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Equational and implicational classes of coalgebras

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Abstract

If F : **Set** $\to$ **Set** is a functor which is bounded and preserves weak generalized pullbacks then a class of F-coalgebras is a covariety, i.e., closed under $\mathscr{H}$ (homomorphic images), $\mathscr{S}$ (sub-coalgebras) and $\sum$ (sums), if and only if it can be defined by a set of "coequations". Similarly, quasi-covarieties, i.e., classes closed under $\mathscr{H}$ and $\sum$, can be characterized by implications of coequations. These results are analogous to the theorems of Birkhoff and of Mal'cev in classical universal algebra. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The recently developed theory of coalgebras under a functor F provides a highly attractive framework for describing the semantics and the logic of various types of transition systems. In contrast to the algebraic semantics of abstract data types where data objects are constructed recursively and equality is proven by induction, coalgebras support definitions by co-recursion and define equivalence by co-induction. This view is appropriate in many contexts, prominently when modelling objects and classes in object-oriented languages [6, 4] or infinite data objects such as processes and streams.

1.1. Transitions and transition systems

A *transition* Θ is nothing but a binary relation on a set S, i.e. $\Theta \subseteq S \times S$. Θ is called *image finite*, if for every $s \in S$, the set $s\Theta = \{t \in S \mid s\Theta t\}$ is finite. Θ is called *deterministic* if it is the graph of a function $\theta : S \to S$, i.e., $\Theta = \{(s, \theta(s)) \mid s \in S\}$.

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A *transition system* is a family $T = (\Theta_a)_{a \in A}$ of transitions on S. Related to this notion is that of an automaton where additionally one considers a set $Q \subseteq S$ of *accepting states*, and perhaps an *output function* $\gamma : S \to B$.

In order to emphasize the dynamical aspect of transitions or transition systems, we describe them by a map α from S to some structured set. Unary relations are be modelled by a map into $Bool = \{true, false\}$ and binary relations $R \subseteq S \times S$ by a map from S into the powerset $\mathscr{P}(S)$. With $\mathscr{P}_{fin}(S)$ we denote the lattice of finite subsets of S.

In particular, a map α of type

$S \to S$	is a deterministic transition,
$S \to \mathscr{P}(S)$	is a nondeterministic transition (binary relation),
$S \to \mathscr{P}_{fin}(S)$	is an image finite nondeterministic transition,
$S \to S^A$	is a deterministic transition system,
$S \to \mathscr{P}_{fin}(S)^A$	models a nondeterministic transition system in which all transitions are image finite,
$S \to B \times S^A$	is an automaton with output, and
$S \to \mathscr{P}_{fin}(S)^A \times Bool$	models an automaton with bounded nondeterminism and an acceptance condition.

In all examples we are given a map from a set S into a set $F(S)$ that is somehow constructed from S. In fact, in each case F is a functor. A coalgebra of type F will be defined as any map $\alpha_S : S \to F(S)$.

The situation is dual to that of universal algebra, where an algebraic structure is given by a map $f^A : F(A) \to A$, where $F(A) = A^{n_1} + A^{n_2} + \cdots + A^{n_k}$, that is a disjoint union of powers of A. Coalgebras as direct dualizations of universal algebras, to be precise, universal algebras in the category $\mathbf{Set}^{op}$, have been investigated by Marvan [5], where special cases of many of the notions and results mentioned in the present paper can be found. In his case, a coalgebra is just a collection of maps $\alpha_i : S \to n_i \cdot S$ from a set into its n_i-fold disjoint union.

For coalgebras relevant in computer science applications, other functors, such as the ones listed above, are needed, for which there is no known theory of *F-algebras* to be dualized. In particular, the functors $\mathscr{P}(-)$ and $\mathscr{P}_{fin}(-)$ are of great importance in applications, but also nonstandard functors such as the "filter functor" $\mathscr{F}(-)$ are of interest, whose coalgebras include all topological spaces (see [2]).

In this note, we shall continue the investigation (started in [7] and continued in [3]) of covarieties, that is classes of coalgebras closed under formation of subcoalgebras, homomorphic images and direct sums. We shall introduce the notion of coequation and prove a theorem analogous to the well-known theorem of Birkhoff, stating that a class of coalgebras of type F is a covariety iff it can be defined by a set of coequations. In analogy to a theorem of Mal'cev, we then introduce quasi-covarieties as classes closed under sums and homomorphic images and we show that quasi-covarieties are precisely the classes of coalgebras which can be specified by co-implications. Most results of this work have been presented at the workshop on Coalgebraic Methods in Computer Science in Lisbon, in March of 1998.

