Inhomogeneous Shearlet Coorbit Spaces

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Shearlets

Shearlets are a new tool in image analysis and especially useful to extract directional information.

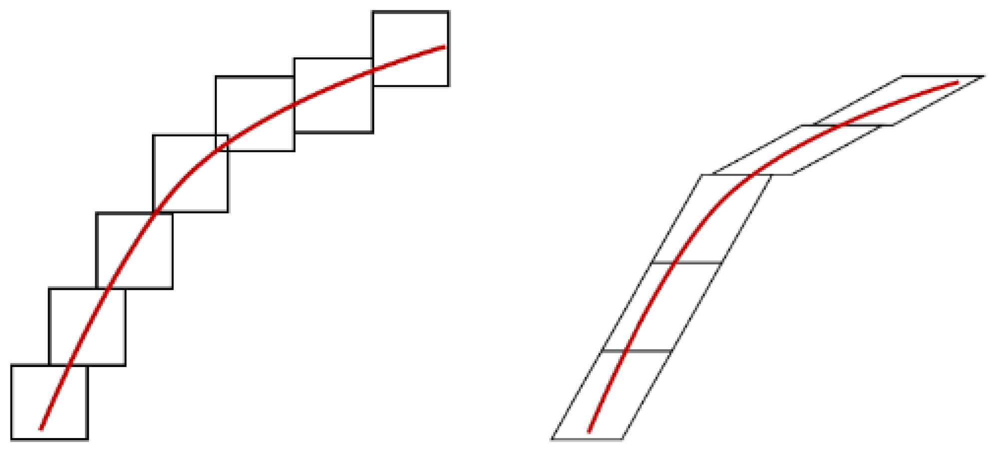

Figure 1: Isotropic atoms (left) and anisotropically scaled and sheared atoms (right) covering curvilinear singularity

First, for $a \in \mathbb{R}^*$ and $s \in \mathbb{R}^{d-1}$ we define the *parabolic dilation matrix* A_a and the *shearing matrix* S_s as follows:

$$A_a = \begin{pmatrix} a & 0_{d-1}^T \\ 0_{d-1} & \operatorname{sgn}(a)|a|^{1/d} I_{d-1} \end{pmatrix}, \qquad S_s = \begin{pmatrix} 1 & s^T \\ 0_{d-1} & I_{d-1} \end{pmatrix}.$$

Together with the *translation* $t \in \mathbb{R}^d$ we obtain the *full shearlet group* by setting $\mathbb{S} = \mathbb{R}^* \times \mathbb{R}^{d-1} \times \mathbb{R}^d$ endowed with the group operation

$$(a,s,t) \circ_{\mathbb{S}} (a',s',t') = (aa', s + |a|^{1-1/d}s', t + S_s A_a t').$$

Then, by setting

$$\psi_{(a,s,t)}(y) := \pi(a,s,t)\Psi(y) := |\det A_a|^{-1/2}\Psi(A_a^{-1}S_s^{-1}(y-t))$$

for any $\Psi \in L_2(\mathbb{R}^d)$, we have a *unitary representation* π of $\mathbb{S}$. This representation is irreducible, integrable, and square-integrable, in particular it follows that there exist *admissable shearlets*. Hence, the classical coorbit theory is applicable, see [1].

Inhomogeneous Shearlet Frame

Let

$$X = \left(\{\infty\} \times \mathbb{R}^{d-1} \times \mathbb{R}^d\right) \cup \left([-1,1]^* \times \mathbb{R}^{d-1} \times \mathbb{R}^d\right)$$

be our parameter space, which is an adaptation of the full shearlet group, similar to [3] with the affine group. We then equip X with the positive Radon measure

$$\int_X F(x)\,\mathrm{d}\mu(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}} F(\infty,s,t)\,\mathrm{d}s\,\mathrm{d}t + \int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}}\int_{-1}^{1} F(a,s,t)\,\frac{\mathrm{d}a}{|a|^{d+1}}\,\mathrm{d}s\,\mathrm{d}t.$$

For odd dimension d we define the following function Φ via its fourier transform:

$$|\hat{\Phi}(\omega_1,\tilde{\omega})| = \omega_1^{\frac{d-1}{2}} \left(\int_{\mathbb{R}\setminus[-|\omega_1|,|\omega_1|]} \frac{|\hat{\Psi}(y,\tilde{\omega})|^2}{|y|^d}\,\mathrm{d}y\right)^{1/2},$$

where Ψ is an admissable shearlet. By defining a family $\mathfrak{F} = \{\psi_x\}_{x\in X}$ via

$$\psi_{(\infty,s,t)}(y) = \Phi(S_s^{-1}(y-t)), \qquad \psi_{(a,s,t)}(y) := |\det A_a|^{-1/2}\Psi(A_a^{-1}S_s^{-1}(y-t)),$$

we obtain the following.

