Coorbit Theorie und ihr Kern-Problem

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What is the goal?

- Define smoothness spaces via representations of groups $\longrightarrow$ coorbit theory
- We intend to to discretize these functions spaces, i.e. $f = \sum_k c_k \varphi_k$ for certain building blocks φ_k
- Applications: e.g. Besov spaces, modulation spaces, development of new spaces
- Today: What is the role of the so-called kernel?

Besov spaces - where is the group structure?

Consider the following characterization of **homogeneous Besov spaces**:

$$f \in \dot{B}^{s-1/2+1/p}_{p,p}(\mathbb{R}) \iff \int_{\mathbb{R}} \int_{\mathbb{R}^*} |\langle L_t D_a \varphi, f \rangle_{L_2}|^p |a|^{-sp} \frac{\mathrm{d}a}{a^2} \mathrm{d}t,$$

φ suitable, $L_t D_a \varphi(x) = a^{-1/2} \varphi(\frac{x-t}{a})$, $s \in \mathbb{R}$, $1 \leq p \leq \infty$. Then, consider the group

$$G = \mathbb{R} \times \mathbb{R}^*, \quad (b,a) \circ (x,y) = (ax+b, ay)$$

with Haar measure $d(x,y) = \frac{\mathrm{d}x\mathrm{d}y}{y^2}$. Then

$$\pi(t,a)\varphi(x) := a^{-1/2} \varphi\left(\frac{x-t}{a}\right)$$

is a unitary representation of G on $L_2(\mathbb{R})$.

Coorbit Theory

Let G be a group with Haar measure dg and $\pi : G \to \mathcal{U}(\mathcal{H})$ a unitary representation of G on a Hilbert space $\mathcal{H}$. Consider the **voice transform**

$$V : \mathcal{H} \to L_\infty(G) \cap C(G), \quad Vv(x) := \langle v, \pi(x)u \rangle_{\mathcal{H}}.$$

We assume that π is irreducible and $u \in \mathcal{H}$ is **admissible**, i.e.

$$V : \mathcal{H} \to L_2(G) \quad \text{is an isometry.}$$

We then call π square-integrable and V is injective and self-adjoint.
An important ingredient is the **kernel function**

$$K(x) := Vu(x) = \langle u, \pi(x)u \rangle_{\mathcal{H}} \in L_2(G),$$

which fulfills

$$\boxed{Vv * K = Vv.}$$

Coorbit theory with integrable kernel

At this point we assume:

$$K \in L_{1,w}(G).$$

Consider the space

$$\mathcal{H}_{1,w} := \{ f \in \mathcal{H} \ : \ Vf \in L_{1,w}(G) \}.$$

Setting $\|f\|_{\mathcal{H}_{1,w}} := \|Vf\|_{L_{1,w}}$, this is a Banach space with $\mathcal{H}_{1,w} \subset \mathcal{H}$ dense. Then we consider the **extended voice transform**

$$V_e T(x) := \langle T, \pi(x) u \rangle_{\mathcal{H}'_{1,w} \times \mathcal{H}_{1,w}}, \quad T \in \mathcal{H}'_{1,w}.$$

We can now define the **coorbit spaces** with respect to $L_{p,m}(G)$ via

$$\boxed{\mathrm{Co}(L_{p,m}) := \{ T \in \mathcal{H}'_{1,w} \ : \ V_e T \in L_{p,m}(G) \},}$$

where $\|T\|_{\mathrm{Co}(L_{p,m})} = \|V_e T\|_{L_{p,m}}$.

Coorbit theory

The weights fulfill $w(gh) \leq w(g)w(h)$ as well as $m(ghk) \leq w(g)m(h)w(k)$.

The most important property is

$$V_e^{-1}F \in \mathrm{Co}(L_{p,m}) \Longleftrightarrow F \in L_{p,m}(G), F * K = F.$$

Furthermore, by **Schur's lemma**, it holds that

$$\|F * K\|_{L_{p,m}} \leq C_K \|F\|_{L_{p,m}},$$

if $K \in L_{1,w}(G)$, what we assumed.

Main idea of the discretization: discretize this convolution!

Diskretisierung

Let $e \in U \subset G$ be compact, consider $\Psi = (\psi_i)_{i \in I}$ and $(x_i)_{i \in I}$ with

- $\bigcup_{i \in I} x_i U = G$,
- $\#\{x_i U \cap x_j U = \emptyset, j \in I\} \leq C$ for all $i \in I$,
- $0 \leq \psi_i \leq 1$,
- $\operatorname{supp}(\psi_i) \subset x_i U$,
- $\sum_{i \in I} \psi_i \equiv 1$.

