Quarklet Frames in Adaptive Numerical Schemes

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AT15, San Antonio, TX

5-23-2016

Joint work with Stephan Dahlke and Thorsten Raasch

Operator equation

- Let H be a Hilbert space with its dual H'. We are interested in the solution $u \in H$ of the operator equation

$$\mathcal{L}u = f,$$

with $\mathcal{L} : H \to H'$ a boundedly invertible operator and $f \in H'$.

- Example: Poisson equation on $\Omega = [-1,1]^2 \setminus (0,1]^2$ with homogeneous Dirichlet boundary conditions,

$$-\Delta u = f \quad \text{in}\, \Omega,$$

with $\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ and $f \in H^{-1}(\Omega)$.

Numerical treatment

- Adaptive space refinement methods with high convergence rate available (FEM and wavelet methods).
- There is some hope that we can beat them.
- Idea: combine adaptivity and space refinement with polynomial enrichment (hp-method).
- Until now hp-methods exist only in the finite element setting.
- Quarklets are the first hp approach based on wavelets.
- In the long run: provable exponential convergence rates.

Discretization

- Discretization of the operator equation via a frame $\Psi = \{\psi_\lambda\}_{\lambda\in\Lambda}$ of H leads to a matrix-vector equation

$$\mathbf{A}\mathbf{u} = \mathbf{f},$$

with $\mathbf{A} = \{\mathcal{L}\psi_\lambda(\psi_\mu)\}_{\lambda,\mu\in\Lambda}$, $\mathbf{f} = \{f(\psi_\lambda)\}_{\lambda\in\Lambda}$ and $u = \Psi^T\mathbf{u}$.

- Note: for a redundant frame the matrix $\mathbf{A}$ is only positive semidefinite.
- Use an adaptive approximation scheme (e.g.: Richardson, steepest descent) to solve this eqution.
- Essential for high convergence rates:
  - fast convergence in respect of best N-term approximation.
  - certain compression properties of the stiffness matrix $\mathbf{A}$.

Construction of quarklets

- To keep it simple: quarklet frames on the real axis.
- CDF basis. Mother wavelet ψ with

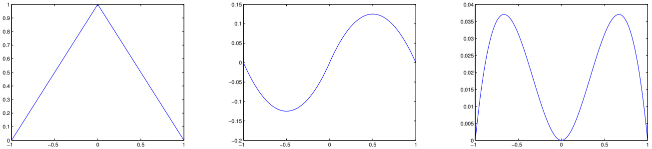
$$\psi(x) = \sum_{k\in\mathbb{Z}} b_k \varphi(2x-k) \quad \text{for all } x \in \mathbb{R},$$

of order d with $\tilde{d}$ vanishing moments and generator function $\varphi = N_d(\cdot + \lfloor \frac{d}{2} \rfloor)$.

Quarks

- Quarks through multiplying the generator function φ with monomes:

$$\varphi_p(x) := \left(\frac{x}{\lceil m/2 \rceil}\right)^p \varphi(x), \quad \text{for all } p \geq 0, x \in \mathbb{R}.$$



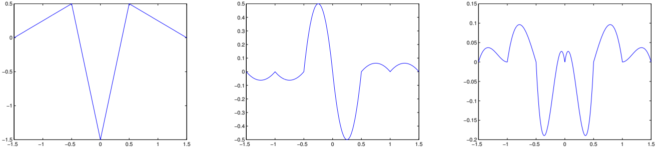
(a) $p = 0$ (b) $p = 1$ (c) $p = 2$

Figure: B-Spline Quarks φ_p of order $d = 2$

Quarklets

Quarklets are defined by

$$\psi_p(x) = \sum_{k\in\mathbb{Z}} b_k \varphi_p(2x-k) \quad \text{for all } p \geq 0, x \in \mathbb{R}.$$



(a) $p = 0$ (b) $p = 1$ (c) $p = 2$

Figure: Quarklets ψ_p of order $d = 2$ with $\tilde{d} = 2$ vanishing moments

For $j_0 \in \mathbb{N}$ we analyze systems of dilated and translated quarklets

$$\psi_{p,j,k} := 2^{j/2}\psi_p(2^j \cdot -k), \quad \text{for all } p \geq 0, j \geq j_0, k \in \mathbb{Z},$$

and quarks

$$\psi_{p,j_0-1,k} := \varphi_{p,j_0,k} := 2^{j_0/2}\varphi_p(2^{j_0}\cdot -k), \quad \text{for all } p \geq 0, k \in \mathbb{Z}.$$

Properties

- Quarklets have the same amount of vanishing moments as the underlying wavelets.
- Properly scaled versions of the quarklet systems build frames in $L_2(\mathbb{R})$ and $H^1(\mathbb{R})$.
- The corresponding Laplacian stiffness matrix is compressible (i.e. it can be well aproximated by matrices with finitely many nontrivial entries).

Compression property

Theorem (Compression of the Laplacian)

Let $d \geq 3$. For $J \in \mathbb{N}_0$, we define the biinfinite matrix $\mathbf{A}_J$ by dropping the entries $a_{\lambda,\lambda'}$ from $\mathbf{A}$ when

$$a \log_2(1 + |p - p'|) + b\,|j' - j| > J, \tag{1}$$

with $a, b > 0$ fulfilling some technical conditions. Then the number of non-zero entries in each row and colum of $\mathbf{A}_J$ is of order 2^J, and

$$||\mathbf{A} - \mathbf{A}_J||_{\mathcal{L}(\ell_2(\Lambda))} \lesssim 2^{-J(d-2)/b}. \tag{2}$$

Numerical results

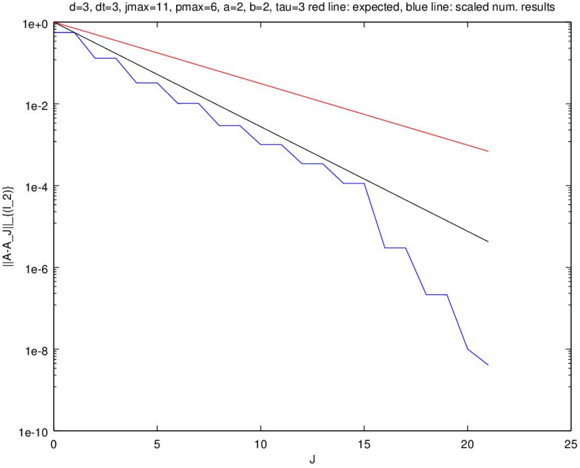


Figure: Compression of the Laplacian **A** for $d = \widetilde{d} = 3$.

Outlook

- Convergence rates
- 2d (L-shaped domain)
- Implementation

Literature

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Thank you very much!