DENSITY CRITERIA IN OPERATOR IDENTIFICATION

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ABSTRACT

Analogous to the density criterion for Gabor frames and Riesz bases in the space of square integrable functions, we develop a necessary density condition for time–frequency localized operators in the realm of operator identification. The developed framework can be seen as a generalization of both, the density result for Gabor frames and Riesz bases, and the identifiability theorem for so-called operator Paley–Wiener spaces.

Keywords— Operator identification, density criteria, Gabor frames, atomic Hilbert–Schmidt operator decompositions.

1. INTRODUCTION

The goal of operator identification is to recover an incompletely known operator taken from a given operator family through observation of a single input / output pair [5, 8]. In general, for normed linear spaces X, Y and $\mathcal{H} \subset \mathcal{L}(X, Y)$, we wish to find an element $f \in X$ such that the evaluation map

$$\Phi_f : \mathcal{H} \to Y, \qquad H \mapsto Hf$$

is bounded and boundedly invertible on its range. Then $\mathcal{H}$ is said to be identifiable by f. Identifiability is important, for example, in mobile radio communications where an a-priori unknown channel operator needs to be identified prior to information transmission through a channel.

In this paper, we consider spaces of Hilbert-Schmidt operators which are defined by atomic decompositions. Then, f identifies the closed linear span $\mathcal{H}_\Lambda$ of $\{H_\lambda\}_{\lambda\in\Lambda}$ if

$$\|Hf\|_{L^2(\mathbb{R})} \asymp \|H\|_{HS}, \quad H \in \mathcal{H}_\Lambda = \overline{\operatorname{span}\{H_\lambda\}_{\lambda\in\Lambda}}. \tag{1}$$

If $\{H_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis sequence in the space of Hilbert–Schmidt operators, then

$$\|H\|_{HS} = \big\| \sum_{\lambda\in\Lambda} c_\lambda H_\lambda \big\|_{HS} \asymp \|\mathbf{c}\|_{\ell^2(\Lambda)}, \quad \{c_\lambda\} \in \ell^2(\Lambda),$$

so identifiability by f is equivalent to establishing

$$\|Hf\|_{L^2(\mathbb{R})} = \big\| \sum_{\lambda\in\Lambda} c_\lambda H_\lambda f \big\|_2 \asymp \|\{c_\lambda\}\|_{\ell^2(\Lambda)}, \quad \{c_\lambda\} \in \ell^2(\Lambda), \tag{2}$$

that is, to showing that $\{H_\lambda f\}_{\lambda\in\Lambda}$ is a Riesz basis sequence in $L^2(\mathbb{R})$. In this paper, we shall focus on establishing condition (2) for given operator families $\{H_\lambda\}_{\lambda\in\Lambda}$.

The operator Riesz basis sequences considered here are defined via Gabor decompositions of the operators' spreading functions, or, equivalently, their Kohn-Nirenberg symbols or time-varying impulse responses. We address the question whether in this setting, identifiability of the operator class $\mathcal{H}_\Lambda$ depends on a notion of density on the time-frequency index set $\Lambda \subseteq \mathbb{R}^4$. See Section 3 for two theorems that motivated this body of work.

We would like to emphasize that the range of the operators considered consists of functions in one variable, while the spreading functions of the operators are bivariate. This dimension mismatch in (1) implies that a single evaluation map Φ_f cannot identify the space of Hlbert–Schmidt operators. Our analysis in Sections 4 and 5 resolves this dimensionality mismatch.

2. PRELIMINARIES

In this section we review some general properties of Gabor Riesz bases and frames for the Hilbert space of square integrable functions $L^2(\mathbb{R}^d)$ and for the space of Hilbert-Schmidt operators on $L^2(\mathbb{R})$.