2. Coalgebras

In this section, we collect definitions and basic results of the general theory of universal coalgebra as developed in the comprehensive exposition of Rutten [7].

Definition 1. Let $F : \mathbf{Set} \to \mathbf{Set}$ be a functor. A *coalgebra of type F* is a pair (A, α_A) consisting of a set A and a map $\alpha_A : A \to F(A)$. A is called the *underlying set* or *carrier* of the coalgebra and α_A is called the *structure map*.

Whenever the structure map is clear from the context, we shall use the same notation for a coalgebra and for its carrier. For most of this paper, the functor F will be kept fixed, that is we shall only consider coalgebras of type F.

We shall make use of the axiom of choice, thus in the category **Set** every epi has a right inverse and every mono, whose domain is nonempty, has a left inverse. Consequently, F preserves epis and it preserves all monos whose domain is nonempty.

2.1. Homomorphisms

Definition 2. A *homomorphism* between coalgebras (A, α_A) and (B, α_B) is a structure-preserving map, that is a map $\varphi : A \to B$ for which the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_B} \\ F(A) & \xrightarrow{F(\varphi)} & F(B) \end{array} .$$

The class of all coalgebras of a fixed type F together with their homomorphisms becomes a category $\mathbf{Set}_F$. From this a number of standard coalgebraic constructions, such as subcoalgebras, homomorphic images, and sums are immediately derived.

It turns out [7], that epimorphisms in $\mathbf{Set}_F$ are surjective and bijective homomorphisms are isomorphisms. We say that A and B are *isomorphic*, in symbols $A \cong B$, if there exists an isomorphism from A to B. A homomorphism from A to A is called an *endomorphism*.

If $\varphi : A \twoheadrightarrow B$ is an epimorphism then we shall call B a *homomorphic image* of A. If A is isomorphic to each of its homomorphic images, then A is called *simple*.

2.2. Subcoalgebras

Definition 3. A coalgebra (S, α_S) is a *subcoalgebra* of (A, α_A) if $S \subseteq A$ and the natural embedding of S into A is a homomorphism.

The structure map α_S on a subcoalgebra S of A is uniquely determined by its carrier set, so we write $S \leqslant A$, if the subset S of A is the carrier set of a subcoalgebra of A. It is straightforward to check that the union of an arbitrary family of subcoalgebras is again a subcoalgebra, in particular, $\emptyset$ is always a subcoalgebra. Given a subset X of a coalgebra A, we denote by $[X]$ the union of all subcoalgebras of A which are contained in X.

Without further assumptions, the set theoretic intersection of subcoalgebras need not be a subcoalgebra. Nevertheless, the set of all subcoalgebras of a given coalgebra A forms a complete lattice where the join is given by set union and the meet of a family $(S_i)_{i \in I}$ is the union of all subcoalgebras contained in their intersection, i.e.,

$$\bigwedge_{i \in I} S_i = \left[\bigcap_{i \in I} S_i \right].$$

A subcoalgebra $S \leqslant A$ is called *invariant in A*, if it is preserved by every endomorphism of A, that is $\varphi(S) \subseteq S$ for each homomorphism $\varphi : A \to A$.

2.3. Sums

Given a family $(A_i, \alpha_i)_{i \in I}$ of coalgebras, let $e_i : A_i \to \sum_{i \in I} A_i$ be the canonical embedding of A_i into the disjoint union of the family $(A_i)_{i \in I}$. The coalgebra structure on $\sum_{i \in I} A_i$ is given by the canonical map that sends an $x \in \sum_{i \in I} A_i$ to $F(e_i)(\alpha_i(x))$, where A_i is the component to which x belongs. This construction yields precisely the sum of the family $(A_i, \alpha_i)_{i \in I}$ in the category $\mathbf{Set}_F$.

What we have here is actually an instance of a more general observation of Barr (see [1]), which states that the forgetful functor $U : \mathbf{Set}_F \to \mathbf{Set}$ creates colimits and every limit which is preserved by F.

A *conjunct sum* of a family $(A_i)_{i \in I}$ is a homomorphic image under some homomorphism φ of a sum $\sum_{i \in I} A_i$ for which the compositions $\varphi \circ e_i$ are monomorphisms.

