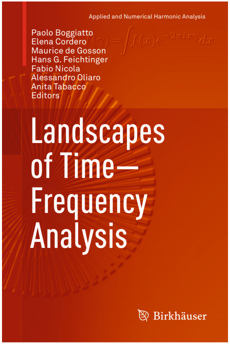
Coorbit Theory and its Kernel Problem

Lukas Sawatzki

Philipps Universität Marburg

Workgroup Numerics und Optimization

Aspects of Time-Frequency Analysis
Turin, Italy
25.06.2019



Joint work with: S. Dahlke, F. De Mari, E. De Vito, G. Steidl, G. Teschke, F. Voigtlaender.

Signal Analysis: Transformation and Discretization

- Transform a signal to handle it, e.g. the STFT:

$$f \mapsto F(\omega, \tau) = \int_{\mathbb{R}} f(t) w(t-\tau) e^{-i\omega t}\, dt;$$

or the wavelet transform:

$$g \mapsto G(a, t) = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} g(x) \overline{\psi\left(\frac{x-t}{a}\right)}\, dx$$

- Discretize a signal, e.g. the Shannon-Whittaker theorem:

$$f(t) = \sum_{n \in \mathbb{Z}} f(n) \cdot \operatorname{sinc}(t-n), \quad \operatorname{supp} \widehat{f} \subset [-\frac{1}{2}, \frac{1}{2}].$$

What is the Goal?

- Define smoothness spaces via transformations of signals $\longrightarrow$ coorbit theory
- Discretize these functions spaces, i.e., $f = \sum_k c_k \varphi_k$ for certain building blocks φ_k
- Applications: e.g. Besov spaces, modulation spaces, development of new spaces
- Today: What is the role of the so-called kernel?

Philipps Universität Marburg

Lukas Sawatzki

AG Numerik und Optimierung

Coorbit Theory

Let G be a group with Haar measure dg and $\pi : G \to \mathcal{U}(\mathcal{H})$ a unitary representation of G on a Hilbert space $\mathcal{H}$. Consider the **voice transform**

$$V : \mathcal{H} \to L_\infty(G) \cap C(G), \quad Vv(x) := \langle v, \pi(x)u \rangle_{\mathcal{H}}.$$

We assume that π is irreducible and $u \in \mathcal{H}$ is **admissible**, i.e.

$$V : \mathcal{H} \to L_2(G) \quad \text{is an isometry.}$$

We then call π square-integrable and V is injective and self-adjoint.
An important ingredient is the **kernel function**

$$K(x) := Vu(x) = \langle u, \pi(x)u \rangle_{\mathcal{H}} \in L_2(G),$$

which fulfills

$$\boxed{Vv * K = Vv.}$$

Coorbit Spaces

At this point we assume:

$$K \in L_{1,w}(G).$$

Consider the space

$$\mathcal{H}_{1,w} := \{f \in \mathcal{H} \;:\; Vf \in L_{1,w}(G)\}.$$

Setting $\|f\|_{\mathcal{H}_{1,w}} := \|Vf\|_{L_{1,w}}$, this is a Banach space with $\mathcal{H}_{1,w} \subset \mathcal{H}$ dense. Then we consider the **extended voice transform**

$$V_e T(x) := \langle T, \pi(x)u\rangle_{\mathcal{H}'_{1,w}\times\mathcal{H}_{1,w}}, \quad T \in \mathcal{H}'_{1,w}.$$

We can now define the **coorbit spaces** with respect to $L_{p,m}(G)$ via

$$\boxed{\mathrm{Co}(L_{p,m}) := \{T \in \mathcal{H}'_{1,w} \;:\; V_e T \in L_{p,m}(G)\},}$$

where $\|T\|_{\mathrm{Co}(L_{p,m})} = \|V_e T\|_{L_{p,m}}$.

Coorbit Theory

The weights fulfill $w(gh) \leq w(g)w(h)$ as well as $m(ghk) \leq w(g)m(h)w(k)$.

The most important property is

$$V_e^{-1}F \in \mathrm{Co}(L_{p,m}) \Longleftrightarrow F \in L_{p,m}(G), F * K = F.$$

Furthermore, by **Schur's lemma**, it holds that

$$\|F * K\|_{L_{p,m}} \leq C_K \|F\|_{L_{p,m}},$$

if $K \in L_{1,w}(G)$, what we assumed.

Main idea of the discretization: discretize this convolution!