Then:

$$\begin{aligned} F(x) = F * K(x) &= \sum_{i \in I} \int_{x_i U} F(y)\psi_i(y) L_y K(x)\, d\mu(y) \\ &\sim \sum_{i \in I} \langle F, \psi_i \rangle L_{x_i} K(x). \end{aligned}$$

We further define the oscillation of a function F:

$$\operatorname{osc}_U(F)(x) := \sup_{y \in U} |F(yx) - F(x)|.$$

Discretization if $K \in L_{1,w}(G)$

Theorem

Under additional conditions on K, we obtain for small U:

- *Every $T \in \mathrm{Co}(L_{p,m})$ can be written as $T = \sum_{i\in I} c_i(T)\pi(x_i)u$ with*

$$\|(c_i(T))_{i\in I}\|_{\ell_{p,m}} \lesssim \|T\|_{\mathrm{Co}(L_{p,m})}.$$

- *For $(d_i)_{i\in I} \in \ell_{p,m}$ we have $T = \sum_{i\in I} d_i\pi(x_i)u \in \mathrm{Co}(L_{p,m})$ with*

$$\|T\|_{\mathrm{Co}(L_{p,m})} \lesssim \|(d_i)_{i\in I}\|_{\ell_{p,m}}.$$

Idea of the proof

Crucial operator:

$$\mathcal{T}_\Psi F := \sum_{i \in I} \langle F, \psi_i \rangle_{L_2} \lambda(x_i) K$$

as an approximation of K. We show the invertibility with a Neumann series argument, i.e.

$$\|\mathcal{T}_\Psi F\|_{L_{p,m}} \lesssim C_{K,U} \|F\|_{L_{p,m}}, \quad C_{K,U} < 1.$$

For that we use

$$\|F * K\|_{L_{p,m}} \leq \|F\|_{L_{p,m}}$$

as well as

$$\|K\|_{L_\infty(xU)} \in L_{1,w}(G) \Longrightarrow \mathrm{osc}_U(K) \in L_{1,w}(G).$$

This presupposes $K \in L_{1,w}(G)$!

What if $K \notin L_{1,w}(G)$?

Before:

$$\mathrm{Co}(L_{p,m}) := \{ T \in \mathcal{H}'_{1,w} \ : \ V_e \in L_{p,m}(G) \},$$

where

$$\mathcal{H}_{1,w} := \{ f \in \mathcal{H} \ : \ Vf \in L_{1,w}(G) \}.$$

In particular we necessarily assume

$$K = Vu \in L_{1,w}(G).$$

What happens, if this condition is not fulfilled, e.g.

$$K \in \bigcap_{1<p<\infty} L_{p,w}(G)?$$

New coorbit spaces

Consider the new space

$$\mathcal{S}_w := \{f \in \mathcal{H} \ : \ Vf \in \bigcap_{1<p<\infty} L_{p,w}(G)\} \subset \mathcal{H},$$

we then extend the **voice transform** to $\mathcal{S}_w'$ via

$$V_e T(x) := \langle T, \pi(x)u \rangle_{\mathcal{S}_w' \times \mathcal{S}_w}, \quad T \in \mathcal{S}_w'.$$

Then the **coorbit space** with respect to $L_{r,m}(G)$ is analogously defined via

$$\boxed{\mathrm{Co}(L_{r,m}) = \{T \in \mathcal{S}_w' \ : \ V_e T \in L_{r,m}(G)\}.}$$

Again we have

$$V_e^{-1} F \in \mathrm{Co}(L_{r,m}) \Longleftrightarrow F \in L_{r,m}(G), F * K = F.$$

Discretization of the new spaces

It holds that $K \notin L_{1,w}(G)$, but let us assume

$\mathrm{RC}_K : L_{r,m}(G) \to L_{r,m}(G), \quad \mathrm{RC}_K F = F * K,$ is continuous,

which is necessary! This is restrictive.

First idea: Take a **finite** subset $X_n = (x_{j,n})_{j \in I_n}$, $\# I_n < \infty$, and $e \in U_n \subset G$ such that

- $X_n \subset X_{n+1}$,
- $\overline{\bigcup_{n\in\mathbb{N}} \bigcup_{j \in I_n} x_{j,n} U_n} = G$.

Consider the projection

$$T_n : L_{r,m}(G) \to L_{r,m}(G), \quad T_n F = \sum_{x \in X_n} \langle F, \psi_{n,x} \rangle_{L_2} L_x K$$

with $\mathrm{Ran}\, T_n = \mathrm{span}\{L_x K \ : \ x \in X_n\} =: V_n$.

Satz

Under technical assumptions on the kernel K we have:

- *For all* $\varepsilon > 0$, $T \in \mathrm{Co}(L_{r,m})$ *there exists a* $n^* = n^*_{T,\varepsilon} \in \mathbb{N}$, *such that for all* $n \geq n^*$ *we have*

$$\|T - \sum_{x \in X_n} c(T)_{n,x} \pi(x) u\|_{\mathrm{Co}(L_{r,m})} \leq \varepsilon,$$

$$\|(c(T)_{n,x})_{x \in X_n}\|_{\ell_{p,m}} \leq C_n (1+\varepsilon) \|T\|_{\mathrm{Co}(L_{r,m})}.$$

- *For* $(d_x)_{x \in Y_n} \in \ell_{q,m}$ *we have* $T = \sum_{x \in Y_n} d_x \pi(x) u \in \mathrm{Co}(L_{r,m})$ *with*

$$\|T\|_{\mathrm{Co}(L_{r,m})} \leq D_n \|(d_x)_{x \in Y_n}\|_{\ell_{q,m}},$$

where $1/q + 1/p = 1 + 1/r$, $p > 1$.

