Learning seminar SoSe 2023,
Philipps-Universität Marburg,

A gentle and coordinate free introduction to the Ricci Flow

The purpose of this seminar is to introduce the audience to the Ricci Flow and to prove Hamilton's remarkable result

Theorem (Hamilton 1982, [1])**.** *Let M be a compact 3-manifold which admits a Riemannian metric with strictly positive Ricci curvature. Them M also admits a metric of constant positive curvature.*

1. **A first look at the Ricci Flow and its properties**, 27.04.2023 – Speaker: Taki
   We will begin with a justification of why the Ricci flow evolution equation is chosen as it is and will calculate first examples. After the definition of the Ricci soliton, we will heuristically discuss the Ricci Flow for surfaces and 3-dimensional manifolds.
   Chapter 1 of [2].
2. **Evolution of geometric quantities** 04.05.2023 – Speaker: Henrik Naujoks
   For $(g_t)_t$ a curve of Riemannian metrics, we will compute how the geometric quantities (like curvatures) derived from g_t are changing with t. Subsequently we will apply the results when g_t fulfills the Ricci flow.
   Chapter 2 of [2].
3. **Maximum Principle** 11.05.2023 – Speaker: Oliver Goertsches
   This maximum principle will allow us to control the volume, Riemannian curvature tensor as well as the scalar curvature. Moreover we will provide a first estimate from above for the maximal time of existence of the Ricci flow.
   Chapter 3 of [2] without Theorem 3.3.3.
4. **Parabolic PDEs and existence theory for the Ricci flow** 25.05.2023 – Speaker: Max Jahnke
   We discuss briefly the important properties of parabolic PDEs for functions, the definition of the symbol of a differential operator and their generalisation to vector bundles. Finally we show that the Ricci flow is not parabolic, due to the diffeomorphism invariance of the equation.
   Chapter 4 and section 5.1 of [2], .
5. **The DeTurck trick** 01.06.2023 – Speaker: Nikolas Wardenski
   The DeTurck trick is used to show short-time existence of the Ricci flow by modifying the Ricci flow equation, such that the modified equation will be parabolic. Then it will be shown that a solution to the latter equation gives a solution to the Ricci flow. Finally we'll show the important fact, that if the flow's existence is finite on a closed manifold, then the norm of the Riemannian curvature tensor must blow-up as the time approaches the maximal time of existence.
   Section 5.2 and section 5.3 of [2], .
6. **Compactness of Ricci flows and blow ups** 15.06.2023 – Speaker: Benjamin Becker
   First we will need to discuss a notion of convergence for Riemannian manifolds and a convergence for Ricci flows. The latter convergence has a compactness theorem if some curvature bound is fulfilled and the injectivity radius in bounded from below for the sequence of Ricci flows. While the curvature condition is easy to establish, the bound on the injectivity radius is a more delicate matter. Therefore we will need the monotonicity of the so called $\mathcal{W}$ functional.
   Section 7, Definition of the $\mathcal{W}$ functional, Definition 8.1.1 and Proposition 8.1.2, Lemma 8.1.8 without proof, everything from [2].

7. **The $\mathcal{W}$-Functional and the no local volume collaps theorem** 22.06.2023 – Speaker: n/a

   Here we prove the monotonicity of $\mathcal{W}$ which is used to prove later the no local collapsing of the volume. This will lead to out desired lower bound for the injectivity radius of a sequence of Ricci flows.

   8.2.1, 8.2.3, 8.2.4, 8.2.6 (without proof), 8.2.7, 8.3.1 (without proof, try to explain how the monotonicity of $\mathcal{W}$ is used here), Sections 8.4 and 8.5, all from [2]

8. **The Uhlenbeck trick and a ODE-PDE theorem** 29.06.2023 – Speaker: Nicolina Istrati

   In dimension 3 it is more convenient to study the evolution equation of the Einstein tensor along the Ricci Flow. The Einstein tensor is a section of the endomorphism bundle, but we would like to see it as a section of the subbundle of symmetric bilinear form with respect to the metric g_t. This subbundle depends on the time parameter and the Uhlenbeck trick gets rid of this dependency in a certain way. Finally we will prove an ODE-PDE Theorem which tells us that if some subset of a vector bundle is preserved under an ODE it is also preserved under a non-homogeneous heat equation. This theorem is central to prove Hamilton's result in dimension 3.

   Sections 9.2 - 9.6 of [2]

9. **Hamilton's theorem** 06.07.2023 – Speaker: n/a

   We apply the ODE-PDE Theorem of the previous talk to the Einstein tensor in 3 dimensions, which will lead to a theorem, where the metrics gets "rounder" along the Ricci flow. Finally, at the end, we will piece all important results of this seminar together to obtain Hamilton's result.

   Sections 9.7 and 10.1 of [2]

References

[1] R. S. Hamilton. "Three-manifolds with positive Ricci curvature". *J. Differential Geometry* 17.2 (1982), pp. 255–306.

[2] P. Topping. *Lectures on the Ricci flow.* Vol. 325. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2006, pp. x+113.