Inhomogeneous Shearlet Coorbit Spaces

Lukas Sawatzki

Philipps-University Marburg

Summer School on Applied Harmonic Analysis
Genova 2017

27.07.2017

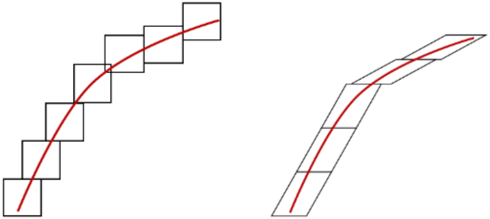
Why Shearlets?

- Goal: Analyze functions $f \in L_2(\mathbb{R}^d)$
- Decompose these functions in suitable blocks
- Wavelets: Isotropic blocks with drawbacks for anisotropic structures
- Workaround: Develop new systems like ridgelets, curvelets, contourlets, etc.
- Or: **Shearlets**

Why Shearlets?

Advantages:

- Anisotropic structure
- Promising numerical results
- Abstract coorbit theory is applicable



Shearlets

Let $a \in \mathbb{R}^*$, $s \in \mathbb{R}^{d-1}$ and $t \in \mathbb{R}^d$, then for a *mother-shearlet* ψ the shearlets are defined as

$$\psi_{(a,s,t)}(x) := |\det A|^{-\frac{1}{2}} \psi(A_a^{-1} S_s^{-1}(x-t)),$$

where

$$A_a := \begin{pmatrix} a & 0_{d-1}^T \\ 0_{d-1} & \operatorname{sign}(a)|a|^{1/d} I_{d-1} \end{pmatrix}$$

and

$$S_s := \begin{pmatrix} 1 & s^T \\ 0_{d-1} & I_{d-1} \end{pmatrix}.$$

Classical Coorbit Theory

Let G be a group, $\mathcal{H}$ a Hilbert space, $\pi : G \to \mathcal{U}(\mathcal{H})$ a unitary representation and ψ an admissable vector. Then the *voice transform* of a function f in $g \in G$ is defined as

$$V_\pi f(g) := \langle f, \pi(g)\psi \rangle_{\mathcal{H}}.$$

For a function space Y on G the *coorbit spaces* are then defined as

$$Co(Y) := \{f \in \mathcal{S}' : \langle f, \pi(\cdot)\psi \rangle_{\mathcal{H}} \in Y\}.$$

Examples

- Taking $G = \mathbb{R}^* \times \mathbb{R}^d$ with the wavelet transform

$$\pi(a,b)\psi(x) = |a|^{-1/2}\psi\left(\frac{x-b}{a}\right)$$

  as well as specific weights yields the *homogeneous Besov spaces* $\dot{B}^{s-1/2-1/p}_{p,p}(\mathbb{R}^d)$.

- The *reduced Heisenberg group* $\mathbb{H}^d_r = \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T}$ with certain specifications leads to the *modulation spaces* $M^s_{p,p}(\mathbb{R}^d)$.

- The *full shearlet group* $\mathbb{R}^* \times \mathbb{R}^{d-1} \times \mathbb{R}^d$ gives us the so-called *shearlet coorbit spaces*.

Generalized Coorbit Theory

Assume we have

- a locally compact Hausdorff space X with measure $\mathrm{d}\mu$,
- a family of functions $\mathfrak{F} = \{\psi_x\}_{x \in X}$ forming a tight continuous frame for $L_2(\mathbb{R}^d)$, i.e.

$$A\|f|L_2\| = \int_X |\langle f, \psi_x \rangle|^2 \mathrm{d}\mu(x)$$

for some $A > 0$.

Then, the associated *voice transform* is defined by

$$V_{\mathfrak{F}} : L_2(\mathbb{R}^d) \to L_2(X, \mu), \qquad V_{\mathfrak{F}} f(x) := \langle f, \psi_x \rangle.$$

Reproducing kernel

The *reproducing kernel* is defined via

$$R_{\mathfrak{F}} : X \times X \to \mathbb{C}, \qquad R_{\mathfrak{F}}(x, y) := V_{\mathfrak{F}}(\psi_y)(x) = \langle \psi_y, \psi_x \rangle.$$

We then have the *reproducing identity*

$$R_{\mathfrak{F}}(V_{\mathfrak{F}} f) = V_{\mathfrak{F}} f$$

for all $f \in L_2(\mathbb{R}^d)$.