Recall that a countable family of vectors $\{g_\lambda\}_{\lambda\in\Lambda}$ in a separable Hilbert space H is called *Riesz basis sequence* if there exist constants $0 < a \leq b$ with

$$a\|c_\lambda\|_{\ell^2(\Lambda)} \leq \| \sum_{\lambda\in\Lambda} c_\lambda g_\lambda \|_H \leq b\|c_\lambda\|_{\ell^2(\Lambda)}, \quad \{c_\lambda\} \in \ell^2(\Lambda).$$

The existence of positive numbers permitting a double inequality such as the one above will be abbreviated in the following by

$$\|c_\lambda\|_{\ell^2(\Lambda)} \asymp \|\sum_{\lambda\in\Lambda} c_\lambda g_\lambda\|_H, \quad \{c_\lambda\} \in \ell^2(\Lambda).$$

A *Riesz basis* is a Riesz basis sequence which is complete in H.

The system $\{g_\lambda\}_{\lambda\in\Lambda}$ is a *frame* for H if

$$\|f\|_{L^2(\mathbb{R}^d)} \asymp \left\|\{\langle f, g_\lambda\rangle\}_{\lambda\in\Lambda}\right\|_{\ell^2(\Lambda)}, \quad f \in H. \qquad (3)$$

If (3) holds only on the closed linear span of $\{g_\lambda\}_{\lambda\in\Lambda}$, that is, $\overline{\operatorname{span}\{g_\lambda\}_{\lambda\in\Lambda}}$, then $\{g_\lambda\}_{\lambda\in\Lambda}$ is a *frame sequence*.

Let $\Lambda = M\mathbb{Z}^{2d} \subset \mathbb{R}^{2d}$ be a (not necessarily full rank) lattice, that is, M is a $2d$ by $2d$ matrix with real entries. A Gabor system $\{g_\lambda\}_{\lambda\in\Lambda} = (g, \Lambda)$ for $L^2(\mathbb{R}^d)$ is the set of all time-frequency shifts of the window function g by $\lambda = (x, \omega) \in \Lambda$, that is

$$(g, \Lambda) := \{g_\lambda = \pi(\lambda)g : \lambda \in \Lambda\},$$

for $\pi(\lambda)g(t) = T_x M_\omega g = g(t-x)e^{2\pi i\langle\omega,t\rangle}$. Then, (g, Λ) is a *Gabor Riesz basis sequence* if

$$\|c_\lambda\|_{\ell^2(\Lambda)} \asymp \|\sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda)g\|_{L^2(\mathbb{R}^d)}, \quad \{c_\lambda\} \in \ell^2(\Lambda),$$

and a *Gabor frame* for $L^2(\mathbb{R}^d)$ if

$$\|f\|^2_{L^2(\mathbb{R}^d)} \asymp \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2, \quad f \in L^2(\mathbb{R}^d).$$

Hilbert-Schmidt operators are those bounded operators on $L^2(\mathbb{R})$ with $L^2(\mathbb{R}^2)$ kernel, that is,

$$Hf(x) = \int \kappa(x, y)f(y)\, dy,$$

with $\kappa \in L^2(\mathbb{R}^2)$, that is, Hilbert-Schmidt operators are defined through finiteness of the Hilbert-Schmidt norm $\|H\|_{HS} = \|\kappa_H\|_{L^2(\mathbb{R}^2)}$.

As operators are in 1-1 correspondents to their kernel, they can also be represented by their time-varying impulse response h, their Kohn-Nirenberg symbol σ and their spreading function η. In fact, formally,

$$\begin{aligned} Hf(x) &= \int h_H(t, x)\, f(x-t)\, dt \\ &= \iint \eta_H(t, \nu)\, e^{2\pi i\nu(x-t)}\, f(x-t)\, d\nu\, dt \\ &= \int \sigma_H(x, \xi)e^{2\pi ix\xi}\widehat{f}(\xi)\, d\xi, \end{aligned}$$

where

$$\begin{aligned} \int \eta_H(t, \nu)e^{2\pi i\nu x}\, d\nu &= h_H(t, x) = \kappa_H(x, x-t) \\ &= \int \sigma_H(x, \xi)e^{2\pi i\xi t}\, d\xi, \end{aligned}$$

and the Fourier transform is normalized as $\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int f(x)e^{-2\pi ix\xi}\, dx$. Clearly,

$$\begin{aligned} \|H\|_{HS} &= \|\kappa_H\|_{L^2(\mathbb{R}^2)} = \|h_H\|_{L^2(\mathbb{R}^2)} \\ &= \|\eta_H\|_{L^2(\mathbb{R}^2)} = \|\sigma_H\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

