Adaptive Wavelet Methods for Stochastic Partial Differential Equations

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Scope

Numerical treatment of SPDEs of parabolic type

(1) $$dU_t = \big(AU_t + F(t,U_t)\big)dt + \Sigma(t,U_t)dW_t, \quad t \in [0,T],$$

on bounded Lipschitz domains $D \subset \mathbb{R}^d$, driven by a (cylindrical) stochastic process W.

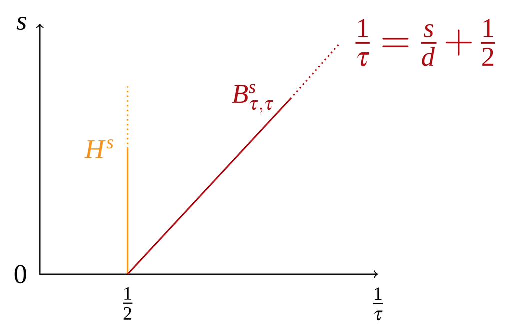
Goals: Adaptive wavelet scheme in time & space ▹ abstract Cauchy problem, Rothe method
Theoretical justification ▹ regularity in Besov spaces
Numerical realization ▹ Marburg software library

State of the art

Adaptive wavelet methods ▹ optimal algorithms for deterministic elliptic operator equations
Regularity theory of SPDEs ▹ usually L_p with $p > 1$, convex domains, Hölder or Lipschitz regularity
Numerics of SPDEs ▹ typically non-adaptive (uniform) space-time approximations

uniform methods ↷ linear approximation
$u \in H^s(D)$ $\quad \|u - u_N\|_{L_2(D)} = O(N^{-s/d})$

adaptive methods ↷ nonlinear approximation
$u \in B^s_{\tau,\tau}(D), \quad \frac{1}{\tau} = \frac{s}{d} + \frac{1}{2}$ $\quad \|u - u_N\|_{L_2(D)} = O(N^{-s/d})$



Adaptive wavelet algorithms for SPDEs: The stationary case

I. Noise representation – sparse wavelet expansions

(2) $$X = \sum_{j=0}^{\infty} \sum_{k \in \nabla_j} Y_{j,k}\, Z_{j,k}\, \psi_{j,k}, \qquad 0 \le \beta \le 1, \qquad \alpha + \beta > 1,$$

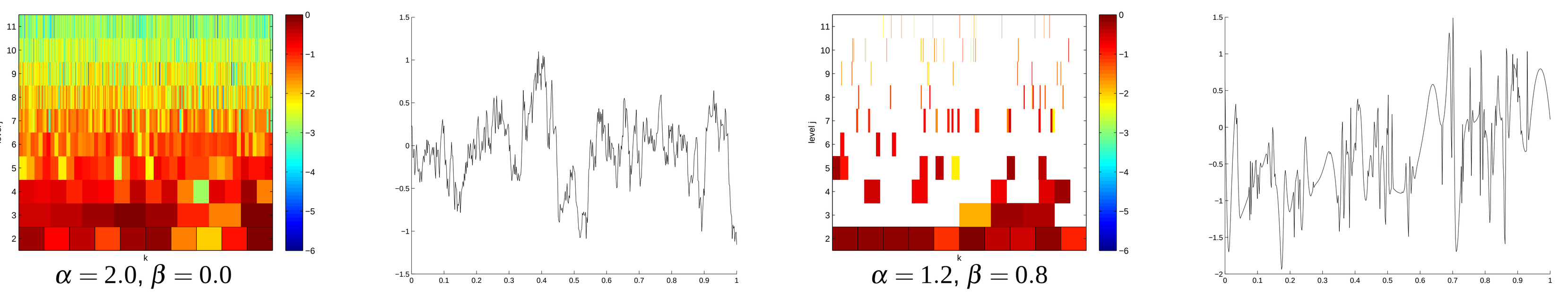
where $Y_{j,k} \sim B(1, 2^{-\beta j d})$, $Z_{j,k} \sim N(0, 2^{-\alpha j d})$ are independent, and $\{\psi_{j,k}\}$ is a wavelet Riesz basis.
Note: If $\beta = 0$ and $\{\psi_{j,k}\}$ is orthonormal, then (2) is the Karhunen-Loève expansion of X.

