

Documentation to software package: *Adaptive spline approximation*

This tool *Adaptive spline approximation* computes approximations to smooth enough functions using step functions or linear functions. In order to run this tool you need a C++ compiler and Matlab. The following files are provided with this software package:

- Adaptive_spline_approximation_Documentation.pdf (this file)
- Adaptive_spline_approximation_picewise_constants.cpp (source code)
- Adaptive_spline_approximation_picewise_linear.cpp (source code)
- Makefile
- Adaptive_spline_approximation.m (user interface)
- Adaptive_spline_approximation.fig (bin for user interface)

The software package *Adaptive spline approximation* is a Beta Version 0.9. If you have any suggestions or remarks please feel free to contact our workgroup. This software package is provided solely for information purposes. Install and run it on your own responsibility, only!

Steps to install and run this tool:

1. Copy this documentary and unpack the archive file into a suitable folder.
2. Compile the source code files. The *make* command uses the provided *Makefile*.
3. Start the user interface *Adaptive_spline_approximation.m* under Matlab.
4. Select desired settings and press the *Start* button on the user interface.
5. A figure displaying the results will open up. For this purpose, the output will be written to a temporary *.m* file.

Let us explain the theoretical background of this tool. As a first point we describe the idea of the algorithm behind the approximation:

1. The algorithm

Consider the set $\mathcal{I} := \{[a, b] \subset [0, 1]\}$. Assume we have a function $\varepsilon : \mathcal{I} \rightarrow [0, \infty)$ with the following properties:

- (1) For all $I \in \mathcal{I}$ the estimation

$$\varepsilon(I) \geq \|f - \tilde{f}\|_{L^\infty(I)} \quad (1)$$

holds, where \tilde{f} is the approximation of f .

- (2) It holds

$$\lim_{h \rightarrow 0} \sup\{\varepsilon(I) : I \in \mathcal{I}, |I| = h\} = 0. \quad (2)$$

More details about the error-estimation are described in the next section. We now start with explaining the approximation with step functions: We fix an error-tolerance δ , this means we want to get $\varepsilon(I) \leq \delta$ for all $I \in \mathcal{I}$. We begin with the unit-interval and compute the error-estimation $\varepsilon([0, 1])$. If this value is smaller than δ we have found an approximation by the function $\tilde{f}(x) = 1/2(\max_{x \in [0, 1]} f(x) + \min_{x \in [0, 1]} f(x))$ for all $x \in [0, 1]$. If $\varepsilon([0, 1]) > \delta$ we separate the interval into the intervals $[0, 1/2]$ and $[1/2, 1]$. Now we have to determine the values $\varepsilon([0, 1/2])$ and $\varepsilon([1/2, 1])$ and compare them to δ . If $\varepsilon([0, 1/2]) < \delta$, we can approximate the function f on $[0, 1/2]$ by $\tilde{f}(x) = (\max_{x \in [0, 1/2]} f(x) + \min_{x \in [0, 1/2]} f(x))$. Otherwise the interval $[0, 1/2]$ has to be separated again and we continue in this way. The same procedure has to be accomplished for the interval $[1/2, 1]$. The main difference when approximating with linear functions is that we do not have to calculate function values. Instead of these value we must determine the sample values. The function values of the approximation in these sample values are the same as the values of the origin function. Between the sample values the approximation is piecewise linear. Furthermore we use another error-estimation. But instead of these differences the principle is the same.

2. Theoretic introduction

Consider a function f on the unit interval $[0, 1]$. Our aim is to find an approximation of f by step functions respectively by linear functions. This leads immediately to the question how to measure the quality of the approximation. We will see, that for our error-estimation we need to require the smoothness of f : For approximation with step functions we need $f \in \mathcal{C}^1([0, 1])$, for approximation with linear functions we have to ensure $f \in \mathcal{C}^2([0, 1])$. We choose for the error-estimation:

- (1) For approximating with step functions:

$$\varepsilon(I) := \int_I |f'(t)| dt. \quad (3)$$

(2) For approximating with linear functions:

$$\varepsilon(I) := \int_I |f''(t)| dt. \quad (4)$$

We will now proof, that the error-estimation in (4) fulfil the conditions (1) and (2). Therefore we define the linear function $L([a, b], f, x)$ which is determined by the values $f(a)$ and $f(b)$:

$$L([a, b], f, x) := f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

Then the following holds:

Proposition: Assume $f \in \mathcal{C}^2([0, 1])$. Then the error-estimation in (4) fulfil the conditions (1),(2).