2.4. Bisimulations

A *bisimulation* between coalgebras A and B is a binary relation $R \subseteq A \times B$ on which a coalgebra structure can be defined so that the canonical projections $\pi_A : R \to A$ and $\pi_B : R \to B$ are homomorphisms. It is easy to check that $\emptyset$ is always a bisimulation and the union of a collection of bisimulations is a bisimulation, so the set of all bisimulations between A and B forms a complete lattice.

Typical representatives of bisimulations are the graphs of homomorphisms, where for $f : A \to B$, its graph is the set $G(f) = \{(x, f(x)) \mid x \in A\}$, in fact, a map $f : A \to B$ is a homomorphism iff its graph is a bisimulation [7].

2.5. Preservation of weak generalized pullbacks

All of the functors mentioned in the introduction satisfy an extra property, which is an important source of additional coalgebraic structure.

Recall that a *pullback* is a limit of two morphisms with a common codomain. By a *generalized pullback* we understand the limit of an arbitrary collection $(\varphi_i)_{i\in I}$ of maps with a common codomain.

The notion of *weak limit* is defined analogous to that of a limit, with the exception that the mediating morphism is not required to be unique. In particular, let $(\varphi_i : A_i \to C)_{i\in I}$ be a collection of morphisms with a common codomain C, then a *weak generalized pullback* consists of an object W and a collection of morphisms $(\pi_i : W \to A_i)_{i\in I}$ so that

(i) $\forall_{i,j\in I}.\ \varphi_i \circ \pi_i = \varphi_j \circ \pi_j$,

(ii) for any object W' and morphisms $(\pi'_i : W' \to A_i)_{i\in I}$ with $\forall_{i,j\in I}.\ \varphi_i \circ \pi'_i = \varphi_j \circ \pi'_j$, there is at least one morphism $\kappa : W' \to W$ satisfying $\pi_i \circ \kappa = \pi'_i$ for all $i \in I$.

It turns out that all functors mentioned in the introduction preserve weak generalized pullbacks, that is, they transform a weak generalized pullback diagram into another weak generalized pullback diagram. In [2] we give a criterion for checking whether a given functor preserves weak (generalized) pullbacks.

For the rest of this paper we shall assume that the functor F preserves weak generalized pullbacks. This has a number of consequences. [1]

Theorem 4 (Rutten). *If F preserves weak generalized pullbacks then*:

(i) *An arbitrary intersection of subcoalgebras is again a subcoalgebra.*

(ii) *In $\mathbf{Set}_F$, monomorphisms are injective maps.*

(iii) *Images and preimages of subcoalgebras under homomorphisms are again subcoalgebras.*

As a consequence of (i), for any set $X \subseteq A$ there is a smallest subcoalgebra of A containing X. This is called *the coalgebra generated by* X and denoted $\langle X\rangle$. For a singleton $\{x\}$ we write $\langle x\rangle$ instead of $\langle\{x\}\rangle$ and call this a *one-generated subcoalgebra*. Every coalgebra A then has a canonical representation as a conjunct sum of its one-generated subcoalgebras.

Correspondingly, the smallest invariant subcoalgebra containing a set X is denoted by $\langle\langle X\rangle\rangle$, (resp. by $\langle\langle x\rangle\rangle$) if $X = \{x\}$. Observe that $\langle\langle X\rangle\rangle = \bigcup\{\varphi(\langle X\rangle) \mid \varphi : A \to A\}$.

3. Covarieties and Quasi-Covarieties

We will particularly be interested in certain subclasses of $\mathbf{Set}_F$ which are called *covarieties*. Here a covariety is a class of F-coalgebras closed under the operators $\mathscr{H}$ (homomorphic images), $\mathscr{S}$ (subcoalgebras), and $\sum$ (sums). Classes closed under $\mathscr{H}$ and under $\sum$ will be called *quasi-covarieties*.

[1] At various places in the literature, authors have erroneously assumed that preservation of weak pullbacks would guarantee existence of 1-generated subcoalgebras. This is false, as we have shown in [4].

It can be easily verified that a class K of coalgebras is a quasi-covariety, iff $K = \mathscr{H}\sum(K)$, and a covariety iff $K = \mathscr{H}\mathscr{S}\sum(K)$ (see [7]). Further descriptions of the covariety generated by a class K of coalgebras (for instance $K = \sum_C \mathscr{H}\mathscr{S}_1(K)$, where $\sum_C$ stands for "conjunct sums") are given in [3].