Kernel spaces

For a kernel K the *associated kernel operator* is defined via

$$K(f)(x) := \int_X K(x,y) f(y)\,\mathrm{d}\mu(y).$$

Now define some kernel spaces via

$$\|K|\mathcal{A}_{q,m}\| = \max\left\{\operatorname*{ess\,sup}_{x\in X} \|K(x,\cdot)m(x,\cdot)|L_q\|, \operatorname*{ess\,sup}_{y\in X} \|K(\cdot,y)m(\cdot,y)|L_q\|\right\}.$$

Lemma (Feise, S. (2017))

Let $\|K|\mathcal{A}_{q,m}\| < \infty$ for every $q > 1$, then we have the continuous embeddings

$$K(L_{p,v}(X,\mu)) \hookrightarrow L_{r,v}(X,\mu)$$

for all $1 < p < r \leq \infty$.

Generalized Coorbit Theory

Assume $R_{\mathfrak{F}} \in \mathcal{A}_{q,m}$ for all $q > 1$.
Then, consider the spaces

$$\mathcal{H}_{\tau,v} := \{f \in L_2(\mathbb{R}^d) : V_{\mathfrak{F}} f \in L_{\tau,v}(X,\mu)\}$$

for $\tau > 1$ and with the norm

$$\|f|\mathcal{H}_{\tau,v}\| := \|V_{\mathfrak{F}} f|L_{\tau,v}\|$$

these spaces are Banach spaces densely embedded in $L_2(\mathbb{R}^d)$ with $\mathfrak{F} = \{\psi_x\}_{x\in X} \subset \mathcal{H}_{\tau,v}$.
Now we extend the *voice transform* by

$$V_{\mathfrak{F},\tau} f(x) = \langle f, \psi_x \rangle_{(\mathcal{H}_{\tau,v})^\sim \times \mathcal{H}_{\tau,v}}$$

to the anti-dual $(\mathcal{H}_{\tau,v})^\sim \supset L_2(\mathbb{R}^d)$.

Definition coorbit spaces

Define the *coorbit spaces* as

$$Co_{\mathfrak{F},\tau}(L_{p,v}) := \{f \in (\mathcal{H}_{\tau,v})^\sim : V_{\mathfrak{F},\tau} f \in L_{p,v}(X,\mu)\}$$

and equipped with the norm

$$\|f | Co_{\mathfrak{F},\tau}(L_{p,v})\| := \|V_{\mathfrak{F},\tau} f | L_{p,v}\|$$

these spaces are Banach spaces.
Furthermore we have an isometric isomorphism

$$Co_{\mathfrak{F},\tau}(L_{p,v}) \longleftrightarrow \{F \in L_{p,v}(X,\mu) : R_{\mathfrak{F}} F = F\}$$

induced by $V_{\mathfrak{F},\tau}$.

Inhomogeneous Shearlet Coorbit Spaces

Consider $X = (\{\infty\} \times \mathbb{R}^{d-1} \times \mathbb{R}^d) \cup ([-1,1]^* \times \mathbb{R}^{d-1} \times \mathbb{R}^d)$ equipped with the measure

$$\int_X F(x)\,\mathrm{d}\mu(x) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}} F(\infty,s,t)\,\mathrm{d}s\mathrm{d}t + \int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}}\int_{-1}^{1} F(a,s,t)\frac{\mathrm{d}a}{|a|^{d+1}}\mathrm{d}s\mathrm{d}t.$$

Furthermore define $\mathfrak{F} = \{\psi_x\}_{x\in X}$ via

$$\psi_{(\infty,s,t)} := \Phi(S_s^{-1}(\cdot - t)),$$
$$\psi_{(a,s,t)} := |\det A_a|^{-\frac{1}{2}}\Psi(A_a^{-1}S_s^{-1}(\cdot - t)),$$

$$A_a := \begin{pmatrix} a & 0_{d-1}^T \\ 0_{d-1} & \operatorname{sign}(a)|a|^{1/d} I_{d-1}\end{pmatrix}, \quad S_s := \begin{pmatrix} 1 & s^T \\ 0_{d-1} & I_{d-1}\end{pmatrix}.$$

Inhomogeneous Shearlet Frame

Under certain assumptions this family imposes a tight frame for $L_2(\mathbb{R}^d)$.

