Inhomogeneous Shearlet Coorbit Spaces and their Atomic Decomposition

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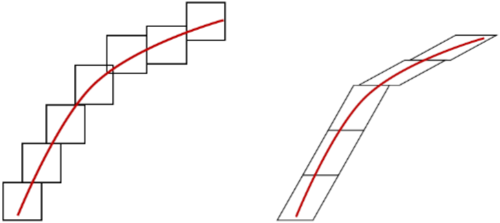
Shearlets

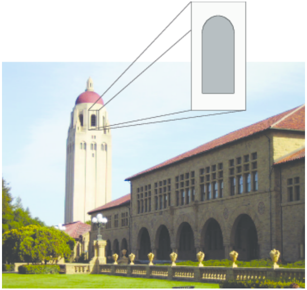
- Goal: Analyze signals by dividing them into suitable building blocks
- Some signals call for anisotropic blocks

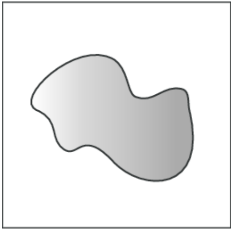


- E.g.: Curvelets, Ridgelets, Bendlets, Contourlets, Shearlets, etc.

Shearlets

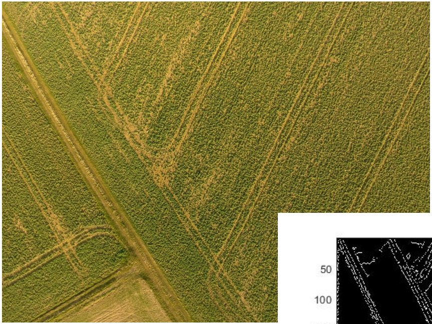
Example: Cartoon-like images

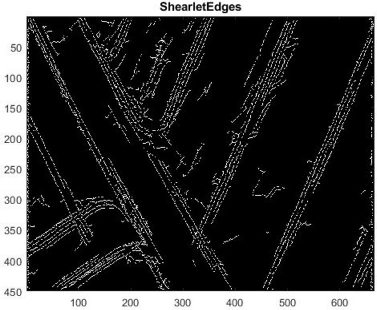




Shearlets

Example: Geographic images





Shearlets: Mathematical Viewpoint

Let $(a, s, t) \in \mathbb{R}^* \times \mathbb{R}^{d-1} \times \mathbb{R}^d$, then for a **mother-shearlet** ψ the **shearlets** are defined as

$$\psi_{(a,s,t)}(x) := |\det A_a|^{-\frac{1}{2}} \psi(A_a^{-1} S_s^{-1}(x - t)),$$

where

$$A_a := \begin{pmatrix} a & 0_{d-1}^T \\ 0_{d-1} & \operatorname{sign}(a)|a|^{1/d} I_{d-1} \end{pmatrix}$$

and

$$S_s := \begin{pmatrix} 1 & s^T \\ 0_{d-1} & I_{d-1} \end{pmatrix}.$$

The **shearlet-transform** of a signal f is

$$\mathcal{ST}_\psi f(a, s, t) = \langle f, \psi_{(a,s,t)} \rangle.$$

Function Spaces

- **Sobolev spaces** $W_p^m(\Omega)$ as natural solution spaces for certain PDEs
- **Besov spaces** $B_{p,q}^s(\Omega)$ closely related to adapative Wavelet algorithms
- **Modulation spaces** $M_m^{p,q}(\Omega)$ related to time-frequency analysis

Classical Coorbit Theory

Ingredients:

- Group G,
- Hilbert space $L_2(\mathbb{R}^d)$,
- Unitary representation $\pi : G \to \mathcal{U}(L_2(\mathbb{R}^d))$,
- Admissable vector ψ,
- **Voice transform** of a signal f:

$$V_\pi f(x) := \langle f, \pi(x)\psi \rangle, \quad x \in G,$$

Then we can define the **coorbit spaces** for a weighted Lebesgue space $L_{p,m}(G)$ as

$$\mathrm{Co}(L_{p,m}) := \{f \in \mathcal{S}' : V_\pi f \in L_{p,m}(G)\}.$$

Coorbit Theory and Function Spaces

$$\text{hom. Besov spaces } \dot{B}^s_{p,q} \longleftrightarrow G = \mathbb{R}^* \times \mathbb{R}^d$$

$$\text{Modulation spaces } M^{p,q}_m \longleftrightarrow G = \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T}$$

$$\text{Shearlet spaces } \mathcal{SC}_{p,m} \longleftrightarrow G = \mathbb{R}^* \times \mathbb{R}^{d-1} \times \mathbb{R}^d$$

A generalized coorbit theory shows:

$$\text{inhom. Besov spaces } B^s_{p,q} \longleftrightarrow X = (\{\infty\} \cup [-1,1]^*) \times \mathbb{R}^d$$

Are there inhomogeneous shearlet spaces similar to this approach?

Inhomogeneous Shearlet Coorbit Spaces

Answer: Yes, but...

Ingredients:

- Measure space
  $X = (\{\infty\} \times \mathbb{R}^{d-1} \times \mathbb{R}^d) \cup ([-1,1]^* \times \mathbb{R}^{d-1} \times \mathbb{R}^d)$,
- Tight continuous frame $\mathfrak{F} = \{\psi_x\}_{x \in X}$ for $L_2(\mathbb{R}^d)$, where

$$\psi_{(\infty,s,t)} := \Phi(S_s^{-1}(\cdot - t)),$$
$$\psi_{(a,s,t)} := |\det A_a|^{-\frac{1}{2}} \Psi(A_a^{-1} S_s^{-1}(\cdot - t)),$$

- Voice transform of a signal f:

$$V_{\mathfrak{F}} f(x) = \langle f, \psi_x \rangle.$$

Inhomogeneous shearlet coorbit spaces:

$$\mathcal{SC}^r_{\mathfrak{F},p} := \mathrm{Co}_{\mathfrak{F}}(L_{p,v_r}(X)) = \{f \in \mathcal{S}' : V_{\mathfrak{F}} f \in L_{p,v_r}(X)\}.$$

Atomic Decomposition

In the classical setting: $T = \sum_{i \in I} c_i \pi(x_i)\psi \in \mathrm{Co}(L_{p,m})$ with

$$\|T\|_{\mathrm{Co}(L_{p,m})} \sim \|(c_i)_{i \in I}\|_{\ell_{p,m}},$$

with the crucial assumption $\langle \psi, \pi(x)\psi \rangle \in L_1(G)$.

However, in our setting we only have

$$K(x) := \langle \psi, \psi_x \rangle \in \bigcap_{1<p<\infty} L_p(X) \nsubseteq L_1(X).$$

What is necessary (and sufficient for a weaker discretization) is

$$f * K \in L_{p,m}(X) \text{ for all } f \in L_{p,m}(X).$$

Thank you for your attention!

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