Analyzing the algorithm for proving the restricted invertibility theorem

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1. Introduction

In a recent important addition to the literature, Spielman and Srivastava [9] surprised the mathematics community by giving an algorithm for proving the Bourgain-Tzafriri Restricted Invertibility Theorem which we will refer to as the S^2-algorithm. They also significantly strengthened the theorem by giving the best possible constants.

Recall that a set of vectors $\{f_i\}_{i\in I}$ has *lower* (respectively, *upper*) *Riesz bound* A (respectively B) if for all scalars $\{a_i\}_{i\in I}$ we have

$$A\sum_{i\in I}|a_i|^2 \le \|\sum_{i\in I} a_i f_i\|^2 \le B\sum_{i\in I}|a_i|^2.$$

The restricted invertibility theorem of Bourgain and Tzafriri has been a major tool in analysis since it appeared in 1987 [1].

Theorem 1.1 (Bourgain-Tzafriri Restricted Invertibility Theorem - Spielman-Srivastava form)**.** *For any $\epsilon > 0$ and any natural number n and any operator $L : \ell_2^n \to \ell_2^n$ with $\|Le_i\| = 1$ for the canonical unit vector basis $\{e_i\}_{i=1}^n$, we can find a subset $\sigma \subset \{1,2,\ldots,n\}$ of cardinality*

$$|\sigma| \ge \epsilon^2 \frac{n}{\|L\|^2},$$

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and for all scalars $\{a_i\}_{i\in\sigma}$ we have

$$\|\sum_{i\in\sigma} a_i Le_i\|^2 \geq (1-\epsilon)^2 \sum_{i\in\sigma} |a_i|^2.$$

The idea of these notes is to examine some of the technicalities in the proof of the theorem as well as some variations of the arguments, some directions for further study and to give a series of examples showing that various results are best possible. We will not reproduce the proof from [9] here and so to read these notes we advise the reader to have [9] in hand to understand what we refer to in each section.

In particular, we address the following topics:

Before outlining the proof of the Bourgain-Tzafriri Restricted Invertibility Theorem given by Spielman and Srivastava in Section 3, we take a close look at a key eigenvalue inequality that Spielman and Srivastava observed. We complement their lemma with the reverse implication and add additional equivalent criteria for their eigenvalue inequality to hold.

Section 4 shows that the parameters chosen by Spielman and Srivastava are optimal for their method of proof. In Section 5 we prove that, in principle, it suffices to show the Bourgain-Tzafriri Restricted Invertibility Theorem for positive operators. This observation may open up new ways for improving the lower Riesz bound guaranteed by Spielman and Srivastava.

In Sections 6 and 7 we ask the question whether the Spielman and Srivastava algorithm also controls the upper Riesz bound, even though it is simply constructed to achieve a good lower Riesz bound. Section 6 develops a dual algorithm used to control the upper Riesz bound, while Section 7 contains an example showing that the upper Riesz bound may indeed blow up when applying the Spielman and Srivastava algorithm. Note that recently, a two-sided algorithm for the restricted invertibility theorem was given [4], this work we will not discuss here further.

Finally, Section 8 addresses the important question whether the Spielman and Srivastava algorithm can also be applied to extend a given Riesz system with good lower Riesz bound to a larger Riesz system.

2. An Eigenvalue Inequality

A fundamental tool in the proof of the S^2-algorithm is an eigenvalue inequality. The algorithm uses only a one directional implication, we extend their result to include the reverse implication.

Theorem 2.1. *Given a Hilbert space $\mathbb{H}_1 \subset \mathbb{H}$, dim $\mathbb{H}_1 = m$, a positive operator $A : \mathbb{H}_1 \to \mathbb{H}_1$ whose smallest eigenvalue is greater than b', and a vector $\omega \in \mathbb{H}$ with $\omega \notin \mathbb{H}_1$ and $\|\omega\| = 1$. Then the following four conditions* **are equivalent***:*

1. *We have*

$$\omega^T (A - b'I)^{-1} \omega < -1.$$

2. We have

$$Tr[(A + \omega\omega^T - b'I)^{-1}] - Tr[(A - b'I)^{-1}] > 0.$$

3. The smallest eigenvalue of $A + \omega\omega^T$ is greater than b'.

4. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ be the eigenvalues of A and let P be the orthogonal projection of $\mathbb{H}$ onto span $\mathbb{H}_1$. Then

$$\sum_{i=1}^{m} \frac{1}{\lambda_i - b'}\omega_i^2 + 1 \leq \frac{\|(I-P)\omega\|^2}{b'}.$$

(Here, ω_i is the i^{th}-coordinate of ω with respect to the eigenbasis of A.)

Or, equivalently,

4'.

$$\sum_{i=1}^{m} \frac{\lambda_i}{\lambda_i - b'}\ \omega_i^2 \leq 1 - b'.$$

Note:

$$\left\{\frac{\lambda_i}{\lambda_i - b'}\right\} \quad \textit{is increasing.}$$

Proof. (1) $\Leftrightarrow$ (2): By the Sherman-Morrison Formula:

$$Tr[(A + \omega\omega^T - b'I)^{-1}] - Tr[(A - b'I)^{-1}] = -\frac{\omega^T(A - b'I)^{-2}\omega}{1 + \omega^T(A - b'I)^{-1}\omega}.$$

Since

$$\omega^T(A - b'I)^{-2}\omega > 0,$$

the result is immediate.

(2) $\Rightarrow$ (3): Let $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_m$ (respectively, $\lambda'_1 \geq \lambda'_2 \geq \cdots \lambda'_{m+1}$) be the eigenvalues of A (respectively, $A + \omega\omega^T$). By the eigenvalue interlacing formula,

$$\lambda'_1 \geq \lambda_1 \geq \lambda'_2 \geq \lambda_2 \cdots \geq \lambda'_m \geq \lambda_m \geq \lambda'_{m+1}.$$

Note that

$$\begin{aligned}
0 &< Tr[(A + \omega\omega^T - b'I)^{-1}] - Tr[(A - b'I)^{-1}] \\
&= \frac{1}{\lambda'_{m+1} - b'} - \frac{1}{0 - b'} + \sum_{i=1}^{m} \frac{1}{\lambda'_i - b'} - \sum_{i=1}^{m} \frac{1}{\lambda_i - b'} \\
&\leq \frac{1}{\lambda'_{m+1} - b'} + \frac{1}{b'} \\
&= \frac{\lambda'_{m+1}}{(\lambda'_{m+1} - b')b'}.
\end{aligned}$$

and since $\lambda'_{m+1} > 0$, this implies $\lambda'_{m+1} > b'$.

(3) $\Rightarrow$ (2): Assume $\lambda'_{m+1} > b'$. Then

$$Tr[(A+\omega\omega^T - b'I)^{-1}] - Tr[(A-b'I)^{-1}]$$

$$\begin{aligned}
&= -\frac{1}{0-b'} + \sum_{i=1}^{m+1} \frac{1}{\lambda'_i - b'} - \sum_{i=1}^{m} \frac{1}{\lambda_i - b'} \\
&= \frac{1}{b'} + \frac{1}{\lambda'_1 - b'} + \sum_{i=1}^{m} \left[\frac{1}{\lambda'_{i+1} - b'} - \frac{1}{\lambda_i - b'} \right] \\
&\geq \frac{1}{b'} + \frac{1}{\lambda'_1 - b'} > 0
\end{aligned}$$

(1) $\Leftrightarrow$ (4): This is immediate since

$$\omega^T (A - b'I)^{-1} \omega = \sum_{i=1}^{m} \frac{1}{\lambda_i - b'} \omega_i^2 - \frac{\|(I-P)\omega\|^2}{b'}.$$

(4) $\Leftrightarrow$ (4$'$): Given (4), we have

$$\sum_{i=1}^{m} \frac{b'}{\lambda_i - b'} \omega_i^2 + b' \leq \|(I-P)\omega\|^2 = 1 - \|P\omega\|^2.$$

