$$

□

Note that another covariety, properly containing the covariety of all topological $\mathcal{F}$-coalgebras could be defined by

$$a \longrightarrow U \Longrightarrow a \in U.$$

5.3. *Examples and Counterexamples*

Our initial motivation for studying the functor $\mathcal{F}$ was to investigate the usefulness of our criterion for the preservation of weak pullbacks, but also to check whether for a satisfactory theory of universal coalgebras it is enough to request that the type functor preserves such weak pullbacks (of finitely many maps) or whether it is necessary to have preservation of *weak generalized pullbacks* (cf. [Rut96]), that is weak limits of arbitrary collections of arrows with a common codomain.

Since in most topological spaces the intersection of arbitrarily many open sets will not be open, they readily provide us with examples of $\mathcal{F}$-coalgebras where the intersection of arbitrarily many subcoalgebras is not a subcoalgebra. Even the intersection of all subcoalgebras containing a given point x need not be a subcoalgebra, so the notion of "1-generated subcoalgebra", which is central in [GS98] and also in [Gum98] is not available, in spite of the fact that the type functor $\mathcal{F}$ preserves weak pullbacks.

We do, of course, have that the intersection of finitely many subcoalgebras is a subcoalgebra. This is a consequence of the fact that $\mathcal{F}$ preserves weak pullbacks. On the other hand one might ask whether this is actually equivalent to the preservation of weak pullbacks. The following proposition gives a negative answer:

PROPOSITION 5.5. *There is a functor $\bar{\mathcal{P}}\bar{\mathcal{P}}$: **Set** $\to$ **Set** which does not preserve weak pullbacks, yet the intersection of finitely many $\bar{\mathcal{P}}\bar{\mathcal{P}}$-subcoalgebras is again a $\bar{\mathcal{P}}\bar{\mathcal{P}}$-subcoalgebra.*

Proof. Let $\bar{\mathcal{P}}\bar{\mathcal{P}}$ be the composition $\bar{\mathcal{P}} \circ \bar{\mathcal{P}}$ of the contravariant powerset functor $\bar{\mathcal{P}}$ with itself. On objects $X \in$ **Set** we have $\bar{\mathcal{P}}\bar{\mathcal{P}}(X) = \mathcal{P}(\mathcal{P}(X))$, that is the set of all collections of subsets of X. Given a map $\varphi : X \to Y$, we obtain $\bar{\mathcal{P}}\bar{\mathcal{P}}(\varphi)(G) = \{V \subseteq Y \mid f^{-1}(V) \in G\}$ for any collection $G \subseteq \mathcal{P}(X)$. In [Rut96] it is shown that this functor does not preserve weak pullbacks.

A $\bar{\mathcal{P}}\bar{\mathcal{P}}$-coalgebra (A, α) associates with every $a \in A$ a collection $\alpha(a) \subseteq \mathcal{P}(X)$ so we use the same convention as introduced in Section 4, writing $a \xrightarrow{\alpha} U$ or simply $a \longrightarrow U$ for $U \in \alpha(a)$. Interestingly, the homomorphism condition is identical to the one already seen for $\mathcal{F}$-coalgebras (Corollary 4.2), that is a map $\varphi : A \to B$ is a homomorphism between $\bar{\mathcal{P}}\bar{\mathcal{P}}$-coalgebras A and B iff for all $a \in A$ and $V \subseteq B$

$$a \longrightarrow \varphi^{-1}(V) \iff \varphi(a) \longrightarrow V.$$

Now (A, α) is a subcoalgebra of (B, β) iff $\subseteq$ is a homomorphism, that is iff for all $a \in A$ and $V \subseteq B$

$$a \xrightarrow{\alpha} A \cap V \iff a \xrightarrow{\beta} V. \tag{*}$$

This formula again shows how the structure map on the subcoalgebra A of B is determined by the structure map on B. The following lemma characterizes those subsets S of a $\bar{\mathcal{P}}\bar{\mathcal{P}}$-coalgebra A which are carriers of subcoalgebras of A, in which case we again write $S \leq A$:

LEMMA 5.6. *A subset $S \subseteq A$ is a subcoalgebra of (A, α) iff for all $s \in S$ and all $V \subseteq A$*

$$s \xrightarrow{\alpha} V \iff s \xrightarrow{\alpha} S \cap V.$$

Proof. If this condition is satisfied, we can define a structure map α_S on S by $\alpha_S(s) = \alpha(s) \cap \mathcal{P}(S)$. For $s \in S$ and $V \subseteq A$ we check condition (*):

$$\begin{aligned} s \xrightarrow{\alpha_S} S \cap V \quad &\iff \quad s \xrightarrow{\alpha} S \cap V \\ &\iff \quad s \xrightarrow{\alpha} V. \end{aligned}$$

Conversely, if S is a subcoalgebra, then we apply condition (*) twice to obtain for $s \in S$ and $V \subseteq A$:

$$\begin{aligned} s \xrightarrow{\alpha} V \quad &\iff \quad s \xrightarrow{\alpha_S} S \cap V \\ &\iff \quad s \xrightarrow{\alpha} S \cap V. \end{aligned}$$

Continuing with the proof of Proposition 5.5, we need to show that the intersection of two subcoalgebras is a subcoalgebra. However, the criterion of the lemma just proven obviously holds for $S_1 \cap S_2$, provided it holds for S_1 and for S_2. $\square$

6. Conclusion

We have given a criterion for checking whether an endofunctor T preserves weak pullbacks. Applying this criterion on the filter functor $\mathcal{F}$ we have found that $\mathcal{F}$ does preserve weak pullbacks, yet $\mathcal{F}$-coalgebras may have some undesirable properties. In particular, intersections of subcoalgebras need not be subcoalgebras.

Thus it becomes apparent that we will need to require that a functor T preserves weak generalized pullbacks ([Rut96]) in order to yield a well behaved theory of coalgebras. Here a generalized pullback is the limit of a set of morphism with a common codomain.

All results from Subsections 2.1 and 2.2 can be formulated (and proved) in exactly the same way with any other diagram $\mathcal{D}$ in place of the one (consisting of two arrows with common target) defining pullbacks. We only have to replace "pullback" and "weak pullback" by "$\mathcal{D}$-limit and "weak $\mathcal{D}$-limit".

In particular, for the case of generalized pullbacks, i.e. limits of a family of maps with common codomain, we get a result corresponding to Theorem 2.8:

PROPOSITION 6.1. *A functor T : **Set** $\to$ **Set** preserves weak generalized pullbacks iff for any family $(f_i : A_i \to C)_{i\in I}$ of maps with common codomain C we have:*

Given any family of elements $(u_i)_{i\in I}$ where $T(f_i)(u_i) = T(f_j)(u_j)$ for all $i, j \in I$, there exists an element $w \in T(\{(x_i)_{i\in I} | \forall_{j,k\in I} . f_j(x_j) = f_k(x_k)\})$ such that $T(\pi_i)(w) = u_i$ for all $i \in I$.

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