Discretization if $K \in L_{1,w}(G)$

Theorem

Under additional conditions on K, we obtain for small U:

- *Every $T \in \mathrm{Co}(L_{p,m})$ can be written as $T = \sum_{i \in I} c_i(T)\pi(x_i)u$ with*

$$\|(c_i(T))_{i \in I}\|_{\ell_{p,m}} \lesssim \|T\|_{\mathrm{Co}(L_{p,m})}.$$

- *For $(d_i)_{i \in I} \in \ell_{p,m}$ we have $T = \sum_{i \in I} d_i \pi(x_i) u \in \mathrm{Co}(L_{p,m})$ with*

$$\|T\|_{\mathrm{Co}(L_{p,m})} \lesssim \|(d_i)_{i \in I}\|_{\ell_{p,m}}.$$

Example: Modulation Spaces

Consider the **reduced Heisenberg group** $\mathbb{H}_r^1 = \mathbb{R} \times \mathbb{R} \times \mathbb{T}$ with group law

$$(x, y, \xi) \circ (x', y', \xi') = (x + x', y + y', \xi\xi' e^{-\pi i(xy' - yx')}).$$

Then the **Schrödinger representation**

$$\rho(x, y, \xi)f(t) = \xi e^{-\pi i xy} e^{2\pi i yt} f(t - x)$$

is a unitary representation of $\mathbb{H}_r^1$ on $L_2(\mathbb{R})$. The associated coorbit spaces are the **modulation spaces**

$$f \in M_{p,p}^s(\mathbb{R}) \quad \Longleftrightarrow \quad \int_{\mathbb{H}_r^1} |\langle \rho(x, y, \xi)g, f\rangle|^p (1 + |y|)^{sp} \, d(x, y, \xi) < \infty$$

Example: Besov Spaces

Consider the group

$$G = \mathbb{R} \times \mathbb{R}^*, \quad (b, a) \circ (x, y) = (ax + b, ay)$$

with Haar measure $d(x, y) = \frac{\mathrm{d}x\mathrm{d}y}{y^2}$. Then

$$\pi(t, a)\varphi(x) := a^{-1/2}\varphi\left(\frac{x-t}{a}\right)$$

is a unitary representation of G on $L_2(\mathbb{R})$.
The associated coorbit spaces are **homogeneous Besov spaces**:

$$f \in \dot{B}^{s-1/2+1/p}_{p,p}(\mathbb{R}) \iff \int_{\mathbb{R}} \int_{\mathbb{R}^*} |\langle L_t D_a \varphi, f\rangle|^p |a|^{-sp} \, \frac{\mathrm{d}a}{a^2} \mathrm{d}t < \infty,$$

φ suitable, $L_t D_a \varphi(x) = \frac{1}{\sqrt{a}}\varphi(\frac{x-t}{a})$, $s \in \mathbb{R}$, $1 \leq p \leq \infty$.

What if $K \notin L_{1,w}(G)$?

Before:

$$\mathrm{Co}(L_{p,m}) := \{ T \in \mathcal{H}'_{1,w} \; : \; V_e \in L_{p,m}(G) \},$$

where

$$\mathcal{H}_{1,w} := \{ f \in \mathcal{H} \; : \; Vf \in L_{1,w}(G) \}.$$

In particular we necessarily assume

$$K = Vu \in L_{1,w}(G).$$

What happens, if this condition is not fulfilled, e.g.

$$K \in \bigcap_{1<p<\infty} L_{p,w}(G)?$$

New Coorbit Spaces

Consider the new space

$$\mathcal{S}_w := \{f \in \mathcal{H} \ : \ Vf \in \bigcap_{1<p<\infty} L_{p,w}(G)\} \subset \mathcal{H},$$

we then extend the **voice transform** to $\mathcal{S}_w'$ via

$$V_e T(x) := \langle T, \pi(x)u\rangle_{\mathcal{S}_w' \times \mathcal{S}_w}, \quad T \in \mathcal{S}_w'.$$

Then the **coorbit space** with respect to $L_{r,m}(G)$ is analogously defined via

$$\boxed{\mathrm{Co}(L_{r,m}) = \{T \in \mathcal{S}_w' \ : \ V_e T \in L_{r,m}(G)\}.}$$

Again we have

$$V_e^{-1} F \in \mathrm{Co}(L_{r,m}) \iff F \in L_{r,m}(G), F * K = F.$$

It holds that $K \notin L_{1,w}(G)$, but let us assume

$$\mathrm{RC}_K : L_{r,m}(G) \to L_{r,m}(G), \quad \mathrm{RC}_K F = F * K, \quad \text{is continuous,}$$

which is necessary! This is restrictive.

Theorem

Under technical assumptions on the kernel K we have:

- *For all $\varepsilon > 0$, $T \in \mathrm{Co}(L_{r,m})$ there exists a $n^* = n^*_{T,\varepsilon} \in \mathbb{N}$, such that for all $n \geq n^*$ we have*

$$\|T - \sum_{x \in X_n} c(T)_{n,x} \pi(x) u\|_{\mathrm{Co}(L_{r,m})} \leq \varepsilon,$$

$$\|(c(T)_{n,x})_{x \in X_n}\|_{\ell_{p,m}} \leq C_n (1 + \varepsilon) \|T\|_{\mathrm{Co}(L_{r,m})}.$$

- *For $(d_x)_{x \in Y_n} \in \ell_{q,m}$ we have $T = \sum_{x \in Y_n} d_x \pi(x) u \in \mathrm{Co}(L_{r,m})$ with*

$$\|T\|_{\mathrm{Co}(L_{r,m})} \leq D_n \|(d_x)_{x \in Y_n}\|_{\ell_{q,m}},$$

where $1/q + 1/p = 1 + 1/r$, $p > 1$.