Theorem 1. *The inhomogeneous shearlet frame $\mathfrak{F}$ is a continuous Parseval frame of $L_2(\mathbb{R}^d)$.*

Now we define the *inhomogeneous shearlet transform* based on $\mathfrak{F}$ as

$$\mathcal{SH}_{\mathfrak{F}} : L_2(\mathbb{R}^d) \to L_2(X,\mu), \qquad f \mapsto \mathcal{SH}_{\mathfrak{F}} f(x) = \langle f, \psi_x \rangle.$$

Integrability of the reproducing kernel

The *reproducing kernel* based on $\mathfrak{F}$ is defined as

$$R_{\mathfrak{F}} : X \times X \to \mathbb{C}, \quad (x,y) \mapsto R_{\mathfrak{F}}(x,y) = \mathcal{SH}_{\mathfrak{F}}(\psi_y)(x) = \langle \psi_y, \psi_x \rangle$$

and for all $f \in L_2(\mathbb{R}^d)$ the *reproducing property* holds:

$$R_{\mathfrak{F}}(\mathcal{SH}_{\mathfrak{F}} f) = \mathcal{SH}_{\mathfrak{F}} f.$$

We are interested in the kernel space

$$\mathcal{A}_{q,1} = \{K : X \times X \to \mathbb{C} : \|K|\mathcal{A}_{q,1}\| < \infty\},$$

$$\|K|\mathcal{A}_{q,1}\| = \max\left\{ \operatorname*{ess\,sup}_{x\in X} \left(\int_X |K(x,y)|^q\,\mathrm{d}\mu(y)\right)^{1/q}, \operatorname*{ess\,sup}_{y\in X} \left(\int_X |K(x,y)|^q\,\mathrm{d}\mu(x)\right)^{1/q} \right\},$$

and we can show the following.

Theorem 2. *We have $R_{\mathfrak{F}} \in \mathcal{A}_{q,1}$ for every $q > 1$.*

- This integrability differs from the setting in [3] and [4] and poses problems regarding the discretization of the spaces defined below.
- We have the continuous embeddings

$$R_{\mathfrak{F}}(L_p(X,\mu)) \hookrightarrow L_r(X,\mu)$$

for all $p < r$.

Inhomogeneous Shearlet Coorbit Spaces

We define the Banach spaces $\mathcal{H}_{q,1} := \{f \in L_2(\mathbb{R}^d) : \mathcal{SH}_{\mathfrak{F}} f \in L_q(X,\mu)\}$ with the natural norms $\|f|\mathcal{H}_{q,1}\| := \|\mathcal{SH}_{\mathfrak{F}} f | L_q(X,\mu)\|$.

- $(\mathcal{H}_{q,1})^\sim$ denotes the anti-dual of $\mathcal{H}_{q,1}$.
- $\mathcal{H}_{q,1}$ is *densely embedded* into $L_2(\mathbb{R}^d)$.
- We have $\mathfrak{F} \subset \mathcal{H}_{q,1}$.
- Hence we can *extend the inhomogeneous shearlet transform* to $(\mathcal{H}_{q,1})^\sim$ and denote it with $\mathcal{SH}_{\mathfrak{F},q}$ to indicate the dependency on the parameter q.
- The *reproducing property* of the kernel $R_{\mathfrak{F}}$ also extends to $(\mathcal{H}_{q,1})^\sim$.

Theorem 3. *For $1 < q < p \leq \infty$ the inhomogeneous shearlet coorbit spaces with respect to the Lebesgue spaces $L_p(X,\mu)$ are defined as*

$$\mathcal{SC}_{\mathfrak{F},p,q} = \{f \in (\mathcal{H}_{q,1})^\sim : \mathcal{SH}_{\mathfrak{F},q} f \in L_p(X,\mu)\}$$

and endowed with the norm

$$\|f|\mathcal{SC}_{\mathfrak{F},p,q}\| = \|\mathcal{SH}_{\mathfrak{F}} f | L_p(X,\mu)\|$$

these spaces are Banach spaces. Furthermore $\mathcal{SH}_{\mathfrak{F},q}$ induces an isometric isomorphism

$$\mathcal{SC}^v_{\mathfrak{F},p,q} \longleftrightarrow \{F \in L_p(X,\mu) : R_{\mathfrak{F}}(F) = F\}.$$

References

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