3.1. Bounded functors and cofree coalgebras

Let X be a set. We refer to the elements of X as "colors" and and to every set map from a coalgebra A to X as a "coloring". A coalgebra $C_K(X)$ together with a coloring $\varepsilon_X : C_K(X) \to X$ is called *cofree over* X, with respect to a class K, if for every coalgebra A in K and for any coloring $\varphi : A \to X$ there exists exactly one homomorphism $\tilde{\varphi} : A \to C_K(X)$ such that $\varphi = \varepsilon_X \circ \tilde{\varphi}$. We write $C(X)$ for $C_{\mathbf{Set}_F}(X)$

$$\begin{tikzcd} & X \\ A \arrow[ur, "\varphi"] \arrow[r, dashed, "\tilde{\varphi}"] & C_K(X) \arrow[u, "\varepsilon_X"'] \end{tikzcd} .$$

There is another way of looking at cofree coalgebras: By an "X-colored F-coalgebra", we shall understand a coalgebra A together with a map $\varphi : A \to X$. That is, an X-colored F-coalgebra is a coalgebra for the functor $X \times F(-)$. A cofree coalgebra $C(X)$ with its coloring ε_X is then nothing but a final object in the category of $X \times F(-)$-coalgebras.

Using this reduction and a result of Barr [1], Rutten [7] shows that cofree coalgebras exist, provided that there is a bound on the cardinality of one-generated F-coalgebras. In this case, the functor F is called *bounded*. It is easily seen that F is bounded if and only if $X \times F(-)$ is bounded. All of the functors mentioned in the introduction, with the exception of $\mathscr{P}(-)$, are bounded.

4. Coequations and coequational classes

In this section, we shall introduce a notion of "coequation" and show that, in analogy to Birkhoff's theorem of Universal Algebra, a class of coalgebras is a covariety if and only if it can be defined by a set of coequations.

Rutten already shows in [7] that every subcoalgebra S of $C(X)$ determines a covariety

$$K(S) = \{A \in \mathbf{Set}_F \mid \forall \varphi : A \to X.\tilde{\varphi}(A) \subseteq S\},$$

and conversely, every covariety arises in this way. Therefore, he considers S a *specification* of the covariety $K(S)$. In [3] it was shown that S may in fact be chosen as an invariant subcoalgebra, and that in this case the correspondence is one to one. More precisely, if F is bounded and preserves weak-generalized pullbacks, then there is a set X so that the invariant subcoalgebras of $C(X)$ are in bijective correspondence with the varieties of F-coalgebras.

4.1. Coequations

The above discussion shows that in order to check whether a coalgebra A belongs to the covariety defined by $S \leqslant C(X)$, one either has to check that every homomorphism $\varphi : A \to X$ factors through S, or, alternatively, that every element of $C(X) - S$ is *avoided*. This suggests the following definition:

Definition 5. (i) A *coequation* is an element of $C(X)$. More precisely, each $e \in C(X)$ is called a *coequation with colors in X*.

(ii) Given any coequation e with colors in X, a coalgebra A, an element $a \in A$ and a coloring map $\varphi : A \to X$, we say *e holds at $a \in A$ under φ* and we write $A, a \models_\varphi e$, if $\tilde{\varphi}(a) \neq e$.

(iii) If $A, a \models_\varphi e$ for every coloring φ, then we say *e holds at $a \in A$*. Finally, we say *e holds in A* and write $A \models e$, if $A, a \models e$ for all $a \in A$.

If $X \subseteq Y$ then the canonical map $\subseteq \circ \varepsilon_X : C(X) \to Y$ extends uniquely to a homomorphism from $C(X)$ to $C(Y)$ which is easily seen to be left cancellative, i.e. an embedding. In fact, $C(-)$ (as well as $C_K(-)$) is a **Set**-functor.

A concrete representation of $C(X)$ as a subcoalgebra of $C(Y)$ can be obtained in the following way:

$$C(X) = [\varepsilon_Y^{-1}(X)].$$

This allows us to speak of “the colors occurring in e” for any coequation e, more precisely:

Lemma 4.1. *For every coequation e with colors in a set Y there is a smallest set $X \subseteq Y$ such that e is a coequation with colors in X, in fact, $X = \varepsilon_Y(\langle e \rangle)$.*

Proof. Let e be a coequation with colors in Y and let $X = \varepsilon_Y(\langle e \rangle)$. Since $e \in \langle e \rangle \subseteq \varepsilon_Y^{-1}(X)$, we have $e \in [\varepsilon_Y^{-1}(X)] = C(X)$.