Hence,

$$\begin{aligned}
1 - b' &\geq \sum_{i=1}^{m} \frac{b'}{\lambda_i - b'} \omega_i^2 + \|P\omega\|^2 \\
&= \sum_{i=1}^{m} \frac{b'}{\lambda_i - b'} \omega_i^2 + \sum_{i=1}^{m} \omega_i^2 \\
&= \sum_{i=1}^{m} \left[\frac{b'}{\lambda_i - b'} + 1 \right] \omega_i^2 \\
&= \sum_{i=1}^{m} \frac{\lambda_i}{\lambda_i - b'}\ \omega_i^2.
\end{aligned}$$

All these steps are reversible to get (4$'$) $\Rightarrow$ (4). □

3. Outline of the Proof

Now we will give an outline of the proof of the S^2-algorithm from [9]. The authors build a matrix $A = \sum_{i\in\sigma} (Le_i)^T (A - bI)^{-1} (Le_i)$ by an iterative process that adds one vector at a time. The main technicality is that we need to guarantee that in each step there is a new vector to be picked to satisfy Theorem 2.1. For this, they introduce a *potential function*

$$\begin{aligned}\Phi_b(A) &= \sum_{i\in\sigma}(Le_i)^T(A-bI)^{-1}(Le_i)\\ &= Tr[L^T(A-bI)^{-1}L],\end{aligned}$$

where the *barrier* b is a real number that will be slightly lowered from step to step.

We let

$$\delta = \frac{1-\epsilon}{\epsilon}\frac{\|L\|^2}{n}.$$

Initially, $A = 0$ and the barrier is at $b = b_0 = 1-\epsilon$. Then

$$\Phi_{b_o}(0) = -\frac{n}{b_0} = -n - \frac{\|L\|^2}{\delta}.$$

We note that the theorem is vacuously true if $\epsilon^2 \frac{n}{\|L\|^2} < 1$. Otherwise, we have that $\delta < b_0$ so we can start. For each step in the process, we add one more rank one projection $\omega\omega^T$ to A where

$$A = \sum_{i\in\sigma}\omega_i\omega_i^T,$$

and show that we can do this without increasing the potential as long as we drop the barrier by δ. i.e.

$$\Phi_{b-\delta}(A+\omega\omega^T) \le \Phi_b(A).$$

Theorem 3.1 (Spielman and Srivastava)**.** *Let $\mathbb{H}$ be a Hilbert space with orthonormal basis $\{v_i\}_{i=1}^n$. Assume $L : \mathbb{H} \to \mathbb{H}$ is a linear operator with $\|Lv_i\| = 1$ for all $i = 1, 2, \ldots, n$, and assume*

$$A = \sum_{i=1}^m Lv_iLv_i^T$$

has m non-zero eigenvalues, all of which are greater than b, and $b' = b-\delta > \delta$. If

$$Tr[L^T(A-bI)^{-1}L] \le -n - \frac{\|L\|^2}{\delta},$$

then there exists a vector $\omega \in \{Lv_i\}_{i=1}^n$ satisfying:

1. $\omega^T(A-b'I)^{-1}\omega < -1$, and hence $\omega = Lv_j$ for some $m < j \le n$.

2. $Tr[L^T(A+\omega\omega^T-b'I)^{-1}L] \le Tr[L^T(A-bI)^{-1}L] \le -n - \frac{\|L\|^2}{\delta}$.

Hence, $A+\omega\omega^T$ has $k+1$ non-zero eigenvalues all greater than b', and by (2) we can start all over with $A+\omega\omega^T$ in the place of A and b' in the place of b and pick another vector.

Proof. **First**: We note that

$$\begin{aligned} Tr[L^T(0-(1-\epsilon)I)^{-1}L] &= -\frac{n}{1-\epsilon} \\ &= -n - \frac{\epsilon}{1-\epsilon}n \\ &= -n - \frac{\epsilon n}{(1-\epsilon)\|L\|^2}\|L\|^2 \\ &= -n - \frac{\|L\|^2}{\delta}. \end{aligned}$$

This starts the algorithm.

Next: We show

$$Tr[L^T(A-bI)^{-1}L] \le -n - \frac{\|L\|^2}{\delta} = -\frac{n}{1-\epsilon},$$

implies

(and this step is where most of the work is involved) there is a vector $\omega \in \{Lv_i\}_{i=1}^n$ satisfying

$$\omega^T(A-b'I)^{-1}LL^T(A-b'I)^{-1}\omega$$
$$\le (Tr[L^T(A-bI)^{-1}L] - Tr[L^T(A-b'I)^{-1}L])(-1-\omega^T(A-b'I)^{-1}\omega)$$

Which in turn **implies both**

(a) $\omega^T(A-b'I)^{-1}\omega < -1$ and $\omega = Lv_j$ for some $m < j \le n$,

and

(b)

$$Tr[L^T(A-b'I)^{-1}L] - \frac{\omega^T(A-b'I)^{-1}LL^T(A-b'I)^{-1}\omega}{1+\omega^T(A-b'I)^{-1}\omega} \le Tr[L^T(A-bI)^{-1}L]$$

And (a) implies

(c) the smallest eigenvalue of $A + \omega\omega^T$ is greater than b'.

while (b) implies

(d) $Tr[L^T(A+\omega\omega^T - b'I)^{-1}L] \le Tr[L^T(A-bI)^{-1}L] \le -n - \frac{\|L\|^2}{\delta}$.

And (d) allows us to start over and pick another ω while replacing b by b'.

$\square$

The following is a corollary of this construction.

Corollary 3.2 (Bourgain-Tzafriri Restricted Invertibility Theorem)**.** *If we iterate the algorithm k times, we get k vectors from $\{Lv_i\}_{i=1}^n$ with lower Riesz bound for the operator A*

$$1 - \epsilon - (k-1)\delta = (1-\epsilon)\left[1 - (k-1)\frac{\|L\|^2}{\epsilon n}\right].$$

Hence,

1. If

$$k = \lceil \frac{\epsilon^2 n}{\|L\|^2} \rceil,$$

then

$$\begin{aligned} (1-\epsilon)\left[1-(k-1)\frac{\|L\|^2}{\epsilon n}\right] &\geq (1-\epsilon)\left[1 - \frac{\epsilon^2 n}{\|L\|^2}\frac{\|L\|^2}{\epsilon n}\right] \\ &= (1-\epsilon)^2, \end{aligned}$$

which is the lower bound in the Bourgain-Tzafriri Restricted Invertibility theorem.

2. If

$$k = \lceil \frac{\epsilon n}{\|L\|^2} \rceil + 1,$$

then

$$\begin{aligned} (1-\epsilon)\left[1-(k-1)\frac{\|L\|^2}{\epsilon n}\right] &\leq (1-\epsilon)\left[1 - \frac{\epsilon n}{\|L\|^2}\frac{\|L\|^2}{\epsilon n}\right] \\ &= (1-\epsilon)0, \end{aligned}$$

and the process stops.

