Tall and monotone complexity one spaces of dimension six

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Content

Hamiltonian T-spaces and their Complexity

- (M^{2n}, ω) : **compact** symplectic manifold
- $T^d \cong (S^1)^d \curvearrowright (M, \omega)$ effective and Hamiltonian

  with moment map $\phi : M \to \left(\mathrm{Lie}(T^d)\right)^*$, i.e., for $\xi \in \mathrm{Lie}(T^d)$

$$\omega(X_\xi, \cdot) = -d\,\langle \phi, \xi \rangle$$

- the **complexity** of this action is $k := n - d$
- (M, ω, T, ϕ) is called a **complexity k space**

Theorem (Atiyah, Guillemin-Sternberg '82)

$\phi(M) = \Delta_M$ is a convex polytope and the fibers of ϕ are connected.

Monotone

Monotone:

- (M, ω) is **monotone** if $c_1(M) = \lambda[\omega]$ for some $\lambda \in \mathbb{R}$.
- Fact: monotone $(M, \omega, T, \phi) \rightsquigarrow$ **positive monotone**, i.e., $\lambda > 0$.
  $\rightsquigarrow$ Assume $c_1(M) = [\omega]$.

Tall

- (Karshon '98): Classification of four-dimensional complexity one spaces
- (Karshon-Tolman '03): Classification of **tall** complexity one spaces

Tall:

- Given a complexity one space (M, ω, T, ϕ), for each $x \in \Delta_M$ $M_x := \phi^{-1}(x)/T$ is a point or a topological surface.
- (M, ω, T, ϕ) is **tall** if M_x is a topological surface for all $x \in \Delta_M$.

Results

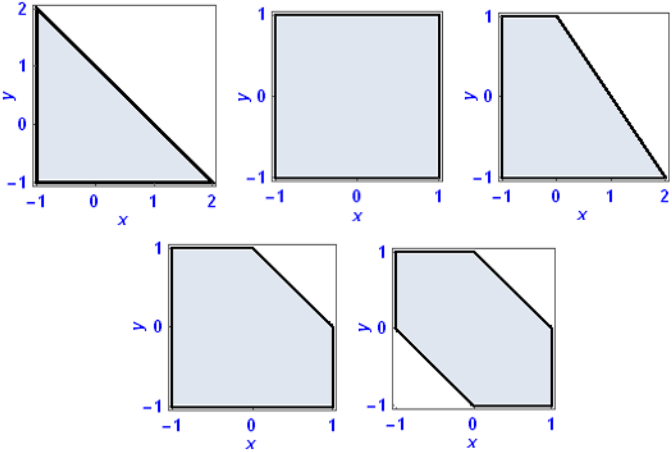
The moment map polytope

Proposition (C.-Sabatini-Sepe)

Let $(M^{2n}, \omega, T^{n-1}, \phi)$ *be a tall complexity one space with* $c_1(M) = [\omega]$. *Then* $\phi(M) = \Delta_M$ *is a reflexive Delzant polytope.*

Fact: There exist just finitely many reflexive (Delzant) polytopes in any dimension.

Two-dimensional reflexive Delzant polytopes



Main Result

Theorem (C.-Sabatini-Sepe)

Let (M^6, ω, T^2, ϕ) be a tall and monotone complexity one space . Then:

- *(Uniqueness) (M, ω, T, ϕ) is determined by its Duistermaat-Heckman function.*
- *(Extension) The T^2-action can be extended to a complexity zero action.*
- *(Finiteness) There exist exactly 20 tall and monotone complexity one spaces of dimension six such that $c_1(M) = [\omega]$.*

Proof of the Main Proposition

The preimage of a vertex

(M, ω, T, ϕ) : complexity one space
v : vertex of $\Delta_M = \phi(M)$

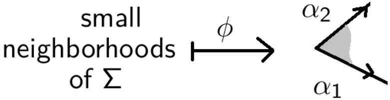
- Then $\phi^{-1}(v)$ is a connected component of M^T, so it is a symplectic submanifold of (M, ω).
- (M, ω, T, ϕ) is tall $\implies$ $\Sigma_v := \phi^{-1}(v)$ is a fixed surface.

Given a surface $\Sigma \subset M^T$ and $p \in \Sigma$,
$\exists$ complex coordinates $(z_1, ..., z_n)$ centered at p, s.t. for $\xi \in \mathrm{Lie}(T)$

$$\exp(\xi)(z_1, ..., z_n) = (e^{2i\pi\alpha_1(\xi)} z_1, ..., e^{2i\pi\alpha_{n-1}(\xi)} z_{n-1},\, z_n)$$

$$\phi(z) = \phi(\Sigma) + \sum_{i=1}^{n-1} \alpha_i \left|z_i\right|^2,$$

where the α_i's are the weights of $T \curvearrowright N\Sigma$.
The α_i's form a $\mathbb{Z}$-basis of the dual lattice $\subset (\mathrm{Lie}(T))^*$.