It is not difficult to see that the space of Hilbert–Schmidt operators on $L^2(\mathbb{R})$, that is, $HS(L^2(\mathbb{R}))$, is not identifiable. Consequently, we will restrict ourselves to closed subspaces $\mathcal{H} \subseteq HS(L^2(\mathbb{R}))$. We shall use the fact that the domain of some subspaces of operators considered below can be extended to include spaces of distributions. For example, choosing $\gamma(x) = e^{-x^2}$, we can define the modulation space $M^\infty(\mathbb{R})$ as space of all distributions with

$$\|f\|_{M^\infty(\mathbb{R})} = \sup_{\lambda\in\mathbb{R}^2} |\langle f, \pi(\lambda)\gamma\rangle| < \infty,$$

and note that, for example, operators in so-called operator Paley–Wiener space

$$OPW_2(M) = \{H \in HS(L^2(\mathbb{R})) : \operatorname{supp}\eta_H \subseteq M\}$$

map boundedly $M^\infty(\mathbb{R})$ to $L^2(\mathbb{R})$ whenever M is a compact set [7].

We now recall the definition of Beurling density. Let $B_d(R)$ denote a ball in $\mathbb{R}^d$ centered at 0 with radius R. Let $\Lambda \subseteq \mathbb{R}^d$. Then the lower and upper Beurling densities of Λ are given by

$$D^-(\Lambda) = \liminf_{R\to\infty} \inf_{z\in\mathbb{R}^d} \frac{|\Lambda \cap \{B_d(R) + z\}|}{\pi R^d},$$

$$D^+(\Lambda) = \limsup_{R\to\infty} \sup_{z\in\mathbb{R}^d} \frac{|\Lambda \cap \{B_d(R) + z\}|}{\pi R^d}.$$

Whenever $D^-(\Lambda) = D^+(\Lambda)$, we speak of the Beurling density of Λ, $D(\Lambda)$.

Clearly, whenever Λ is a lattice, the Beurling density is the inverse of the Lebesgue measure of any measurable fundamental domain of Λ.

3. MOTIVATING EXAMPLES

The work presented here is motivated by two important density phenomena in time-frequency analysis, namely, the density criterion for Gabor frames and Riesz bases (here, $H_\lambda = \pi(\lambda)$, $\lambda \in \Lambda$, which is not Hilbert-Schmidt and therefore formally not part of the here described theory), and a necessary identifiability condition for operators in the operator Paley-Wiener space $OPW_2(M)$ (where $H_\lambda = \pi(\lambda)H_0\pi(\lambda)^*$, $\lambda \in \Lambda$ and H_0 chosen appropriately).

Theorem 1. *Let $\Lambda \subseteq \mathbb{R}^2$ be a lattice. If there exists $f \in L^2(\mathbb{R})$ with $\{\pi(\lambda)f\}_{\lambda\in\Lambda}$ being a Riesz basis sequence in $L^2(\mathbb{R})$, then $D(\Lambda) \leq 1$.*

Note that Theorem 1 can be adjusted to fit the framework of Hilbert–Schmidt operators by replacing $H_\lambda = \pi(\lambda)$ with $H_\lambda = \pi(\lambda) \circ H_0$ with H_0 being any fixed Hilbert–Schmidt operator. Note that in this case, we can also consider f to be in a distributional space (for example, $f \in M^\infty(\mathbb{R})$ as long as $H_0 f \in L^2(\mathbb{R})$ holds).

The second result motivating this paper is the following [5, 8].