$$C_n \sim_w |U_n|^{1-1/r} \|T_n^{-1}\|$$
$$D_n \sim_w |U_n|^{1/q-1} \cdot \|\mathrm{osc}_{U_n}(K) + |K|\|_{L_{p,w}}.$$

Example

Consider the group $G = (\mathbb{R}, +)$ and $\Omega \subset \mathbb{R}$, and the Hilbert space $B^2_\Omega = \{f \in L_2(\mathbb{R}) \ : \ \mathrm{supp}\widehat{f} \subset \Omega\}$. The representation π is given via

$$\pi(t)f(x) = f(x-t)$$

and as an admissable vector we choose

$$u = K = \mathcal{F}^{-1}\chi_\Omega \in B^2_\Omega.$$

It holds that $K \notin L_1(\mathbb{R})$, but $K \in \bigcap_{1<p<\infty} L_p(\mathbb{R})$ (under certain conditions). However the operator

$$\mathrm{RC}_K : L_p(\mathbb{R}) \to L_p(\mathbb{R}), \quad 1 < p < \infty,$$

is not always bounded! But there exists a $W \in L_1(\mathbb{R})$ with $K * W = K$, s.t. $\chi_\Omega \cdot \mathcal{F}W = \chi_\Omega$.

Discretization under additional conditions

Previous idea: discretize the convolution RC_K. Assume there is a $W \in L_{1,w}(G)$ with $K * W = K$, then we have

$$F * W = F \quad \text{für alle} \quad V_e^{-1} F \in \mathrm{Co}(L_{r,m}).$$

Therefore: discretize RC_W instead!

Theorem

Under certain conditions on K and W we have:

- *Each $T \in Co(L_{r,m})$ can be written as $T = \sum_{i \in I} c_i(T)\pi(x_i)u$ with*

$$\|(c_i(T))_{i \in I}\|_{\ell_{r,m}} \lesssim \|T\|_{Co(L_{r,m})}.$$

- *For $(d_i)_{i \in I} \in \ell_{r,m}$ we have $T = \sum_{i \in I} d_i \pi(x_i)u \in Co(L_{r,m})$ with*

$$\|T\|_{Co(L_{r,m})} \lesssim \|(d_i)_{i \in I}\|_{\ell_{r,m}}.$$

Example

Consider again the group $G = (\mathbb{R}, +)$ and $\Omega \subset \mathbb{R}$, and the Hilbert space $B^2_\Omega = \{f \in L_2(\mathbb{R}) \;:\; \operatorname{supp}\widehat{f} \subset \Omega\}$. The representation π is given via

$$\pi(t)f(x) = f(x-t)$$

and as an admissable vector we choose

$$u = K = \mathcal{F}^{-1}\chi_\Omega \in B^2_\Omega.$$

Then there exists W as described above. Let $x_k = k/2\pi$, $k \in \mathbb{N}$, and $U = [-1/4\pi, 1/4\pi)$. Then we conclude

$$\mathcal{M}_r \ni f = \sum_{k\in\mathbb{Z}} \langle f, K(\cdot - k/2\pi)\rangle_{L_2} K(\cdot - k/2\pi)$$
$$\iff (\langle f, K(\cdot - k/2\pi)\rangle_{L_2})_{k\in\mathbb{Z}} \in \ell_r(\mathbb{Z}),$$

see also Nyquist-Shannon sampling theorem.

However: The existence of such a $W \in L_{1,w}(G)$ is not ensured; there are counterexamples!

Generalization of the theory

We replace the group G with a σ-finte measure space (X, μ), and consider a **Parseval frame** $\{\psi_x\}_{x\in X}$ on $\mathcal{H}$, i.e

$$\|f\|_{\mathcal{H}} = \int_X |\langle f, \psi_x\rangle_{\mathcal{H}}|^2 \, d\mu(x) \quad \text{for all } f \in \mathcal{H}.$$

The **voice transform** is then given by

$$V : \mathcal{H} \to L_2(X, \mu), \quad Vf = \langle f, \psi_x\rangle_{\mathcal{H}}.$$

We can now define coorbit spaces similarly to the construction before!
Application: characterization vof **inhomogeneous Besov spaces** through

$$B^{s-1/2+1/d}_{p,p} \cong \mathrm{Co}(L_{p,s}).$$

These spaces can also be considered and discretized as above for $K \notin L_{1,w}(X, \mu)$.

Summary

- Coorbit theory connects smoothness spaces with an underlying group structure and gives a universal access as well as possible discretizations
- Normally we assume $K \in L_{1,w}(G)$, but the weaker case $K \in \bigcap_{1<p<\infty} L_{p,w}(G)$ also leads to meaningful smoothness spaces
- However, any discretization is only possible if RC_K is bounded on $L_{r,m}(G)$
- A proper discretization is only possible if there exists an additional kernel $W \in L_{1,w}(G)$ with $K * W = W$
- These additional assumptions need to be checked individually!

Thank you!

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