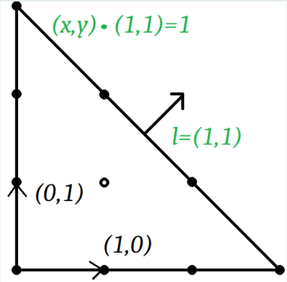
Since the fibers of ϕ are connected:

- $\Sigma = \phi^{-1}(v)$ for a vertex v of Δ_M
- the edges at v are of the form $v + t\alpha_i$, $t > 0$ and $i = 1, ..., n-1$, where the α_i's the weights of $T \curvearrowright N\Sigma$ $\rightsquigarrow$ the primitive direction vectors (pointing into the polytope) (**the weights at** v) are the same as those of $T \curvearrowright N\Sigma$.
- the preimages of these edges are symplectic submanifolds of dimension 4

Lemma

The moment map polytope of a tall complexity one space is Delzant and $v \mapsto \phi^{-1}(v)$ defines a bijection between the vertices of Δ_M and the fixed surfaces. The weights of $T \curvearrowright N\Sigma_v$ are the same the ones of the vertex v.

Example



This polytope satisfies the **vertex-Fano-condition,** i.e., $v = -$sum of the weights at v for all vertices.

Fact: For Delzant polytopes : reflexive $\iff$ vertex-Fano-condition

Proof of the Main-Proposition.

Given a tall complexity one space (M, ω, T, ϕ), such that $c_1(M) = [\omega]$.

- Δ_M is Delzant.
- By results of (Kirwan '84) in equivariant cohomology $\implies$ ϕ satisfies the **weight sum formula**, i.e., $\phi(F)$ is equal to minus the sum of the weights along F for all $F \subset M^T$
- weight sum formula $\implies$ vertex-Fano-condition

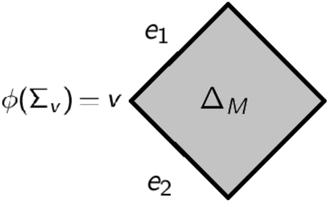
□

Ideas behind the Main Theorem

We need to understand the Duistermaat-Heckman function and the behavior of the isolated fixed points!

The DH-function near its minimum

(M, ω, T, ϕ) : tall complexity one space
$DH_M \colon \Delta_M \to \mathbb{R}_{>0}$: DH-function
v : vertex of Δ_M with edges $e_i \subset v + t\alpha_i$



$c_1(L_i)$: first Chern class of the normal bundle of Σ_v in $\phi^{-1}(e_i)$

near v :

$$DH_M(v + t_1\alpha_1 + \ldots + t_{n-1}\alpha_{n-1}) = -\sum_{i=1}^{n-1} t_i \cdot c_1(L_i)[\Sigma_v] + \int_{\Sigma_v} \omega$$

Lemma

Assume (M, ω, T, ϕ) to be monotone and that DH_M attains its minimum at v, then either,

- $c_1(L_i)[\Sigma_v] = 0$ *for all* $i = 1, ..., n-1$ *or*
- $c_1(L_j)[\Sigma_v] = -1$ *for one* $j = 1, ..., n-1$ *and* $c_1(L_i)[\Sigma_v] = 0$ *for* $i \neq j$.

Proof.

- DH_M attains its minimum at $v \implies \forall$ i: $c_1(L_i)[\Sigma_v] \leq 0$
- Monotonicity $\implies$
  $1 \leq c_1(M)[\Sigma_v] = c_1(\Sigma_v)[\Sigma_v] + \sum_i c_1(L_i)[\Sigma_v]$
- (Sabatini-Sepe '19) $\implies$ Σ_v is 2-sphere, so $c_1(\Sigma_v)[\Sigma_v] = 2$

□

Consequence:

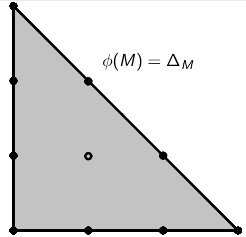
Given a tall complexity one space (M, ω, T, ϕ) with $c_1(M) = [\omega]$ and **without isolated fixed points**. Then:

- All points in the interior of Δ_M are regular values of ϕ.
- (Duistermaat-Heckman '82) $\implies$ DH_M is a polynomial of degree ≤ 1
- (Cho-Kim '12) $\implies$ DH_M is log-concave $\implies$ DH_M attains its minimum at a vertex of Δ_M.