Theorem 2. *Let $\Lambda \subseteq \mathbb{R}^2$ be a lattice and H be a Hilbert–Schmidt operator with smooth and compactly supported spreading function. If there exists $f \in M^\infty(\mathbb{R})$ with $\{\pi(\lambda)H\pi(\lambda)^* f\}_{\lambda\in\Lambda}$ being a Riesz basis sequence in $L^2(\mathbb{R})$, then $D(\Lambda) \leq 1$.*

Note that Theorem 2 is proven similarly to the result that, with M being a fundamental domain of Λ, $OPW_2(M)$ is identifiable if and only if the Lebesgue measure of M is less than or equal to one, a condition which is equivalent to the density of Λ being less than or equal to one.

In the following, we shall try to better understand the relationship of the results above by means of finding a common generalization.

4. MAIN RESULT

Hilbert-Schmidt operators on $L^2(\mathbb{R})$ are characterized by $\eta_H \in L^2(\mathbb{R}^{2d})$. Hence, we can obtain operator expansions through Gabor Riesz basis sequence or Gabor frame expansions of η_H.

Proposition 3. *For $H_0 \in HS(L^2(\mathbb{R}))$, we write $\eta_0 = \eta_{H_0}$. Then the operator $T_A M_B T_{-C} H_0 T_C M_D$ has spreading function*

$$\eta_{T_A M_B T_{-C} H_0 T_C M_D} = T_{A,B+D} M_{B,C} \eta_0, \quad A,B,C,D \in \mathbb{R}.$$

Hence, if

$$\eta_H = \sum_{k,l,m,n} c_{k,l;m,n} T_{am,bn} M_{ck,dl} \eta_0$$

with convergence in L^2-norm, then

$$H = \sum_{k,l,m,n} c_{k,l;m,n} T_{am} M_{ck} T_{-dl} H_0 T_{dl} M_{bn-ck}.$$

with convergence in $HS(L^2(\mathbb{R}))$.

Note that Theorem 1 corresponds to choosing H_0 to be the identity operator and by restricting ourselves to two parameter coefficient sequences, namely, to coefficient sequences with $c_{k,l;m,n} = \delta(k)\delta(l)\widetilde{c}_{m,n}$. Similarly, for Theorem 2 we consider H_0 with η_H smooth and compactly supported, and coefficient sequences $c_{k,l;m,n} = \delta(m)\delta(n)\widetilde{c}_{k,l}$.

For $\lambda = (x_1, x_2; \omega_1, \omega_2) \in \mathbb{R}^4$ and H_0 Hilbert–Schmidt with spreading function η_0, we define H_λ by

$$\eta_{H_\lambda} = \pi(\lambda)\eta_0 = M_{(\omega_1,\omega_2)} T_{(x_1,x_2)} \eta_0. \tag{4}$$

Then, Theorem 1 corresponds to testing condition (2) for $\{H_\lambda\}_{\lambda\in\Lambda}$ with

$$\Lambda = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix}^T \mathbb{Z}^2,$$

while for Theorem 2 we choose

$$\Lambda = \begin{pmatrix} 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}^T \mathbb{Z}^2.$$

This observation leads to the following question.

Question 4. *Can we define a density $\widetilde{D}$ on lattices $\Lambda = M\mathbb{Z}^2 \subseteq \mathbb{R}^4$, M a 4 by 2 real matrix, so that for some $C > 0$ we have*

$$\{H_\lambda f\}_{\lambda\in\Lambda} \text{ is a Riesz basis sequence} \Rightarrow \widetilde{D}(\Lambda) \leq C.$$

By analogy we shall define a 'Beurling-type' 2-density for sets of points Λ lying within general 2 dimensional subspaces $\mathbb{S}$ of $\mathbb{R}^4$

Definition 5. *The "2-dimensional" upper and lower Beurling densities (or for short 2-density) of $\Lambda \subseteq \mathbb{R}^4$ are given by*

$$D^+_{(2)}(\Lambda) = \liminf_{R\to\infty} \inf_{z\in\mathbb{S}} \frac{|\Lambda \cap \{B_4(R) + z\}|}{\pi R^2},$$
$$D^-_{(2)}(\Lambda) = \limsup_{R\to\infty} \sup_{z\in\mathbb{S}} \frac{|\Lambda \cap \{B_4(R) + z\}|}{\pi R^2}.$$

If $D^+_{(2)}(\Lambda) = D^-_{(2)}(\Lambda)$, then Λ has uniform 2-Beurling density $D_{(2)}(\Lambda) = D^-_{(2)}(\Lambda)$.