Let now $X' \subseteq Y$ with $e \in C(X') = [\varepsilon_Y^{-1}(X')]$, then there is a subcoalgebra $S \leqslant C(Y)$ with $e \in S \subseteq \varepsilon_Y^{-1}(X')$. Consequently, $\langle e \rangle \subseteq \varepsilon_Y^{-1}(X')$, so $X \subseteq X'$.

Also, we can “rename” the colors, in a coequation without affecting its validity, provided, the recoloring is injective. Identifying colors makes the coequation harder to be satisfied. That is, given a coequation e with colors in X and any mapping $f : X \to Y$, then $e' = (\widetilde{f \circ \varepsilon_X})(e)$ is a coequation with colors in Y and for every $a \in A \in \mathbf{Set}_F$ we have

$$A, a \models e' \text{ implies } A, a \models e. \qquad \square$$

4.2. Coequational classes

Definition 6. Let E be a set of coequations and K a class of F-coalgebras.
(i) The *coequational class* defined by E is

$$\mathbf{Mod}(E) := \{A \in \mathbf{Set}_F \mid \forall e \in E.A \models e\}.$$

(ii) Let X be a set(of colors). Then the set of all *coequations defined by* K is

$$\mathbf{CoEq}(K) := \{e \in C(X) \mid \forall A \in K.A \models e\}.$$

Clearly, any homomorphism $\psi : A \to C(X)$ arises from a coloring, specifically $\psi = \tilde{\varphi}$ for $\varphi = \varepsilon_X \circ \psi$. Thus, for a coequation whose covariables are amongst X, we have that $A \models e$ iff there does not exist any homomorphism $\psi : A \to C(X)$ with $e \in \psi(A)$. From this remark it follows immediately that $\mathbf{Mod}(E)$ is closed under homomorphic images and sums. To show that $\mathbf{Mod}(E)$ is also closed under subcoalgebras, note that a coloring of a subcoalgebra can always be extended to a coloring of the whole algebra, and, as a consequence, a homomorphism from a subcoalgebra $B \leqslant A$ to $C(X)$ can be extended to the whole coalgebra A, see [3]. These remarks prove the following lemma:

Lemma 4.2. *Let E be a set of coequations, then* $\mathbf{Mod}(E)$ *is closed under* $\mathscr{H}$, $\mathscr{S}$ *and* $\sum$, *i.e., a covariety.*

But the converse turns out to be true too. This is the coequational version of Birkhoff's theorem:

Theorem 4.3. *Covarieties are the same as coequational classes, specifically, for any class K of coalgebras,*

$$\mathbf{Mod}(\mathbf{CoEq}(K)) = \mathscr{H}\mathscr{S}\sum(K).$$

Proof. Clearly, $K \subseteq \mathbf{Mod}(\mathbf{CoEq}(K))$, hence $\mathscr{H}\mathscr{S}\sum(K) \subseteq \mathbf{Mod}(\mathbf{CoEq}(K))$ by Lemma 4.2.

For the converse inclusion, suppose $A \in \mathbf{Mod}(\mathbf{CoEq}(K))$. Let X be a bound for F and $a \in A$. Let $\imath : A \to X$ be a map that is injective on $\langle a \rangle \cdot \imath$ extends to a homomorphism $\tilde{\imath} : A \to C(X)$. Let $e = \tilde{\imath}(a)$, then $\langle a \rangle$ is isomorphic to $\langle e \rangle$.