Theorem [1]. *(Besov regularity of the noise.) Let $p,q > 0$.*
$$X \in B^s_{p,q}(D) \text{ P-a.s.} \iff \left(\frac{1}{p} - 1\right)_+ < \frac{s}{d} < \frac{\alpha - 1}{2} + \frac{\beta}{p}.$$

Corollary [1]. *(Regularity in the adaptivity scale.) Let $s > 0$.*
$$X \in B^s_{\tau,\tau}(D) \text{ P-a.s.} \quad \text{with} \quad \frac{1}{\tau} = \frac{s}{d} + \frac{1}{2} \quad \text{if} \quad \frac{1}{2} < \frac{(1-\beta)s}{d} < \frac{\alpha - 1}{2} + \frac{\beta}{2}.$$

II. Noise approximation – error $(\mathbb{E}\|X - \widehat{X}\|^2_{L_2(D)})^{1/2}$

$e_{\text{lin}} \asymp N^{-\frac{\alpha+\beta-1}{2}}$ linear approximation rate $\qquad e_{N\text{-term}} \preceq N^{-\frac{\alpha+\beta-1}{2(1-\beta)}}$ average best N-term rate



$\alpha + \beta$ determines Sobolev regularity $\qquad \frac{1}{1-\beta}$ enhances Besov regularity & sparsity

III. Rothe method – first in time, then in space (stiff problem ↷ implicit discretization in time)

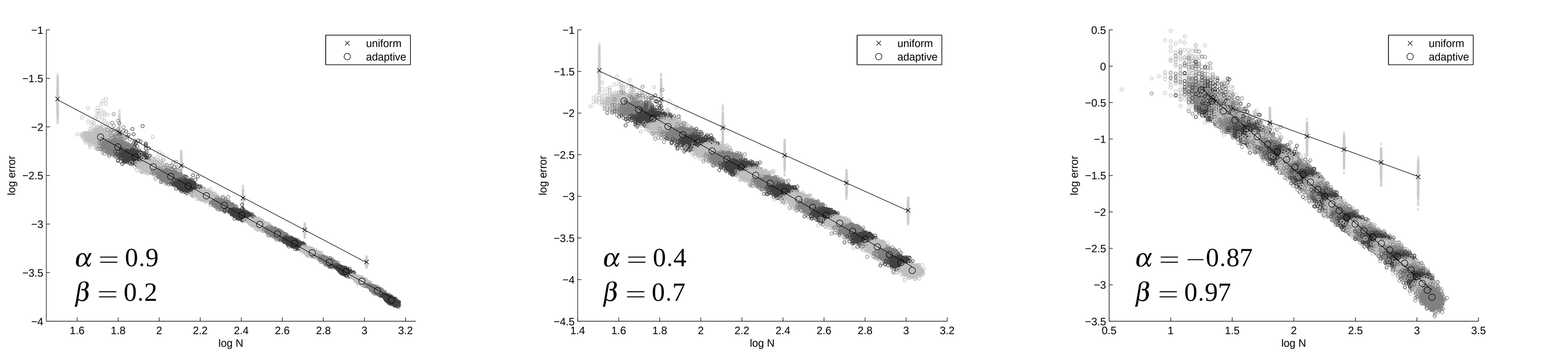
$$\big(I - (t_{n+1} - t_n)A\big)U_{t_{n+1}} = U_{t_n} + (t_{n+1} - t_n)F(t_n, U_{t_n}) + \Sigma(t_n, U_{t_n})\big(W_{t_{n+1}} - W_{t_n}\big)$$