$$C_n \sim_w |U_n|^{1-1/r} \|T_n^{-1}\|$$
$$D_n \sim_w |U_n|^{1/q-1} \cdot \|\mathrm{osc}_{U_n}(K) + |K|\|_{L_{p,w}}.$$

Example

Consider the group $G = (\mathbb{R}, +)$ and $\Omega \subset \mathbb{R}$, and the Hilbert space $B^2_\Omega = \{f \in L_2(\mathbb{R}) \ : \ \operatorname{supp} \widehat{f} \subset \Omega\}$. The representation π is given via

$$\pi(t)f(x) = f(x - t)$$

and as an admissible vector we choose

$$u = K = \mathcal{F}^{-1}\chi_\Omega \in B^2_\Omega.$$

It holds that $K \notin L_1(\mathbb{R})$, but $K \in \bigcap_{1<p<\infty} L_p(\mathbb{R})$ (under certain conditions). However the operator

$$\mathrm{RC}_K : L_p(\mathbb{R}) \to L_p(\mathbb{R}), \quad 1 < p < \infty,$$

is not always bounded! But there exists a $W \in L_1(\mathbb{R})$ with $K * W = K$, s.t. $\chi_\Omega \cdot \mathcal{F}W = \chi_\Omega$.

Discretization under Additional Conditions

Previous idea: discretize the convolution RC_K. Assume there is a $W \in L_{1,w}(G)$ with $K * W = K$, then we have

$$F * W = F \quad \text{für alle} \quad V_e^{-1} F \in \mathrm{Co}(L_{r,m}).$$

Therefore: discretize RC_W instead!

Theorem

Under certain conditions on K and W we have:

- *Each $T \in Co(L_{r,m})$ can be written as $T = \sum_{i \in I} c_i(T)\pi(x_i)u$ with*

$$\|(c_i(T))_{i \in I}\|_{\ell_{r,m}} \lesssim \|T\|_{Co(L_{r,m})}.$$

- *For $(d_i)_{i \in I} \in \ell_{r,m}$ we have $T = \sum_{i \in I} d_i \pi(x_i)u \in Co(L_{r,m})$ with*

$$\|T\|_{Co(L_{r,m})} \lesssim \|(d_i)_{i \in I}\|_{\ell_{r,m}}.$$

Example

Consider again the group $G = (\mathbb{R}, +)$ and $\Omega \subset \mathbb{R}$, and the Hilbert space $B^2_\Omega = \{f \in L_2(\mathbb{R}) \ : \ \operatorname{supp} \widehat{f} \subset \Omega\}$. The representation π is given via

$$\pi(t)f(x) = f(x - t)$$

and as an admissible vector we choose

$$u = K = \mathcal{F}^{-1}\chi_\Omega \in B^2_\Omega.$$

Then there exists W as described above. Let $x_k = k/2\pi$, $k \in \mathbb{N}$, and $U = [-1/4\pi, 1/4\pi)$. Then we conclude

$$\mathcal{M}_r \ni f = \sum_{k\in\mathbb{Z}} c_k K(\cdot - k/2\pi) \Longleftrightarrow (c_k)_{k\in\mathbb{Z}} \in \ell_r(\mathbb{Z}),$$

see also Nyquist-Shannon sampling theorem.

Summary

- Coorbit theory connects smoothness spaces with an underlying group structure and gives a universal access as well as possible discretizations
- Normally we assume $K \in L_{1,w}(G)$, but the weaker case $K \in \bigcap_{1<p<\infty} L_{p,w}(G)$ also leads to meaningful smoothness spaces
- However, discretization is only possible if RC_K is bounded on $L_{r,m}(G)$
- A proper discretization is only possible if there exists an additional kernel $W \in L_{1,w}(G)$ with $K * W = W$
- These additional assumptions need to be checked individually!
- This can be extended to a generalized coorbit theory without group structure.

Thank you!

References

S. Dahlke, F. De Mari, E. De Vito, D. Labate, G. Steidl, G. Teschke, S.Vigogna.
Coorbit spaces with voice in a Fréchet space, Journal of Fourier Analysis and Applications 23(1), 2016

S. Dahlke, F. De Mari, E. De Vito, L. Sawatzki, G. Steidl, G. Teschke, F. Voigtlaender.
Landscapes of Time-Frequency Analysis, On the atomic decompositions of coorbit spaces with non-integrable kernel, Springer, 2019

S. Dahlke, F. De Mari, P. Grohs, D. Labate.
Harmonic and Applied Analysis. Applied and Numerical Harmonic Analysis, Birkhäuser, 2015, ISBN 978-3-319-18862-1.

F. Feise, L. Sawatzki.
Inhomogeneous shearlet coorbit spaces, International Journal of Wavelets, Multiresolution and Information Processing, Vol. 16, No. 04, 2018.

M. Fornasier, H. Rauhut.
Continuous frames, function spaces, and the discretization problem. J. Fourier Anal. Appl., 11(3):245-287, 2005.