Obviously, $A \not\models e$, hence there is some $B_e \in K$ so that $B_e \not\models e$, i.e., there is a homomorphism $\varphi_e : B_e \to C(X)$ with $e \in \varphi_e(B_e)$. $\varphi_e^{-1}(\langle e \rangle)$ is a subcoalgebra of B_e whose homomorphic image $\langle e \rangle$ is isomorphic to $\langle a \rangle$. Hence, $\langle a \rangle \in \mathscr{H}\mathscr{S}(K)$. Since A is a conjunct sum of its 1-generated subcoalgebras, it follows that $A \in \mathscr{H}\sum\mathscr{H}\mathscr{S}(K) \subseteq \mathscr{H}\mathscr{S}\sum(K)$. $\square$

4.3. A consequence relation

Let A be a coalgebra and $f \in C(X)$ a coequation. Instantly from the definition of "$\models$" we have

(i) $A \not\models f$ implies $A \not\models g$ for all $g \in \langle f \rangle$,
(ii) $A \not\models f$ implies $A \not\models \varphi(f)$ for every endomorphism φ of $C(X)$.
These two properties can be combined, referring to the notion of "invariant" subcoalgebra, introduced in Section 2.2:

$$A \not\models f \text{ implies } A \not\models g \text{ for all } g \in \langle\langle f \rangle\rangle.$$

We now redefine "$\models$" as a consequence relation on equations

$$E \models f :\Leftrightarrow \forall_{A \in \mathbf{Set}_F}. \ (\forall_{e \in E}. \ A \models e) \Rightarrow A \models f.$$

An algebraic characterization of this relation can be given as follows:

Theorem 4.4. *$E \models f$ if and only if $\langle\langle f \rangle\rangle \cap E \neq \emptyset$.*

Proof. Clearly, $\langle f \rangle \not\models f$, so

$$\begin{aligned}(E \models f) &\Rightarrow \exists_{e \in E}.\langle f \rangle \not\models e \\ &\Rightarrow \exists_{e \in E}.\exists_{\varphi:\langle f \rangle \to C(X)}.e \in \varphi(\langle f \rangle) \\ &\Rightarrow \exists_{e \in E}.\exists_{\bar{\varphi}:\mathscr{C}(X) \to C(X)}.e \in \bar{\varphi}(\langle f \rangle) \\ &\Rightarrow \langle\langle f \rangle\rangle \cap E \neq \emptyset.\end{aligned}$$

Conversely, assume $\langle\langle f \rangle\rangle \cap E \neq \emptyset$, then for some $e \in E$ and some endomorphism $\psi : C(X) \to C(X)$ we have $e \in \psi(\langle f \rangle)$. To show $E \models f$, consider a coalgebra A with $A \not\models f$, then there is a homomorphism $\varphi : A \to C(X)$ with $f \in \varphi(A)$. It follows $\langle f \rangle \leqslant \varphi(A)$ and $e \in \psi(\langle f \rangle) \leqslant \psi(\varphi(A))$, hence $A \not\models e \in E$. □

For the special case $E = \{e\}$ we obtain

Corollary 7. $e \models f \Leftrightarrow e \in \langle\langle f \rangle\rangle$.

5. Co-implications

Definition 8. If E is a set of coequations and f a single coequation then the expression $(E \Rightarrow f)$ is called a *co-implication*. Let X be the set of colors occurring in E or in f. We say that $(E \Rightarrow f)$ *holds* in some coalgebra A if for all colorings $\varphi : A \to X$ we have

$$(\forall_{e \in E}.A \models_\varphi e) \Rightarrow A \models_\varphi f.$$

Again, it is easy to check from this definition:

Lemma 5.1. *Let Q be a set of co-implications, then* $\mathbf{Mod}(Q)$ *is closed under $\mathscr{H}$ and $\sum$, i.e., a quasi-covariety.*

In fact, we shall see that quasi-covarieties are precisely the classes definable by co-implications. To this end, define for any set Q of co-implications $\mathbf{Mod}(Q)$ as the class of all coalgebras satisfying all co-implications in Q. Similarly, let $\mathbf{CoImp}(K)$ be the set of all co-implications satisfied in all members of K, then we have:

Theorem 5.2. *Let K be any class of F-coalgebras, then*

$$\mathscr{H}\sum(K) = \mathbf{Mod}(\mathbf{CoImp}(K)).$$

Proof. Let $A \in \mathbf{Mod}(\mathbf{CoImp}(K))$, choose a set X with $|X| \geqslant |A|$ and an injective map $\iota : A \to X$. Then A is isomorphic to $\tilde{\iota}(A) \leqslant C(X)$ Put $E = C(X) - \tilde{\iota}(A)$, then for every $e \in \tilde{\iota}(A)$ the co-implication $(E \Rightarrow e)$ fails to hold in A. Hence, there must be some $B_e \in K$ with $B_e \not\models (E \Rightarrow e)$. This means that for every $e \in \tilde{\iota}(A)$ there is some $B_e \in K$ and a homomorphism $\varphi_e : B_e \to \tilde{\iota}(A)$ with $e \in \varphi_e(B_e)$. We now obtain a surjective homomorphism $\psi : \sum_{e \in \tilde{\iota}(A)} B_e \to \tilde{\iota}(A)$, so $A \cong \tilde{\iota}(A) \in \mathscr{H}\sum(K)$.