4. Their use of $1-\epsilon$

The S^2-algorithm starts with $b = 1-\epsilon$. This seems like a **waste** since for our first choice of a vector ω, the operator $A = \omega\omega^T$ has only one non-zero eigenvalue, and that eigenvalue is equal to one. So it appears that any $b < 1$ would have worked perfectly well for the starting point. However, it turns out that $b = 1-\epsilon$ **is optimal** for this argument. To see this, assume we start the algorithm with $b = 1 - \mu$ instead. Then for the first case we get:

$$\begin{aligned} Tr[L^T(A-bI)^{-1}L] &= \frac{-n}{1-\mu} \\ &= -n - \frac{\mu}{1-\mu}n \\ &= -n - \|L\|^2 \frac{\mu n}{\|L\|^2(1-\mu)} \\ &= -n - \frac{\|L\|^2}{\delta}, \end{aligned}$$

where

$$\delta = \frac{(1-\mu)\|L\|^2}{\mu n}.$$

Now, the algorithm works in every detail where we subtract δ at each step. After

$$\frac{\epsilon^2 n}{\|L\|^2} - iterations,$$

we end up with a lower Riesz bound of:

$$\begin{aligned} 1 - \mu - \frac{\epsilon^2 n}{\|L\|^2}\delta &= 1 - \mu - \frac{\epsilon^2 n}{\|L\|^2}\frac{(1-\mu)\|L\|^2}{\mu n} \\ &= 1 - \mu - \epsilon^2 \frac{1-\mu}{\mu} \\ &= 1 + \epsilon^2 - (\mu + \frac{\epsilon^2}{\mu}). \end{aligned}$$

If we let

$$f(x) = x + \frac{\epsilon^2}{x},$$

then f is minimized at $x = \epsilon$ and so

$$1 + \epsilon^2 - (\mu + \frac{\epsilon^2}{\mu}) \text{ is maximized when } \epsilon = x, \text{ i.e. their case,}$$

and in this case, we get

$$1 + \epsilon^2 - (\mu + \frac{\epsilon^2}{\mu}) = 1 + \epsilon^2 - (\epsilon + \frac{\epsilon^2}{\epsilon}) = 1 + \epsilon^2 - 2\epsilon = (1-\epsilon)^2.$$

Remark 4.1. In the S^2-algorithm, there is an **implicit** restriction on ϵ. That is, if we are working in $\mathbb{H}_n$, then in order to have vectors to pick, we need to have:

$$\frac{\epsilon^2 n}{\|L\|^2} \geq 1,$$

and hence

$$\epsilon > \frac{\|L\|}{\sqrt{n}}.$$

Motivated in part by the computations above, as well as the remark above, we take a closer look at the possible optimality in their algorithm.

Theorem 4.2. *Given $L : \mathbb{H}_n \to \mathbb{H}_n$ with $\|Le_i\| = 1$, for al $i = 1, , \ldots, n$, we have:*

1. If we choose a Riesz sequence of K vectors, then the optimal lower Riesz bound achieved by the S^2-algorithm is

$$\tilde{b} = (1-\epsilon)^2 = (1 - \sqrt{\frac{K}{n}}\|L\|)^2.$$

This optimal lower Riesz bound is based on choosing

$$\epsilon = \sqrt{\frac{K}{n}}\|L\| \quad \text{and} \quad \delta = \frac{\|L\|^2}{n}\frac{1-\epsilon}{\epsilon} = \frac{\|L\|}{\sqrt{nK}}\Big(1 - \sqrt{\frac{K}{n}}\|L\|\Big).$$

2. Given a Riesz bound $\tilde{b}$, the largest number of vectors K we can pick leaving $\tilde{b}$ as the lower Riesz bound is

$$K = \Big\lfloor (1 - \sqrt{\tilde{b}}\,)^2 \frac{n}{\|L\|^2} \Big\rfloor.$$

Proof. In the following, $b_1 = b_1(\epsilon) = 1 - \epsilon$, $0 < \epsilon < 1$. To satisfy the conditions for the algorithm, we need

$$\delta \geq \frac{\|L\|^2}{n}\frac{1-\epsilon}{\epsilon}; \quad \delta \geq \frac{\|L\|^2}{n}(1-\epsilon).$$

As $\epsilon \in (0,1)$, we have that the two inequalities above are satisfied if and only if the first one is satisfied. For any ϵ, we choose

$$\delta = \delta(\epsilon) = \frac{\|L\|^2}{n}\frac{1-\epsilon}{\epsilon}.$$

1. We have $b_K = (1-\epsilon) - K\delta$ and we will try to find ϵ, δ to maximize b_K. To this end, set

$$f(\epsilon) = (1-\epsilon) - \frac{K\|L\|^2}{n}\frac{1-\epsilon}{\epsilon} = (1-\epsilon)(1 - \frac{K\|L\|^2}{n}\epsilon^{-1})$$

and

$$f'(\epsilon) = -(1 - \frac{K\|L\|^2}{n}\epsilon^{-1}) + (1-\epsilon)\frac{K\|L\|^2}{n}\epsilon^{-2}.$$

We compute $f'(\epsilon) = 0$ if and only if

$$(1 - \frac{K\|L\|^2}{n}\epsilon^{-1}) = (1-\epsilon)\frac{K\|L\|^2}{n}\epsilon^{-2}$$

iff

$$\epsilon^2 - \frac{K\|L\|^2}{n}\epsilon = (1-\epsilon)\frac{K\|L\|^2}{n}$$

iff

$$\epsilon^2 = \frac{K\|L\|^2}{n}$$

that is $\epsilon = \sqrt{\frac{K}{n}}\|L\|$,

$$\delta = \frac{\|L\|^2}{n} \frac{1-\sqrt{\frac{K}{n}}\|L\|}{\sqrt{\frac{K}{n}}\|L\|} = \frac{\|L\|}{\sqrt{nK}}(1-\sqrt{\frac{K}{n}}\|L\|)$$

and, therefore,

$$b_K = (1-\epsilon) - K\delta = (1-\sqrt{\frac{K}{n}}\|L\|) - \sqrt{\frac{K}{n}}\|L\|(1-\sqrt{\frac{K}{n}}\|L\|) = (1-\sqrt{\frac{K}{n}}\|L\|)^2$$

2. Follows from 1. As if $K = \lfloor(1-\sqrt{\widetilde{b}})^2\frac{n}{\|L\|^2}\rfloor$, then the optimal Riesz bound for K vectors is larger or equal then $\widetilde{b}$. If in turn a choice of $K\lfloor(1-\sqrt{\widetilde{b}})^2\frac{n}{\|L\|^2}\rfloor+1$ would always be possible, then the optimality of $\widetilde{b}$ in (1) would be contradicted. □

5. The operator L in the Theorem

We now observe that in the BT-Restricted Invertibility Theorem, we may assume that the operator L is a positive operator.

Proposition 5.1. *Given an orthonormal basis $\{e_i\}_{i=1}^n$ for $\mathbb{H}_n$ and an operator $L : \mathbb{H}_n \to \mathbb{H}_n$ with $\|Le_i\| = 1$ for all $i = 1, 2, \ldots, n$, there is a positive operator $S : \mathbb{H}_n \to \mathbb{H}_n$ satisfying:*

1. *$\|Se_i\| = 1$, for all $i = 1, 2, \ldots, n$.*
2. *$\|S\| = \|L\|$.*
3. *For any family of scalars $\{a_i\}_{i=1}^n$ we have*

$$\|\sum_{i=1}^n a_i Le_i\| = \|\sum_{i=1}^n a_i Se_i\|.$$

Proof. Let $S = (L^*L)^{1/2}$. Now we check our properties.

1. For any e_i,

$$\|Se_i\|^2 = \langle (L^*L)^{1/2}e_i, (L^*L)^{1/2}e_i\rangle = \langle L^*Le_i, e_i\rangle = \langle Le_i, Le_i\rangle = \|Le_i\|^2 = 1.$$

2. We compute

$$\begin{aligned}
\|(L^*L)^{1/2}\|^2 &= \sup_{\|x\|=1} \|(L^*L)^{1/2}x\|^2 \\
&= \sup_{\|x\|=1} \langle (L^*L)^{1/2}x, (L^*L)^{1/2}x\rangle \\
&= \sup_{\|x\|=1} \langle (L^*L)x, x\rangle \\
&= \sup_{\|x\|=1} \langle Lx, Lx\rangle \\
&= \sum_{\|x\|=1} \|Lx\|^2 \\
&= \|L\|^2.
\end{aligned}$$

3. We compute

$$\begin{aligned}
\|\sum_{i=1}^n a_i(L^*L)^{1/2}e_i\|^2 &= \langle\sum_{i=1}^n a_i(L^*L)^{1/2}e_i, \sum_{i=1}^n a_i(L^*L)^{1/2}e_i\rangle \\
&= \sum_{i,j=1}^n a_i\overline{a_j}\langle (L^*L)^{1/2}e_i, (L^*L)^{1/2}e_j\rangle \\
&= \sum_{i,j=1}^n a_i\overline{a_j}\langle((L^*L)e_i, e_j\rangle \\
&= \sum_{i,j=1}^n a_i, \overline{a_j}\langle Le_i, Le_j\rangle \\
&= \langle\sum_{i=1}^n a_iLe_i, \sum_{i=1}^n a_iLe_i\rangle \\
&= \|\sum_{i=1}^n a_iLe_i\|^2.
\end{aligned}$$

Hence, from now on in the Restricted Invertibility Theorem we may as well assume that L is a positive operator. □

It turns out that the operators we construct in the algorithm have a very special form.