(3) leads to elliptic subproblem: $\quad -\Delta V = X$ in D, $\quad V = 0$ on ∂D $\quad$ (model problem)

Theorem [1]. *The rate of best N-term wavelet approximation of the solution of (3) in $H^1(D)$ is at least*
$$e_{N\text{-term}} \preceq N^{-r+\varepsilon}, \qquad r = \min\left\{\frac{1}{2(d-1)}, \frac{\alpha+\beta-1}{6} + \frac{2}{3d}\right\}, \qquad d > 1, \forall \varepsilon > 0.$$

Note: On general Lipschitz domains uniform approximation only achieves the weaker rate $N^{-1/(2d)}$.

IV. Approximation rates of solutions to elliptic subproblems – uniform vs. adaptive



Regularity of SPDEs in Besov spaces: Spatial regularity

I. Parabolic model equation – linear equation, additive noise

(4) $$dU_t = AU_t\,dt + \sum_{m=1}^{\infty} g^m(t)\,dw^m_t \text{ on } [0,T] \times D, \qquad U(0,\cdot) = u_0 \text{ on } D,$$

A 2[nd] order elliptic PDO, $g = (g^m) \in L_p(\Omega_T; H^{\gamma-1}_{p,\theta}(D,\ell_2))$, $u_0 \in L_p(\Omega; H^{\gamma-2/p}_{p,\theta+2-p}(D))$, $p \ge 2$, $\gamma, \theta \in \mathbb{R}$

For a $\kappa \in (0,1)$ and $\theta \in (d-\kappa, d+\kappa+p-2)$ there is a unique $U \in L_p(\Omega_T; H^{\gamma}_{p,\theta-p}(D))$ solving (4), i.e.

$$\langle U_t, \varphi \rangle = \langle u_0, \varphi \rangle + \int_0^t \langle AU_s, \varphi \rangle ds + \sum_{m=1}^{\infty} \int_0^t \langle g^m(s,\cdot), \varphi \rangle dw^m_s, \quad \forall \varphi \in C_0^\infty(D).$$

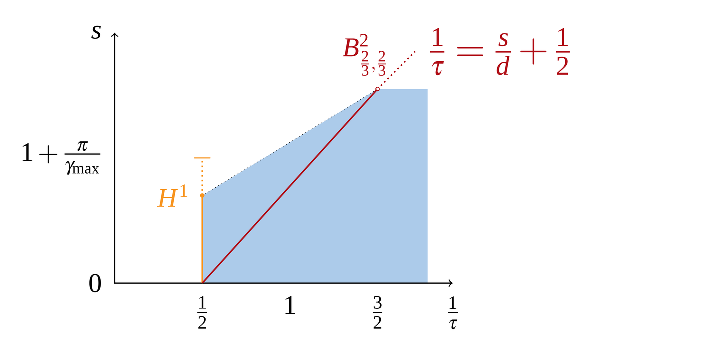
Theorem [2]. *Let U be the solution of (4) for some $\gamma \in \mathbb{N}$. If $U \in L_p(\Omega_T; B^s_{p,p}(D))$ with $0 < s \le \gamma \wedge (1 + \frac{d-\theta}{p})$, then*
$$U \in L_\tau(\Omega_T; B^\alpha_{\tau,\tau}(D)), \qquad \frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}, \qquad 0 < \alpha < \gamma \wedge \frac{sd}{d-1},$$
and $$\|U\|_{L_\tau(\Omega_T; B^\alpha_{\tau,\tau}(D))} \le C\Big(\|U\|_{L_p(\Omega_T; B^s_{p,p}(D))} + \|g\|_{L_p(\Omega_T; H^{\gamma-1}_{p,\theta}(D,\ell_2))} + \|u_0\|_{L_p(\Omega; H^{\gamma-2/p}_{p,\theta+2-p}(D))}\Big).$$