6. Coequations, patterns, and two examples

We do not know of any "syntactical" representation for coequations that would work for arbitrary functors. However, we can think of coequations as *patterns* that are to be avoided. To be precise let us define an *X-pattern* as a triple (u, U, φ) consisting of a coalgebra U generated by the element $u \in U$, i.e., $U = \langle u \rangle$, and a (coloring) map $\varphi : U \to X$ so that U with coloring φ is simple, considered as $X \times F(-)$-coalgebra.

Given a coalgebra A and $a \in A$, we say that A *matches* the X-pattern $p = (u, U, \varphi)$ at a, if there exists a coloring $\psi : \langle a \rangle \to X$ so that as X-colored coalgebras we have $(\langle a \rangle, \psi)/\sim_X \; \cong (U, \varphi)$, and the isomorphism associates a with u. Here $\sim_X$ is the largest bisimulation on $\langle a \rangle$, that respects the coloring. Otherwise, we say that A *avoids* p at a.

Lemma 6.1. *Let e be a coequation with colors in X. Let A be a coalgebra and $a \in A$. Then*

$$A, a \models e \;\;\Leftrightarrow\;\; A \text{ avoids the pattern } (e, \langle e \rangle, \varepsilon_X).$$

Proof. Let e be a coequation and $a \in A \in \mathbf{Set}_F$. First note that $p = (e, \langle e \rangle, \varepsilon_X)$ is indeed a pattern. This follows from the fact that $C(X)$ as X-colored coalgebra is final, hence simple. Subcoalgebras of simple coalgebras are simple, so $\langle e \rangle$ with coloring ε_X is simple as X-colored coalgebra.

If $A, a \not\models e$ then there is a homomorphism $\varphi : A \to C(X)$ with $\varphi(a) = e$. It follows that $\varphi(\langle a \rangle) = \langle e \rangle$. The kernel of φ is a bisimulation on $\langle a \rangle$, which also respects the coloring if this is defined as $\varepsilon_X \circ \varphi$. The image of φ is simple as an X-colored coalgebra, hence, the kernel of φ is the largest bisimulation $\sim_X$ on the X-colored coalgebra $\langle a \rangle$. Hence, as X-colored coalgebras, $\langle a \rangle/\sim_X \; \cong \langle e \rangle$. Obviously, the isomorphism carries a to e, so A matches p at a.

Conversely, assume that A matches p at a, then we have a color-preserving homomorphism $\pi_\sim : \langle a \rangle \to \langle e \rangle$ with $\pi_\sim(a) = e$. This can be extended to a homomorphism $\bar{\pi}_\sim : A \to C(X)$, since $C(X)$ has the extension property [3]. It follows that $A, a \not\models e$.

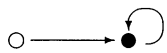
Thus, a coequation gives rise to a pattern. Conversely, of course, each pattern arises from a coequation. That is, given a pattern $p = (a, \langle a \rangle, \varphi)$, we obtain a homomorphism $\tilde{\varphi} : \langle a \rangle \to C(X)$ which is color preserving and injective, since $\langle a \rangle$ with coloring φ is simple.

This gives us a method to represent equations and to check a given equation on a coalgebra A. For any a we need to check all colorings of $\langle a \rangle$, each time factoring by the largest bisimulation and comparing the resulting pattern with $(e, \langle e \rangle, \varepsilon_X)$. $\square$

6.1. A coequation

First, we elaborate a simple example of a covariety. Let $\mathscr{I}$ be the identity functor on **Set**. An $\mathscr{I}$-coalgebra is a map $\alpha : S \to S$. Consider the subclass K of $\mathbf{Set}_{\mathscr{I}}$ consisting of all (S, α_S) such that $\forall s \in S.\ \exists n \in \mathbf{N}.\ \alpha^n(s) = s$. It is easy to check that K is a covariety, i.e., it is closed under $\mathscr{H}$, $\mathscr{S}$, and $\sum$.

K can be described by the coequation given by the following pattern:

f :  .