Proposition 5.2. *Given a positive operator $L : \mathbb{H}_n \to \mathbb{H}_n$ with $\|Le_i\| = 1$ for all $i = 1, 2, \ldots, n$, if we let $\omega_i = Le_i$,*

$$A = \sum_{i=1}^m \omega_i\omega_i^T,$$

and let P_m be the orthogonal projection of $\mathbb{H}_n$ onto span $\{e_i\}_{i=1}^m$. Then $A = LP_mL$.

Proof. We observe that for any $f \in span\ \{\omega_i\}_{i=1}^m =: K_m$ we have

$$\begin{aligned} Af &= \sum_{i=1}^m \langle f, \omega_i \rangle \omega_i \\ &= \sum_{i=1}^m \langle f, Le_i \rangle Le_i \\ &= L\left(\sum_{i=1}^m \langle Lf, e_i \rangle e_i \right) \\ &= L(P_m Lf). \end{aligned}$$

□

6. Duality for the Algorithm

Given that $\{Lv_i\}_{i=1}^m$ is a Riesz sequence with frame operator A and lower Riesz bound b', then $\{A^{-1}Lv_i\}_{i=1}^m$ is the dual sequence for $\{Lv_i\}_{i=1}^m$. That is,

$$\langle A^{-1}Lv_i, Lv_j \rangle = \delta_{ij}.$$

Also, the lower Riesz bound of $\{Lv_i\}_{i=1}^m$ is $\geq b'$ if and only if the upper Riesz bound of $\{A^{-1}Lv_i\}_{i=1}^m$ is $\leq \frac{1}{b'}$. Also, $\{A^{-1/2}Lv_i\}_{i=1}^m$ is an orthonormal sequence.

We would like to prove a two sided version of BT Restricted Invertibility where we get control of both the upper and lower Riesz bounds. This does not seem to be available at this time. Below we give the standard *duality argument* and see that it does not resolve this problem.

Theorem 6.1. *Let $\{e_i\}_{i=1}^n$ be an orthonormal basis for a n-dimensional Hilbert space $\mathbb{H}_n$ and let L be a bounded operator on $\mathbb{H}_n$ satisfying $\|Le_i\| = 1$ for all $i = 1, 2, \ldots, n$. Given $0 < \epsilon < 1$, choose a set $I \subset \{1, 2, \ldots, n\}$ with*

$$|I| \geq \epsilon^2 \frac{n}{\|L\|^2},$$

and the lower Riesz bound for $\{Le_i\}_{i\in I}$ is $(1-\epsilon)^2$. Let $\{f_i\}_{i\in I}$ be the dual functionals for the Riesz sequence $\{Le_i\}_{i\in I}$. Then there is a subset $J \subset I$ with

$$|J| \geq \epsilon^4 (1-\epsilon)^2 \frac{n}{\|L\|^2},$$

and $\{f_i\}_{i\in J}$ has Riesz bounds $(1-\epsilon)^2,\ \frac{1}{(1-\epsilon)^2}$.

Proof. It follows from our assumptions that the dual functionals $\{f_i\}_{i\in I}$ satisfy:

1. The upper Riesz bound of $\{f_i\}_{i\in I}$ is

$$\frac{1}{(1-\epsilon)^2}.$$

Hence, if $Te_i = f_i$, then

$$\|T\|^2 \leq \frac{1}{(1-\epsilon)^2}.$$

2. We have

$$1 \leq \|f_i\| \leq \frac{1}{1-\epsilon}.$$

Now we apply the S^2-algorithm (the general form for non-norm one vectors) to get a $J \subset I$ with

$$\begin{aligned}
|J| \geq \frac{\epsilon^2 \|T\|_F^2}{\|T\|^2} &\geq \frac{\epsilon^2 |I|}{\|T\|^2} \\
&\geq \epsilon^2 (1-\epsilon)^2 |I| \\
&\geq \epsilon^2 (1-\epsilon)^2 \frac{\epsilon^2 n}{\|L\|^2} \\
&\geq \epsilon^4 (1-\epsilon)^2 \frac{n}{\|L\|^2},
\end{aligned}$$

so that its lower Riesz bound is

$$\begin{aligned}
\frac{(1-\epsilon)^2 \|T\|_F^2}{|I|} &\geq \frac{(1-\epsilon)^2}{\frac{\epsilon^2 n}{\|L\|^2}} \frac{\epsilon^2 n}{\|L\|^2} \\
&= (1-\epsilon)^2.
\end{aligned}$$

That is, $\{f_i\}_{i \in J}$ is a Riesz sequence with Riesz bounds

$$(1-\epsilon)^2, \ \frac{1}{(1-\epsilon)^2}.$$

□

Remark 6.2. We would like to conclude from above that $\{Le_i\}_{i \in J}$ is a Riesz sequence with Riesz bounds

$$(1-\epsilon)^2, \ \frac{1}{(1-\epsilon)^2}.$$

However, this conclusion is false as we will see in the next section. The problem here is that the vectors $\{Le_i\}_{i \in J}$ are not the dual functionals for $\{f_i\}_{i \in J}$ since they do not lie in the dual space.

Remark 6.3. The percentage of the vectors we are getting is not really good since it has both ϵ and $1-\epsilon$ in it and hence if ϵ is close to zero or one - this deteriorates. Perhaps it could be improved to get bounds $(1-\epsilon)^2, \ (1-\epsilon)^{-2}$ as we got above.

7. Upper versus Lower Riesz Bounds

In this section, we will show that using the S^2- algorithm to find a subset of our vectors which has a good lower Riesz bound, does not guarantee that we get a good upper Riesz bound or even that we can find a further subset of the proper size which has a good upper Riesz bound.. i.e. We may end up choosing a subset of the vectors with lower Riesz bound $(1-\epsilon)^2$ but upper Riesz bound arbitrarily large. In particular, the upper Riesz bound does not compare to $1/(1-\epsilon)^2$.

Example: Here, we use a modified version of the unitary discrete Fourier transform matrix (DFT). We fix $0 < \epsilon < 1$, $n \in \mathbb{N}$ and choose $K \in \mathbb{N}$ so that

$$\frac{K-1}{n-1} < \epsilon.$$

We will take an $n \times n$ DFT and multiply its columns by $\{\sqrt{\lambda_j}\}_{j=1}^n$ where

$$\lambda_1 = K,\ \lambda_j = 1 - \frac{K-1}{n-1}, \quad \text{for all } j = 2, 3, \ldots, n.$$

Note that

$$\sum_{j=1}^{n} \lambda_j = n.$$

That is, our matrix is:

$$A = \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ \sqrt{\lambda_1} & \sqrt{\lambda_2} & \cdots & \sqrt{\lambda_n} \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix}$$

Note that the square sums of the rows of this matrix add up to 1. i.e. The rows are unit norm vectors. Also, since the columns of this matrix are orthogonal, it follows that the unit vectors $\{e_j\}_{j=1}^n$ are the eigenvectors for the frame operator for this frame with respective eigenvalues $\{\lambda_j\}_{j=1}^n$. Now we add to this set $K-1$ copies of the vectors $\{e_j\}_{j=2}^n$. It follows that altogether we have

$$n + (K-1)(n-1) = Kn - (K-1) \ \text{ unit vectors},$$

with their frame operator having $\{e_j\}_{j=1}^n$ as its eigenvectors with eigenvalues,

$$\{K,\ 1 - \frac{K-1}{n-1} + (K-1), \ldots, 1 - \frac{K-1}{n-1} + (K-1)\} =$$
$$\{K, K - \frac{K-1}{n-1}, \ldots, K - \frac{K-1}{n-1}\}.$$