Theorem (Feise, S. (2017))

Let $\Psi \in L_1 \cap L_2$ be an admissable shearlet and let $\Phi \in L_1 \cap L_2$ be such that

$$\int_{\mathbb{R}^{d-1}} \frac{|\hat{\Phi}(y,\sigma)|^2}{|y|^{d-1}} \mathrm{d}\sigma + \int_{\mathbb{R}^{d-1}} \int_{-|y|}^{|y|} \frac{|\hat{\Psi}(\xi_1,\tilde{\xi})|^2}{|\xi_1|^d} \mathrm{d}\xi_1 \mathrm{d}\tilde{\xi} = 1$$

for almost every $y \in \mathbb{R}$, then $\mathfrak{F}$ is a continuous Parseval frame for $L_2(\mathbb{R}^d)$, i.e.

$$\int_X |\langle f, \psi_x \rangle|^2 \,\mathrm{d}\mu(x) = \|f|L_2(\mathbb{R}^d)\|^2, \qquad f \in L_2(\mathbb{R}^d).$$

Conditions on the Reproducing Kernel

Remember the *reproducing kernel*

$$R_{\mathfrak{F}}(x, y) = \langle \psi_y, \psi_x \rangle.$$

Theorem (Feise, S. (2017))

Let $\hat{\Phi}$ and $\hat{\Psi}$ have specific compact supports, then we have

$$R_{\mathfrak{F}} \in \mathcal{A}_{q,m_v}$$

for all $q > 1$.

Hence, the generalized coorbit theory is applicable.

Inhomogeneous shearlet coorbit spaces

The *inhomogeneous shearlet transform* is defined as

$$\mathcal{SH}_{\mathfrak{F}} f(x) = \langle f, \psi_x \rangle.$$

Then, for $1 \leq p < \infty$ and $1 < \tau \leq 2$ with $p < \tau'$ we define the *inhomogeneous shearlet coorbit spaces* as

$$\mathcal{SC}^r_{\mathfrak{F},\tau,p} := Co_{\mathfrak{F},\tau}(L_{p,v_r}) = \{ f \in (\mathcal{H}_{\tau,v_r})^\sim : \mathcal{SH}_{\mathfrak{F}} f \in L_{p,v_r}(X,\mu) \}$$

and equipped with the norm

$$\| f | \mathcal{SC}^r_{\mathfrak{F},\tau,p} \| := \| \mathcal{SH}_{\mathfrak{F}} f | L_{p,v_r}(X,\mu) \|$$

these spaces are Banach spaces.

Properties

Under certain assumptions the following properties hold:

- $\mathcal{SC}^r_{\mathfrak{F},\tau,p} \subset \mathcal{SC}^r_{\mathfrak{F},\tau,q}$ for $p < q$,
- $\mathcal{SC}^r_{\mathfrak{F},\tau,p} \subset \mathcal{SC}^s_{\mathfrak{F},\tau,p}$ for $r < s$,
- $\mathcal{SC}^r_{\mathfrak{F},\tau,p} = \mathcal{SC}^r_{\mathfrak{G},\tau,p}$ if for the *Gramian kernel* it holds $G(\mathfrak{F},\mathfrak{G}) \in \mathcal{A}_{1,m_{v_r}}$,
- $\mathcal{SC}^r_{\mathfrak{F},\tau,p} = \mathcal{SC}^r_{\mathfrak{F},\sigma,p}$, for $p < \sigma', \tau' < \infty$.

Thank you for your attention!

References

S. Dahlke, F. De Mari, P. Grohs, D. Labate.
Harmonic and Applied Analysis. Applied and Numerical Harmonic Analysis, Birkhäuser, 2015, ISBN 978-3-319-18862-1.

M. Fornasier, H. Rauhut.
Continuous frames, function spaces, and the discretization problem. J. Fourier Anal. Appl., 11(3):245-287, 2005.

H. Rauhut, T. Ullrich.
Generalized coorbit space theory and inhomogeneous function spaces of Besov-Lizorkin-Triebel type. J. Funct. Anal., 260(11):3299-3362, 2011.

F. Feise, L. Sawatzki.
Inhomogeneous shearlet coorbit spaces, preprint 2017.