Universität Bremen

Coorbit Spaces and Banach Frames on Homogeneous Spaces

Classical Setting:

Let $\mathcal{N}$ be a differentiable manifold and let $\mathcal{G}$ be a separable Lie group, with (right) Haar measure μ. A *continuous representation* of $\mathcal{G}$ in $\mathcal{H} = L_2(\mathcal{N})$ is defined as a mapping $U : \mathcal{G} \longrightarrow \mathcal{L}_U(\mathcal{H})$ of $\mathcal{G}$ into the unitary operators on $\mathcal{H}$. U is said to be *square-integrable* if it is irreducible and there exists a nonzero $\psi \in \mathcal{H}$ such that

$$\int_{\mathcal{G}} |\langle \psi, U(g)\psi \rangle|^2 d\mu(g) < \infty .$$

Such a vector ψ is called *admissible*. In this case, the *wavelet transform* $V_\psi f(h) := \langle f, U(g^{-1})\psi \rangle$ is well–defined. By H_1' we denote the space of all continuous linear functionals on

$$H_1 := \{f \in L_2(\mathcal{N}) : \quad V_\psi f \in L_1(\mathcal{G})\}.$$

Then we may define the so–called *coorbit spaces* by setting

$$M_p := \{f \in H_1' : \ V_\psi f \in L_p(\mathcal{G})\} .$$

Classical examples are given by the *modulation spaces* and the *Besov spaces*. To handle these spaces numerically, it is necessary to construct some kind of basis for them. The famous Feichtinger/Gröchenig–theory ensures that judiciously chosen sets $(x_i)_{i \in I}$ in $\mathcal{G}$ give rise to so–called *Banach frames*, i.e., every $f \in M_p$ has an expansion

$$f = \sum_{i \in \mathcal{I}} c_i (U(x_i^{-1})\psi ,$$

$$A_1 \|f\|_{M_p} \le \|(c_i)_{i \in I}\|_{\ell_p} \le A_2 \|f\|_{M_p} .$$

Extension to Coset Spaces:

There are cases in which square-integrable representations are not available. One way out is to restrict U to a convenient quotient space $\mathcal{G}/\mathcal{P}$, where $\mathcal{P}$ is a closed subgroup of $\mathcal{G}$. Then a suitable wavelet transform can be defined by

$$V_\psi f(h) := \langle f, U(\sigma(h)^{-1})\psi \rangle ,$$

where $\sigma : \mathcal{G}/\mathcal{P} \to \mathcal{G}$ is a Borel section of the fiber bundle $\Pi : \mathcal{G} \to \mathcal{G}/\mathcal{P}$. By means of the following three step procedure, we can establish Banach frames for the associated coorbit spaces.

- Fix a section σ and a suitable set $(x_i)_{i \in I}$, $x_i \in \mathcal{G}$, $(h_i)_{i \in I} \in \mathcal{G}/\mathcal{P}$. Define an operator T on $L_2(\mathcal{G}/\mathcal{P})$ by
$$T(F)(h) := \sum_{i \in I} c_{i,\sigma}(F) R(h_i, h) ,$$
$$R(h_i, h) := V_\psi(U(\sigma(h_i)^{-1})\psi)(h)$$
- Show that T is invertible, this yields:
$$F = TT^{-1}F = \sum_{i \in I} c_{i,\sigma}(T^{-1}F) R(h_i, h)$$
- Apply the expansion to $F = V_\psi(f)$:
$$f = \sum_{i \in I} c_{i,\sigma}(T^{-1} V_\psi(f)) U(\sigma(h_i)^{-1})\psi ,$$
and establish the frame bounds.

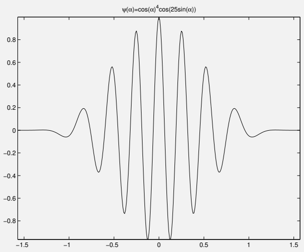
Application to the Spheres:

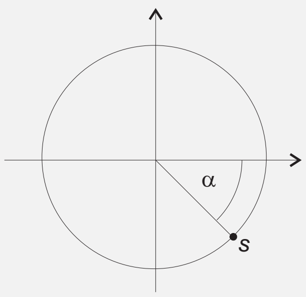
The whole construction may be used to construct generalized windowed Fourier transforms, the associated modulation spaces and the corresponding Gabor frames on the spheres $S^{n-1} \subset \mathbf{R}^n$. Indeed, the Euklidean group $\mathcal{G} := E(n) = SO(n) \ltimes \mathbf{R}^n$ admits the representation

$$(U(R,p))f(s) := e^{i<s,p>} f(R^{-1}s) ,$$

where $s \in S^{n-1}$, which is square–integrable when restricted to $\mathcal{G}/\mathcal{P}$ for some suitable stability group $\mathcal{P}$.

As an example, on S^1 an admissible wavelet has to satisfy the condition $0 < \int_{-\pi/2}^{\pi/2} |\psi(\alpha)|^2 / \cos(\alpha) d\alpha < \infty$.





Gabriele Steidl
Fakultät Mathematik und Informatik
Universität Mannheim
D-68131 Mannheim

Stephan Dahlke
Philipps–Universität Marburg
Fachbereich Mathematik und Informatik
D-35032 Marburg

Gerd Teschke
Zentrum für Technomathematik
Universität Bremen
D-28334 Bremen