In particular, the norm of the frame operator is K. If we apply the S^2-algorithm to this family, we should be able to choose

$$\begin{aligned} \epsilon^2 \frac{Kn-(K-1)}{\|L\|^2} &= \epsilon^2 \frac{Kn-(K-1)}{K} \\ &= \epsilon^2 (n - \frac{K-1}{K}) \\ &\geq \epsilon^2 (n-1), \quad \text{vectors,} \end{aligned}$$

yielding a Riesz sequence with lower Riesz bound $(1-\epsilon)^2$. However, the algorithm could have picked all its vectors from our DFT set. This is true since it will pick its m^{th}-vector as long as we will have a lower Riesz bound of $1-\epsilon-(m-1)\delta$. But, $1-\epsilon > 1-\epsilon-(m-1)\delta$, and so we need to check that if we start choosing vectors from our DFT set, then whatever we pick, we will have a lower Riesz bound $\geq 1-\epsilon$. This follows from the eigenvalue interlacing theorem. i.e. If we could choose vectors $\{\omega_i\}_{i=1}^m$ and discover the lower Riesz bound is $< 1-\epsilon$, then no matter what elements you pick from the remaining vectors in this set, the lower Riesz bound will just go down. Hence, if we pick all of the DFT vectors then we would get a lower Riesz bound $< 1-\epsilon$. We claim this is a contradiction since if we pick all the DFT vectors then we get a lower Riesz bound of

$$1 - \frac{K-1}{n-1} \geq 1-\epsilon,$$

by our choice of K, n. Now, suppose the algorithm has picked all its vectors from our DFT set. So we have taken $\epsilon^2(n-1)$ vectors from the DFT set, call them $\{f_i\}_{i\in I}$. Then

$$\sum_{i\in I} |\langle f_i, e_1\rangle|^2 = \epsilon^2 \lambda_1 \frac{1}{n} = \epsilon^2 K \frac{n-1}{n}.$$

That is, our upper Riesz bound is at least

$$\epsilon^2 K \frac{n-1}{n},$$

and so is not on the order of

$$\frac{1}{(1-\epsilon)^2},$$

for K large.

Remark 7.1. If we compare this to the theorem in the previous section, we see that we have ourselves in serious trouble if we want to get a Riesz sequence with upper Riesz bound $\leq \frac{1}{(1-\epsilon)^2}$. In particular, there is no percentage subset of our family $\{f_i\}_{i\in I}$ which has an upper Riesz bound on that order, since even if we pick a subset of the rows of the DFT on the order of

$$\epsilon^4(1-\epsilon)^2(n-1),$$

the calculation above shows that the upper Riesz bound is larger than

$$\epsilon^4(1-\epsilon)^2 K \frac{n-1}{n},$$

which is not of the order of $\frac{1}{(1-\epsilon)^2}$. This shows that if you apply S^2 to a set to get a good lower Riesz bound, this set may not have a percentage which has a small upper Riesz bound. That is, the moment you applied S^2, you can no longer solve the problem of getting both upper and lower bounds.

Remark 7.2. The above remark gives a strong motivation for developing a version of S^2 which gives upper Riesz bounds. Because then you can get both upper and lower Riesz bounds. i.e. If we can find a set $I \subset \{1, 2, \ldots, n\}$ so that

$$|I| \geq \epsilon^2 \frac{n}{\|L\|^2},$$

with upper Riesz bound $\frac{1}{(1-\epsilon)^2}$, then we can apply the S^2-algorithm to this to get a set $J \subset I$ with

$$|J| \geq \epsilon^2 \frac{|I|}{\frac{1}{(1-\epsilon)^2}} = \epsilon^4(1-\epsilon)^2 \frac{n}{\|L\|^2},$$

with lower Riesz bound $(1-\epsilon)^2$.

This is especially interesting since in our example above, whatever form of S^2 which gives good upper bounds, it will not pick out any percentage of vectors from the DFT matrix.

In particular, if we have both an upper and lower Riesz version of the S^2-algorithm then we cannot apply the lower version first followed by the upper version to get a Riesz sequence with simultaneously good upper and lower bounds. But, if we apply upper first followed by lower, then we might be able to get such a set.

8. The Starting Point of the Algorithm

We should be able to first pick a good Riesz sequence out of the $\{Le_i\}_{i=1}^n$ and **then start** the algorithm to continue to the largest Riesz sequence with a given bound. To be able to do this, we would do the following. Assume we have chosen $\{Le_i\}_{i=1}^m$ so that the lower Riesz bound is b. Then, choose $\epsilon > 0$ so that

$$b = 1 - \epsilon - (m-1)\delta.$$

We can choose such an ϵ since letting

$$\begin{aligned} f(\epsilon) &= 1 - \epsilon - (m-1)\delta \\ &= 1 - \epsilon - \frac{1-\epsilon}{\epsilon} \frac{(m-1)\|L\|^2}{n} \\ &= -\epsilon - \frac{c}{\epsilon} + 1 + c, \quad \text{where } c = \frac{(m-1)\|L\|^2}{n}. \end{aligned}$$

Then

$$f'(\epsilon) = -1 + \frac{c}{\epsilon^2} < 0,$$

if and only if

$$\epsilon^2 \geq \frac{(m-1)\|L\|^2}{n}.$$

But this is immediate since

$$m - 1 \leq \frac{\epsilon^2 n}{\|L\|^2}.$$

So f is decreasing in this range and this is all we need.

Now, to get the algorithm to start up we need to know that

$$Tr[L^T(A-(b-\delta)I)^{-1}L] \leq -n - \frac{\|L\|^2}{\delta}.$$

The question is: Can we show this?

Note that we can choose some δ' which works since letting $\delta \to b$ in the above inequality, the right hand side converges to the constant

$$-n - \frac{\|L\|^2}{b},$$

while the left hand side goes to $-\infty$ (since the smallest eigenvalue of A is b and so this eigenvalue on the left hand side becomes $1/0$ when we reach b). That is, there does exist a δ' so that

$$Tr[L^T(A-(b-\delta')I)^{-1}L] \leq -n - \frac{\|L\|^2}{\delta'}.$$

The problem here is that we have no idea what the δ' is.

Here is a proposition which might help us start the algorithm after m-vectors have already been picked.

Theorem 8.1. *Assume we have picked $i = 1, 2, \ldots, m$ of the $Lv_i = \omega_i$ to form our operator $A = \sum_{i=1}^m \omega_i \omega_i^T$ and assume that the smallest eigenvalue of A is greater than b and we have*

$$Tr[L^T(A-bI)^{-1}L] \leq -(n-m) - \frac{2\|L\|^2}{\delta}. \tag{1}$$

If $b' = b - \delta$, then we can pick another vector ω so that

$$Tr[L^T(A+\omega\omega^T - b'I)^{-1}L] = \tag{2}$$

$$Tr[L^T(A-b'I)^{-1}L] - \frac{\omega^T(A-b'I)^{-1}LL^T(A-b'I)^{-1}\omega}{1+\omega^T(A-b'I)^{-1}\omega} \leq Tr[L^T(A-bI)^{-1}L].$$

Moreover,

$$\omega^T(A-b'I)^{-1}\omega < -1.$$

Hence, the smallest eigenvalue of $A+\omega\omega^T$ *is greater than* b'*, and by Equation 2 we have*

$$\begin{aligned} Tr[L^T(A+\omega\omega^T-b'I)^{-1}L] &\leq Tr[L^T(A-bI)^{-1}L] \\ &\leq -(n-m)-\frac{2\|L\|^2}{\delta} \\ &= -n+m-\frac{2\|L\|^2}{\delta} \\ &\leq -n+m+1-\frac{2\|L\|^2}{\delta} \\ &= -(n-(m+1))-\frac{2\|L\|^2}{\delta} \end{aligned}$$

.

and so we can start over to pick another ω*.*

Proof. We do this in steps.