The figure represents a simple 2-colored one-generated $\mathscr{I}$-coalgebra which cannot be obtained as a color preserving homomorphic image of any 2-colored coalgebra in K. Conversely, if $A \notin K$, there exists $a \in A$ such that $a \notin B_a$ where B_a is defined as $\{\alpha^n(a) \mid 0 < n \in \mathbf{N}\}$. For a coloring φ, painting every element in B_a black and painting a white, we shall obtain the coequation as $\tilde{\varphi}(a)$.

6.2. A co-implicational class

Again, consider coalgebras of the identity functor $\mathscr{I}$. Let K consist of all coalgebras (S, α_S) where $\alpha_S : S \to S$ is surjective. It is easy to check that K is closed under homomorphic images and sums. However, K is not closed under subcoalgebras. This is shown by the following example: Consider $\alpha : \mathbb{N} \to \mathbb{N}$ defined by

$$\alpha(n) = \begin{cases} n - 1 & \text{if } n > 0, \\ 0 & \text{if } n = 0. \end{cases}$$

α is onto, but for every $k \in \mathbb{N}$, the set $\{0, \ldots, k\}$ is the carrier of a subcoalgebra of $(\mathbb{N}, \alpha)$ in which the coalgebra operation is not surjective.

Given that K is a closed under homomorphic images and sums, Theorem 5.2 tells us that it must be definable by a set of co-implications. In general, such a set might become rather large and unwieldy, so that the proof is not constructive in any practical sense. However, in the current example we are lucky, for we can actually exhibit a single co-implication, defining K. Let f be the coequation from the previous example and let e_1 and e_2 be defined as

$$e_1 : \circ \longrightarrow \circ \longrightarrow \bullet \circlearrowleft$$

$$e_2 : \bullet \longrightarrow \circ \longrightarrow \bullet \circlearrowleft$$

then we have:

Proposition 6.2. *The class of all $\mathscr{I}$-coalgebras whose structure map is surjective is defined by the co-implication*

$$\{e_1, e_2\} \Rightarrow f.$$

Before entering the proof, let us give the intuition behind this co-implication and the patterns occurring therein. Clearly, an $\mathscr{I}$-coalgebra excluding the pattern f is a member of the covariety K from the previous example, hence its structure map is onto. Otherwise, if pattern f does appear, then surjectivity of α requires its generating element to have an α-preimage which leads to the occurrence of one of the patterns e_1 or e_2.

Proof. First notice that e_1, e_2, and f are elements of the cofree 2-colored $\mathscr{I}$-coalgebra (using the colors *black* and *white*). Denote this coalgebra by C and its structure map by α_C, then $\alpha_C(e_1) = \alpha_C(e_2) = f$ and there is no other $x \in C$ with $\alpha_C(x) = f$.

Let now $\alpha_A : A \to A$ be surjective and let φ be a $\{black, white\}$-coloring of A. Assume that $A, a \not\models_\varphi f$, that is $\tilde{\varphi}(a) = f$. Since α_A is onto, there must be some $b \in A$ with $\alpha_A(b) = a$, hence $\alpha_C(\tilde{\varphi}(b)) = \tilde{\varphi}(\alpha_A(b)) = \tilde{\varphi}(a) = f$. Depending on the color of b we either have $\tilde{\varphi}(b)$ equal to e_1 or to e_2, hence, one of the premises e_1 or e_2 was violated under the coloring φ.

For the other direction, assume that $A \models \{e_1, e_2\} \Rightarrow f$. Given any $a \in A$ we have to find some $b \in A$ with $\alpha_A(b) = a$. If there exists $n \in \mathbb{N}$ with $\alpha_A^n(a) = a$ then we are done; otherwise, let φ be the coloring painting a white and every other element of A black. Then $\tilde{\varphi}(a) = f$, hence $A, a \not\models_\varphi f$. Consequently, there must be an element $b \in A$ such that $A, b \not\models_\varphi e_1$ or $A, b \not\models_\varphi e_2$, that is $\tilde{\varphi}(b) = e_1$ or $\tilde{\varphi}(b) = e_2$. In any case, $\tilde{\varphi}(\alpha_A(b)) = \alpha_C(\tilde{\varphi}(b)) = f$. Since $\tilde{\varphi}$ is not only an $\mathscr{I}$-homomorphism, but also color preserving, we conclude that $\alpha_A(b)$ is *white*, hence equal to a. $\square$

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