$\longrightarrow$ **Finitely many DH-functions!**

Example

(M^6, ω, T^2, ϕ) tall complexity one space with $c_1(M) = [\omega]$, without isolated fixed points and



W.l.o.g. assume that DH_M attains its minimum at the vertex $(-1, -1)$.

Then: $DH_M = 2$ or $DH_M = y + 2$

About isolated fixed points

Rigidity of the isolated fixed points

Given $(M^{2n}, \omega, T^{n-1}, \phi)$ tall and monotone, s.t. ϕ satisfies the weight sum formula. Then for $p \in M^T$ isolated

$$\phi(p) = -\alpha_1 - ... - \alpha_n \in \Delta_M \cap \mathbb{Z}^{n-1},$$

where $\alpha_1, ..., \alpha_n$ are the weights of $T \curvearrowright T_p M$.

$\longrightarrow$ Rigidity of $\phi(p)$ and of the weights of $T \curvearrowright T_p M$

An upper bound for # p$\in M^T$ isolated

Key Lemma

Let (M, ω, T, ϕ) *be a tall and monotone complexity one space of dimension six. Then:*

$$\#\text{vertices of } \Delta_M \;\geq\; \#p \in M^T \text{isolated}$$

Ideas of the Proof.

- (Godinho-Sabatini '12) and (Lindsay-Panov '18) $\Longrightarrow$ $\int_M c_1(M)c_2(M) = 24$
- Use the ABBV localization formula due to (Atiyah-Bott '84) and (Berline-Vergne '82) to express $\int_M c_1(M)c_2(M) = 24$ as a sum of contributions coming from the connected components of M^T.

□

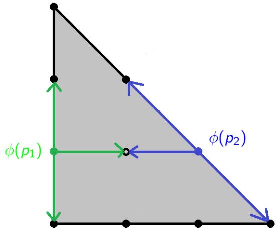
We can recover the number of $p \in M^T$ isolated exactly.

# vertices of Δ_M	# isolated $p \in M^T$
3	0 or 2
4	0, 2 or 4
5	0
6	0

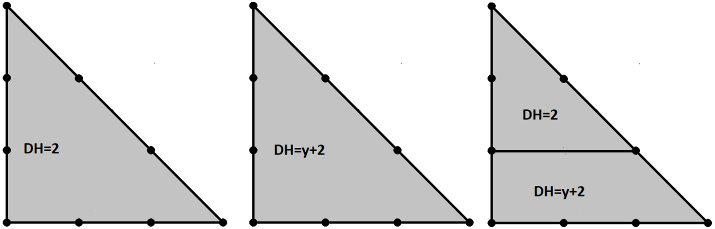
We also know $\phi(p)$ and the weights of $T \curvearrowright T_pM$ for all $p \in M^T$ isolated.

Verification in an example

Example:
Let (M^6, ω, T^2, ϕ) be a tall complexity one space with $c_1(M) = [\omega]$ and such that Δ_M is a triangle. This space has no or 2 isolated fixed points $p_1, p_2 \in M^T$:



The Duistermaat-Heckman function is one of the following:

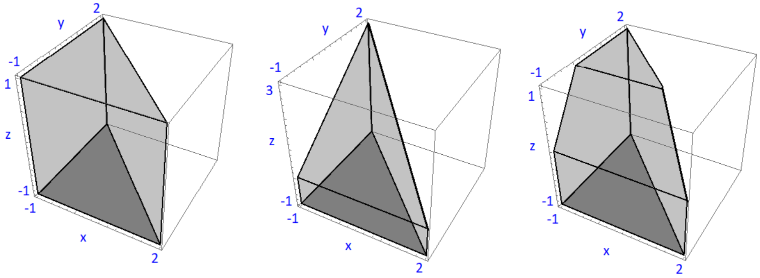


In particular, the Duistermaat-Heckmann function determines the space of exceptional orbits M_{exc}, which is as topological space the empty set or a compact interval.

Uniqueness:

- Since monotonicity implies that the genus is zero (Sabatini-Sepe '19):
- (Karshon-Tolman '03) $\implies$ $\exists$ at most 3 tall and monotone complexity one spaces of dimension six, such that $c_1(M) = [\omega]$ and Δ_M is a triangle. These spaces are determined by their DH-functions.

Existence and Extension:
Consider the monotone symplectic toric manifolds of dimension six (M, ω, T^3, ϕ) given by the following Delzant polytopes:



By forgetting the last S^1-factor, we obtain three different tall complexity one spaces of dimension six, such that $c_1(M) = [\omega]$ and such that Δ_M is a triangle.

Thank you!