Step 1: We show:

$$Tr[L^T(A-b'I)^{-1}L] \leq Tr[L^T(A-bI)^{-1}L].$$

Proof: As in our notes on the proof, we have

$$\frac{-1}{b}-\frac{-1}{b'} \geq \frac{\delta}{2(b')^2},$$

and

$$\frac{1}{\lambda_i-b}-\frac{1}{\lambda_i-b'} \geq \frac{\delta}{2(\lambda_i-b')^2}.$$

Hence,

$$\begin{aligned} Tr[L^T(A-bI)^{-1}L-L^T(A-b'I)^{-1}L] &= Tr[L^T(A-bI)^{-1}-(A-b'I)^{-1}L] \\ &\geq \frac{\delta}{2}Tr[L^T(A-b'I)^{-2}L] > 0. \end{aligned}$$

Step 2: We note that for $\omega=\omega_i$, for $i=1,2,\ldots,m$, we have

$$\omega^T(A-b'I)^{-1}LL^T(A-b'I)^{-1}\omega > 0,$$

and

$$\omega^T(A-b'I)^{-1}\omega > 0.$$

Step 3: We show that

$$-(n-m)-\sum_{i=1}^{n}\omega_i^T(A-b'I)^{-1}\omega_i \leq -(n-m)-\sum_{i=m+1}^{n}\omega_i^T(A-b'I)^{-1}\omega_i,$$

and

$$\sum_{i=m+1}^{n} \omega_i^T (A-b'I)^{-1} LL^T (A-b'I)^{-1}\omega_i \leq \sum_{i=1}^{n} \omega_i (A-b'I)^{-1} LL^T (A-b'I)^{-1}\omega_i.$$

Proof: For the first inequality, by Step 2 we have

$$\omega_i^T (A-b'I)^{-1}\omega_i > 0, \quad \text{for } i = 1, 2, \ldots, m.$$

Hence,

$$\begin{aligned} \sum_{i=1}^{n} \omega_i^T (A-b'I)^{-1}\omega_i &= \sum_{i=1}^{m} \omega_i^T (A-b'I)^{-1}\omega_i + \sum_{i=m+1}^{n} \omega_i^T (A-b'I)^{-1}\omega \\ &\geq \sum_{i=m+1}^{n} \omega_i^T (A-b'I)^{-1}\omega_i. \end{aligned}$$

Hence,

$$-\sum_{i=1}^{n} \omega_i^T (A-b'I)^{-1}\omega_i \leq -\sum_{i=m+1}^{n} \omega_i^T (A-b'I)^{-1}\omega_i.$$

The first result follows by adding $-(n-m)$ to both sides of the inequality.

For the second inequality, by Step 2,

$$\sum_{i=1}^{m} \omega_i^T (A-b'I)^{-1} LL^T (A-b'I)^{-1}\omega_i > 0.$$

Hence,

$$\sum_{i=m+1}^{n} \omega_i^T (A-b'I)^{-1} LL^T (A-b'I)^{-1}\omega_i \leq$$

$$\sum_{i=1}^{m} \omega_i^T (A-b'I)^{-1} LL^T (A-b'I)^{-1}\omega_i + \sum_{i=m+1}^{n} \omega_i^T (A-b'I)^{-1} LL^T (A-b'I)^{-1}\omega_i =$$

$$\sum_{i=1}^{n} \omega_i^T (A-b'I)^{-1} LL^T (A-b'I)^{-1}\omega_i.$$

Step 4: We show:

$$Tr[L^T (A-b'I)^{-1} LL^T (A-b'I)^{-1} L] \leq$$

$$\left(Tr[L^T (A-bI)^{-1} L] - Tr[L^T (A-b'I)^{-1} L]\right)\left(-(n-m) - Tr[L^T (A-b'I)^{-1} L]\right).$$

Proof: Since,

$$Tr[L^T (A-b'I)^{-1} L] \leq Tr[L^T (A-bI)^{-1} L] \leq -(n-m) - \frac{2\|L\|^2}{\delta},$$

we have that

$$\|L\|^2 \leq \frac{\delta}{2}\left(-(n-m) - Tr[L^T(A-b'I)^{-1}L]\right).$$

Hence,

$$Tr[L^T(A-b'I)^{-1}LL^T(A-b'I)^{-1}L] \leq$$
$$\|L\|^2 Tr[L^T(A-b'I)^{-2}L] \leq$$
$$\frac{\delta}{2} Tr[L^T(A-b'I)^{-2}L]\left(-(n-m) - Tr[L^T(A-b'I)^{-1}L]\right) \leq$$
$$\left(Tr[L^T(A-bI)^{-1}L] - Tr[L^T(A-b'I)^{-1}L]\right)\left(-(n-m) - Tr[L^T(A-b'I)^{-1}L]\right)$$

Finally, we put this all together.

$$\sum_{i=m+1}^{n} \omega_i^T(A-b'I)^{-1}LL^T(A-b'I)^{-1}\omega_i \leq$$
$$\sum_{i=1}^{n} \omega_i(A-b'I)^{-1}LL^T(A-b'I)^{-1}\omega_i =$$
$$Tr\left[L^T(A-b'I)^{-1}LL^T(A-b'I)^{-1}L\right] \leq$$
$$\left(Tr[L^T(A-bI)^{-1}L] - Tr[L^T(A-b'I)^{-1}L]\right)\left(-(n-m) - Tr[L^T(A-b'I)^{-1}L]\right) =$$
$$\left(Tr[L^T(A-bI)^{-1}L] - Tr[L^T(A-b'I)^{-1}L]\right)\left(-(n-m) - \sum_{i=1}^{n}\omega_i^T(A-b'I)^{-1}\omega_i\right) \leq$$
$$\left(Tr[L^T(A-bI)^{-1}L] - Tr[L^T(A-b'I)^{-1}L]\right)\left(-(n-m) - \sum_{i=m+1}^{n}\omega_i^T(A-b'I)^{-1}\omega_i\right).$$

It follows that there exists a $m+1 \leq j \leq n$ satisfying:

$$0 < \omega_j^T(A-b'I)^{-1}LL^T(A-b'I)^{-1}\omega_j \leq$$
$$\left(Tr[L^T(A-bI)^{-1}L] - Tr[L^T(A-b'I)^{-1}L]\right)\left(-1 - \omega_j^T(A-b'I)^{-1}\omega_j\right).$$

It follows that

$$-1 - \omega_j^T(A-b'I)^{-1}\omega_j > 0,$$

and so by our results, the lowest eigenvalue of the new operator is $\geq b'$ and by this and the Sherman-Morrison formula,

$$Tr[L^T(A+\omega_j\omega_j^T - b'I)^{-1}L] = Tr[L^T(A-b'I)^{-1}L] - \frac{\omega_j^T(A-b'I)^{-1}LL^T(A-b'I)^{-1}\omega_j}{1+\omega_j^T(A-bI)^{-1}\omega_j}$$
$$\leq Tr[L^T(A-bI)^{-1}L].$$

So we can iterate this. □

Remark 8.2. The main question now is: When can we actually start up the algorithm after we have been given m vectors? The proposition works for any choice of δ as long as we have Inequality 1. We would hope that the correct choice of δ is around

$$\delta = \frac{b\|L\|^2}{\epsilon(n-m)}.$$

Then if we iterate the algorithm

$$k = \frac{\epsilon^2(n-m)}{\|L\|^2} - times,$$

we get a lower Riesz bound of

$$b - k\delta = b - \frac{\epsilon^2(n-m)}{\|L\|^2}\frac{b\|L\|^2}{\epsilon(n-m)} = b - b\epsilon = b(1-\epsilon).$$

Unfortunately, this is not the case. If we are given a very bad choice of vectors, to get he algorithm to start again we may need delta on the order of

$$\delta = \frac{b\|L\|^2 m}{\epsilon(n-m)}.$$

We will first show that this order actually works to start up the algorithm, and afterwards give an example to show that the *unfortunate* m term in the numerator of δ is necessary in general.