Proof: We have to show that $\varepsilon([a, b]) \geq \|f - L([a, b], f, \cdot)\|_{L_\infty([a, b])}$ for all $[a, b] \in \mathcal{I}$. For $x \in [a, b]$ we obtain:

$$|f(x) - L([a, b], f, x)| = \left| f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a) \right|.$$

It exists a $\tau_1 \in (a, b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(\tau_1)$, so we get

$$\begin{aligned} |f(x) - L([a, b], f, x)| &= |f(x) - f(a) - f'(\tau_1)(x - a)| \\ &= |x - a| \left| \frac{f(x) - f(a)}{x - a} - f'(\tau_1) \right| \\ &\leq \left| \frac{f(x) - f(a)}{x - a} - f'(\tau_1) \right|. \end{aligned}$$

We find a $\tau_2 \in (a, x)$ such that $\frac{f(x)-f(a)}{x-a} = f'(\tau_2)$. This leads to

$$|f - L([a, b], f, x)| \leq |f'(\tau_1) - f'(\tau_2)|.$$

Because of

$$|f'(z_1) - f'(z_2)| = \left| \int_{z_1}^{z_2} f''(t) dt \right| \leq \int_{z_1}^{z_2} |f''(t)| dt = \varepsilon([z_1, z_2]) \text{ for all } z_1, z_2 \in [0, 1]$$

we obtain

$$|f - L([a, b], f, x)| \leq \varepsilon([\tau_1, \tau_2]) \leq \varepsilon([a, b]).$$

So we have

$$\|f - L(I, f, \cdot)\|_{L^\infty([a,b])} \leq \varepsilon([a, b]).$$

To show the condition (2) we use the estimation

$$\int_a^b |f''(t)| dt \leq (b - a) \|f''\|_{L^\infty([0,1])}.$$

Because f'' is continuous and $[0, 1]$ is compact the term $\|f''\|_{L^\infty([0,1])}$ is bounded independent of a, b . We get

$$\sup_{\substack{[a,b] \in \mathcal{I} \\ b-a=h}} \left\{ \int_a^b |f''(t)| dt \right\} \leq \sup_{\substack{[a,b] \in \mathcal{I} \\ b-a=h}} \{(b - a) \|f''\|_{L^\infty([0,1])}\} = h \|f''\|_{L^\infty([0,1])} \xrightarrow{h \rightarrow 0} 0.$$

□

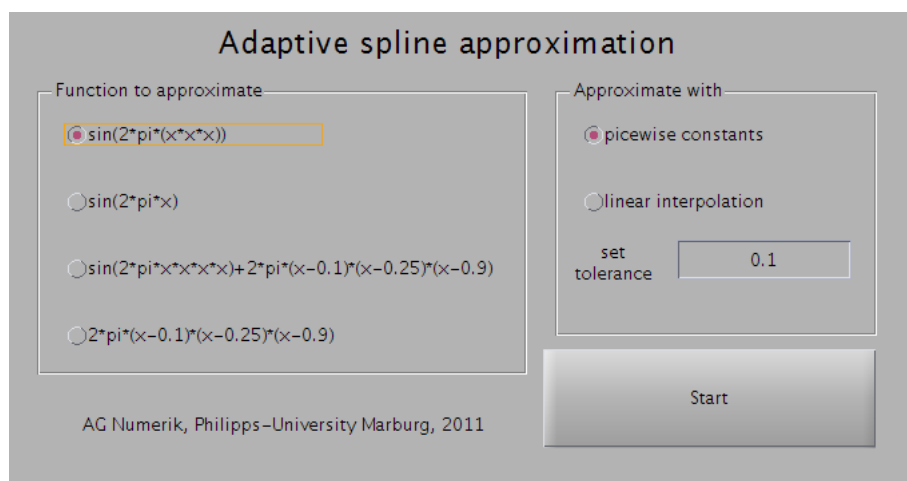
With the notations $c_f([a, b]) = 1/2(\max_{x \in [a,b]} f(x) + \min_{x \in [a,b]} f(x))$ we formulate the corresponding statement for the approximation with step functions:

Proposition: Assume $f \in \mathcal{C}^1([0, 1])$. Then the error-estimation in (3) fulfil the conditions (1),(2).

The proof is left to the reader as an exercise.

3. Explanation of the program

When you run *Adaptive_spline_approximation.m* in Matlab the following window will open:



You can choose one of the four functions through flagging the radiobutton. Then you have to choose the way of approximating the function: You can select between 'picewise constant' and 'linear interpolation'. Then type in your error-tolerance. Now press the

Start-Button and the approximation-process can be observed in a new figure.

In the background: The Matlab program calls the appropriate C++ program which computes the approximation as described in the first section. The C++ program creates a new (temporary) output *.m* file, which is executed and opens up the results in a graphical figure.

Important:

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