II. Generalizations – linear equation, multiplicative noise

$$dU_t = \big(A(t)U_t + F(t)\big)dt + \sum_{m=1}^{\infty}\sum_{i=1}^{d}\Big(\sigma^{im}(t)\frac{\partial U_t}{\partial x^i} + v^m(t)U_t + g^m(t)\Big)dw^m_t \text{ on } [0,T] \times D, \quad U(0,\cdot) = u_0 \text{ on } D$$

III. Sobolev regularity – limited by re-entrant corners

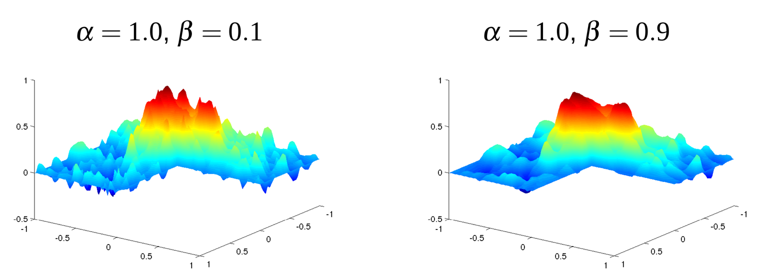
Theorem [4]. *Let $\gamma_{\max}$ be the largest interior angle of a polygonal D. For certain equations of type (4) one has*
$$U = \underbrace{U_{regular}}_{\in \Lambda^2} + \underbrace{U_{singular}}_{\notin \Lambda^s,\ s > 1 + \pi/\gamma_{\max}}$$
and $\Lambda^s \subseteq L_2(\Omega; H^{-1/2-\varepsilon}((0,\infty); H^s(D)))$.



↷ $U_\cdot(\omega) \notin L_2([0,T]; H^s(D)),\ s > 1 + \pi/\gamma_{\max}$

Numerical realization

Elliptic subproblem on L-shaped domain



▹ singularities in solution due to noise and domain
▹ adaptive wavelet frame approach
▹ constructed by overlapping domain decomposition, parametric images of unit-cube
▹ additive and multiplicative Schwarz methods
▹ based on Marburg software library ▹ MaRC

Objectives and work schedule

Adaptive wavelet algorithms for SPDEs

I. Noise representation

∘ time dependent extension of X using independent scalar Brownian motions $(w^{j,k}_t)_{t\in[0,T]}$

$$W_t = \sum_{j=0}^{\infty}\sum_{k \in \nabla_j} Y_{j,k}\, w^{j,k}_t\, \psi_{j,k}$$

∘ characterization of spaces of negative smoothness ∘ characterization by frames
∘ approximation in Sobolev norms ∘ anisotropic spaces

II. Setup of fully adaptive scheme

∘ regularity estimates in time direction ▹ rigorous error estimates for uniform time discretization
∘ fully adaptive wavelet scheme in space and time ∘ convergence and optimality analysis
∘ evaluation of nonlinear functionals in the stochastic setting ∘ tensor wavelet schemes

Regularity of SPDEs in Besov spaces

I. Nonlinear problems

∘ as perturbation of linear setting ∘ use of variational techniques ▹ establishing Besov regularity

II. Regularity in space and time

∘ Hölder and Sobolev regularity in time ∘ weighted Sobolev estimates in time and space
∘ full space-time Besov regularity

III. Extensions

∘ allow non-local semigroup generators A ∘ non-Markovian or non-continuous driving noises

Numerical realization

I. Setup of Rothe method

∘ uniform time discretization first ∘ adaptive time-step control ∘ comparison to existing schemes

II. Parallel algorithms

∘ on the level of the SPDE solver ∘ for Monte Carlo simulation experiments ∘ added efficiency

III. Implementation of fully adaptive algorithm in space and time

∘ evaluation of nonlinear functionals ∘ application to problems on polygonal, polyhedral domains
∘ based on Marburg software library

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