Corollary 8.3. *We can start the algorithm in Theorem 8.1 as long as*

$$\delta \geq \frac{2b}{1-b}\|L\|^2\left[\frac{m+1}{n-m}\right].$$

Proof. We check the required trace as

$$\begin{aligned}
Tr[L^T(A-bI)^{-1}L] &= \sum_{i=1}^{n}\omega_i^T(A-bI)^{-1}\omega_i \\
&= \sum_{j=1}^{m}\frac{1}{\lambda_j - b}\sum_{i=1}^{n}\omega_{ij}^2 - \frac{n-m}{b} + \frac{1}{b}\sum_{i=m+1}^{n}\|Q_m\omega_i\|^2 \\
&\leq \sum_{j=1}^{m}\frac{1}{\lambda_j - b}\|L\|^2 - \frac{n-m}{b} + \frac{1}{b}m\|L\|^2 \\
&\leq \frac{m\|L\|^2}{\delta} - \frac{n-m}{b} + \frac{m\|L\|^2}{b} \\
&= m\|L\|^2\left[\frac{1}{\delta} + \frac{1}{b}\right] - \frac{n-m}{b} \\
&\leq \frac{2m\|L\|^2}{\delta} - \frac{n-m}{b}.
\end{aligned}$$

So the algorithm will start up if

$$\frac{2m\|L\|^2}{\delta} - \frac{n-m}{b} \leq -(n-m) - \frac{2\|L\|^2}{\delta}.$$

Rearranging terms, we need

$$\begin{aligned}\frac{2\|L\|^2}{\delta}(m+1) &\leq \frac{n-m}{b} - (n-m)\\ &= \left(\frac{1}{b} - 1\right)(n-m)\\ &= \frac{1-b}{b}(n-m).\end{aligned}$$

Hence, the algorithm will start up as long as

$$\delta \geq \frac{2b}{1-b}\|L\|^2 \left[\frac{m+1}{n-m}\right].$$

□

We will now give an example to show that the corollary is essentially best possible. The problem with this example is that it is a set that the algorithm would never have picked. Applying the algorithm, we have that the smallest eigenvalue drops by δ each time. So the eigenvalues after m iterations are on the order of

$$\lambda_i \approx 1 - \epsilon - i\delta.$$

i.e. Each eigenvalue is approximately δ smaller than the previous. In our example, the eigenvalues are the *worst case senerio* in that they are on the order of

$$\lambda_i = 1 - \epsilon - \frac{m}{2}\delta, \quad \text{for all } i = \tfrac{m}{2}+1, \tfrac{m}{2}+2, \ldots, m.$$

Example: We fix m, $0 < b' < 1$ (and to avoid having to work with fractions, we will work with $2m$ in the place of m) and any large $n > 2m$. Now pick a set of eigenvectors $\{e_i\}_{i=1}^n$ and a corresponding set of eigenvalues

$$\lambda_i = \begin{cases} 2 - b' & \text{if } \; i = 1, 2, \ldots, m \\ b' & \text{if } \; i = m+1, m+1, \ldots, 2m \\ 0 & \text{if } \; i = 2m+1, 2m+2, \ldots, n \end{cases}$$

So

$$\sum_{i=1}^{2m} \lambda_i = m(2-b') + mb' = 2m.$$

By a result of Casazza and Tremain [7], there exists a set of $2m$ norm-one vectors $\{\omega_i\}_{i=1}^{2m}$ so that the corresponding frame operator

$$A = \sum_{i=1}^{2m} \omega_i \omega_i^T,$$

has precisely these eigenvectors and eigenvalues. Since we have exactly $2m$ non-zero vectors supported in a $2m$-dimensional space, these vectors form a Riesz basis for span $\{e_i\}_{i=1}^{2m}$ with Riesz bounds $b, 2-b$. Now, pick new eigenvalues

$$\lambda_i' = \begin{cases} 0 & \text{if } \ i = 1, 2, \ldots, m \\ 2(1-b') & \text{if } \ \ i = m+1, m+1, \ldots, 2m \\ 1 - \frac{m(2-2b')}{n-m} & \text{if } \ \ i = 2m+1, 2m+2, \ldots, n \end{cases}$$

Our choice of λ_i' guarantees that

$$\sum_{i=1}^{n} \lambda_i' = n - m.$$

Again by [7], we can pick norm one vectors $\{\omega_i'\}_{i=m+1}^{n}$ so that their frame operator has eigenvalues $\{\lambda_i'\}_{i=2m+1}^{n}$. Since we have $n-m$ vectors supported in an $n-m$-dimensional space, these vectors form a Riesz basis for span $\{e_i\}_{i=m+1}^{n}$ with Riesz bounds

$$1 - \frac{m(2-2b')}{n-m}, \quad 2(1-b').$$

For n large, we should be picking the vectors $\{\omega_i'\}_{i=m+1}^{n}$ as a large Riesz set. But suppose someone picked the vectors $\{\omega_i\}_{i=1}^{2m}$ and asked us to start up the algorithm again? As we will see, this is a problem for our choice of δ. Let us note that in this example the frame operator of the vectors $\{\omega_i\}_{i=1}^{2m} \cup \{\omega_i'\}_{i=m+1}^{n}$ has eigenvalues

$$\lambda_i + \lambda_i' = \begin{cases} 2-b' & \text{if } \ i = 1, 2, \ldots, m \\ 2-b' & \text{if } \ \ i = m+1, m+1, \ldots, 2m \\ 1 - \frac{m(2-2b')}{n-m} & \text{if } \ \ i = 2m+1, 2m+2, \ldots, n \end{cases}$$

Hence, if $L : \mathbb{H}_n \to \mathbb{H}_n$ is given by

$$Le_i = \begin{cases} \omega_i & \text{if } \ i = 1, 2, \ldots, 2m \\ \omega_{m+j}' & \text{if } \ \ i = 2m+j \end{cases}$$

then we have that $\|L\|^2 = 2 - b$ and

$$\sum_{i=1}^{2m} \omega_{ij}^2 + \sum_{i=m+1}^{n} (\omega_{ij}')^2 = \lambda_j + \lambda_j', \quad \text{for all } j = 1, 2, \ldots, n. \tag{3}$$

For the algorithm to start up, we must have

$$Tr[L^T(A - bI)^{-1}L] \le -(n-2m) - \frac{2\|L\|^2}{\delta}.$$

We have for $b = b' - \delta$ (b - and hence δ - to be defined)

$$Tr[L^T(A - bI)^{-1}L] = \sum_{j=1}^{2m} \frac{1}{\lambda_j - b} \sum_{i=1}^{n} \omega_{ij}^2 - \frac{n-2m}{b} + \frac{1}{b} \sum_{i=2m+1}^{n} \|Q_m \omega_i\|^2,$$

where in our case (from above) we have

$$\sum_{i=1}^{2m}\omega_{ij}^2+\sum_{i=2m+1}^{n}(\omega'_{ij})^2=2-b',\quad \text{for}\quad j=m+1,m+2,\dots,2m,$$

and

$$\frac{1}{\lambda_j-b}=\frac{1}{\delta},\quad \text{for}\quad j=m+1,m+2,\dots,2m.$$

Now, using the above and throwing away a lot of information, we have a necessary condition for the algorithm to start as

$$\begin{aligned}
-(n-2m)-\frac{2\|L\|^2}{\delta} &= -(n-2m)-\frac{2(2-b')}{\delta}\\
&\geq Tr[L^T(A-bI)^{-1}L]\\
&= \sum_{j=1}^{2m}\frac{1}{\lambda_j-b}\sum_{i=1}^{n}\omega_{ij}^2-\frac{n-2m}{b}+\frac{1}{b}\sum_{i=2m+1}^{n}\|Q_m\omega_i\|^2\\
&\geq \sum_{j=m+1}^{2m}\frac{1}{\lambda_j-b}(2-b')-\frac{n-2m}{b}\\
&= \frac{m(2-b')}{\delta}-\frac{n-2m}{b}.
\end{aligned}$$

That is, we need

$$\begin{aligned}
\frac{2-b'}{\delta}(2+m) &= \frac{m(2-b')}{\delta}+\frac{2(2-b')}{\delta}\\
&\leq \frac{n-2m}{b}-(n-2m)\\
&= \frac{1-b}{b}(n-2m).
\end{aligned}$$

Thus, we need

$$\delta\geq\frac{(2-b')b}{1-b}\left[\frac{2+m}{n-2m}\right]=\frac{b}{1-b}\|L\|^2\left[\frac{2+m}{n-2m}\right].$$

So our condition to start up the algorithm in Corollary 8.3 is necessary.

9. Remarks

Remark 9.1. Note that this algorithm actually establishes much more than just pulling a Riesz sequence out of the $\{Le_i\}_{i=1}^n$. It is actually establishing a **hierarchy** of vectors where the first m-vectors $\{Le_i\}_{i=1}^m$ have lower Riesz bound

$$1-\epsilon-(m-1)\delta,$$

which is overall much better than the final Riesz bound we will get in the end.

Remark 9.2. If we have two Riesz bases, the BT selection theorem will pick out one of the Riesz bases. But the S^2-algorithm has the chance to pick from both of the Riesz bases to get a new Riesz sequence.

Remark 9.3. Note that we do not have to stop the algorithm after picking

$$m = \frac{\epsilon^2 n}{\|L\|^2} - vectors.$$

This stopping point just gives us a lower Riesz bound of $(1-\epsilon)^2$ for our family. But we can continue picking vectors until the lower Riesz bound equals zero. This point is shown below.

Remark 9.4. By our earlier results, given $\{Lv_i\}_{i=1}^m$ with frame operator A having eigenvalues $\{\lambda_1 \geq \lambda_2 \cdots \geq \lambda_m$, we can add a vector Lv_{m+1} to our set and get a new frame operator $A + \omega\omega^T$ with eigenvalues $\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_{m+1}$ satisfying

$$\lambda'_{m+1} \geq b',$$

if and only if

$$\sum_{i=1}^m \frac{\lambda_i}{\lambda_i - b'} \omega_i^2 \leq 1 - b'.$$

But, we also have that

$$\sum_{i=1}^m \lambda_i = m, \quad \sum_{i=1}^{m+1} \lambda'_i = m + 1.$$

So

$$1 = \sum_{i=1}^m (\lambda'_i - \lambda_i) + \lambda'_{m+1},$$

and so

$$\sum_{i=1}^m (\lambda'_i - \lambda_i) = 1 - \lambda'_{m+1}.$$

That is,

$$\lambda'_{m+1} \geq b' \text{ if and only if } \sum_{i=1}^m (\lambda'_i - \lambda_i) \leq 1 - b' \text{ if and only if}$$

$$\sum_{i=1}^m \frac{\lambda_i}{\lambda_i - b'} \omega_i^2 \leq 1 - b'.$$

This indicates that there should be equality between the sums above if b' is optimal. i.e. There is some relationship between these sums.

10. Problems

There are several significant problems surrounding the restricted invertibility theorem which are still left open. The first is whether we can get control of both the upper and lower Riesz bounds from the S^2-algorithm.

Problem 10.1. *Find a variation of the S^2-algorithm which produces both upper and lower Riesz bounds. That is, given $0 < \epsilon < 1$ and any natural number n and any operator $L : \ell_2^n \to \ell_2^n$ with $\|Le_i\| = 1$ for the canonical unit vector basis $\{e_i\}_{i=1}^n$, then we can find a subset $\sigma \subset \{1, 2, \ldots, n\}$ of cardinality*

$$|\sigma| \geq \epsilon^2 \frac{n}{\|L\|^2},$$

and for all scalars $\{a_i\}_{i\in\sigma}$ we have

$$(1-\epsilon)^2 \sum_{i\in\sigma} |a_i|^2 \leq \| \sum_{i\in\sigma} a_i L e_i \|^2 \leq (1-\epsilon)^{-2} \sum_{i\in\sigma} |a_i|^2.$$

Recently [4] the above problem was answered by developing a two sided algorithm for proving the Restricted Invertibility Theorem. Keep in mind that in Section 7 we saw that we cannot get the optimal bounds by applying the S^2-algorithm and then finding a further subset which has a good upper Riesz bound.

Another problem left open in the work of Bourgain-Tzafriri [1] is the infinite dimensional version of the theorem. To state this problem, we will need a definition.

Definition 10.2. *For $J \subseteq \mathbb{N}$, the* **lower and upper density** *of J are given, respectively, by*

$$D^-(J) = \liminf_{R\to\infty} \inf_{k\in\mathbb{N}} \frac{|B_R(k)\cap J|}{|B_R(k)|}, \qquad (4)$$

$$D^+(J) = \limsup_{R\to\infty} \sup_{k\in\mathbb{N}} \frac{|B_R(k)\cap J|}{|B_R(k)|}, \qquad (5)$$

where $|\cdot|$ denotes the cardinality of the set and $B_R(k)$ denotes the subset of $\mathbb{N}$

$$B_R(k) = \{k, k+1, \ldots, k+R\}$$

is an interval of length R starting at K.

If $D^-(J) = D^+(J)$, then we say that J is of **uniform density**.

Problem 10.3. *Find universal constants A, B so that the following holds: Let $\{e_i\}_{i\in\mathbb{N}}$ be the unit vector basis for ℓ_2 and $L : \ell_2 \to \ell_2$ be a linear operator with $\|Le_i\| = 1$, for all $i \in \mathbb{N}$. Then there is a subset $J \subset \mathbb{N}$ of uniform density so that*

(1) $D(J) \geq \frac{A}{\|L\|^2}$,

(2) For all scalars $\{a_i\}_{i\in J}$ *we have*

$$\|\sum_{i\in J} a_i L e_i\|^2 \geq B \sum_{i\in J} |a_i|^2.$$

Of course we would like to have for any $\epsilon > 0$ to have $A = \epsilon^2$ and $B = (1-\epsilon)^2$. Casazza and Pfander [3] have shown that Problem 10.3 has a positive solution for ℓ_1-localized operators. Vershynin [10] has shown that Problem 10.3 has a positive solution for restrictions of exponentials to subsets of the torus.

Finally, we should keep in mind that all this work is directed towards solving the famous, intractible 1959 Kadison-Singer Problem [8] (See also [5, 2, 6]. It is known [5] that this problem is equivalent to the Bourgain-Tzafriri Conjecture which grew out of the Restricted-Invertibility Theorem.

Problem 10.4 (Bourgain-Tzafriri Conjecture)**.** *There is a universal constant* $A > 0$ *so that for every* $B > 1$ *there is a natural number* $r = r(B)$ *satisfying: For any natural number* n*, and any linear operator* $L : \ell_2^n \to \ell_2^n$ *with* $\|L\| \leq B$ *and* $\|Te_i\| = 1$*, for all* $i = 1, 2, \ldots, n$ *(*$\{e_i\}_{i=1}^n$ *the unit vector basis of* ℓ_2^n*), then there is a partition* $\{A_j\}_{j=1}^r$ *of* $\{1, 2, \ldots, n\}$ *so that for all* $j = 1, 2, \ldots, r$ *and all choices of scalars* $\{a_i\}_{i\in A_j}$ *we have*

$$A \sum_{i\in A_j} |a_i|^2 \leq \|\sum_{i\in A_j} a_i L e_i\|^2.$$

It is now known that this conjecture is equivalent to a number of famous problems in pure mathematics, applied mathematics and engineering [5, 2].

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