Note that for this definition of density, it is easy to construct examples implying that there exist no $c > 0$ with

$$\widetilde{D}(\Lambda) < c \Rightarrow \{H_\lambda f\}_{\lambda\in\Lambda} \text{ is not a Riesz basis sequence for all } f.$$

In short, we cannot guarantee identifiability of $\mathcal{H}_\Lambda$ by simply requiring $\{H_\lambda\}_{\lambda\in\Lambda}$ being a Riesz basis sequence in the space of Hilbert–Schmidt operators and requiring a high density. Additional remarks regarding positive identification results are included in Section 5.

As we shall restrict our attention to lattices which define a 2-dimensional plane in $\mathbb{R}^4$, we observe that with

$$\Lambda = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} \mathbb{Z}^2 = \begin{pmatrix} a_1 m + a_2 n \\ b_1 m + b_2 n \\ c_1 m + c_2 n \\ d_1 m + d_2 n \end{pmatrix} : m, n \in \mathbb{Z},$$

we have

$$\begin{aligned} D_{(2)}(\Lambda) =& [(a_1 b_2 - a_2 b_1)^2 + (a_1 c_2 - a_2 c_1)^2 \\ &+ (a_1 d_2 - a_2 d_1)^2 + (b_1 c_2 - b_2 c_1)^2 \\ &+ (b_1 d_2 - b_2 d_1)^2 + (c_1 d_2 - c_2 d_1)^2]^{-1/2}. \end{aligned}$$

Hence, for

$$\Lambda = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix}^T \mathbb{Z}^2.$$

we have $D_{(2)}(\Lambda) = |ab|$ and for

$$\Lambda = \begin{pmatrix} 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}^T \mathbb{Z}^2$$

we have $D_{(2)}(\Lambda) = |cd|$.

Theorem 6. *Let $\Lambda = M\mathbb{Z}^2$, let H_0 have a smooth and compactly supported spreading function and let $\{H_\lambda\}_{\lambda\in\Lambda}$ be defined by (4). If there exists $f \in M^\infty(\mathbb{R})$ with $\{H_\lambda f\}_{\lambda\in\Lambda}$ being a Riesz basis sequence in $L^2(\mathbb{R})$, then $D_{(2)}(\Lambda) \leq \sqrt{2}$,.*

Consequently, if $\{H_\lambda\}_{\lambda\in\Lambda}$ is Riesz in the space of Hilbert–Schmidt operators, then $D_{(2)}(\Lambda) > \sqrt{2}$ implies that $\mathcal{H}_\Lambda$ is not identifiable.

We conclude this section by presenting the central ideas behind the proof of Theorem 6.

Proof. For $m, n \in \mathbb{Z}$ and $\lambda = M(m,n)^T$, we observe that

$$\begin{aligned} H_\lambda &= T_{(a_1-d_1)m+(a_2-d_2)n} M_{c_1 m + c_2 n} \\ &\quad H_0 T_{d_1 m + d_2 n} M_{(b_1-c_1)m+(b_2-c_2)n}. \end{aligned}$$

The condition η_{H_0} smooth and compactly supported implies that for any $f \in M^\infty(\mathbb{R})$,

$$f_\lambda = H_0 T_{d_1 m + d_2 n} M_{(b_1-c_1)m+(b_2-c_2)n} f$$

is time–frequency localized at 0. Hence, for any $f \in M^\infty(\mathbb{R})$,

$$\{T_{(a_1-d_1)m+(a_2-d_2)n} M_{c_1 m + c_2 n} f_{M(m,n)^T}\}_{m,n\in\mathbb{Z}} \subseteq L^2(\mathbb{R})$$

is a system of Gabor molecules localized with respect to the lattice

$$\Lambda' = \begin{pmatrix} a_1 - d_1 & a_2 - d_2 \\ c_1 & c_2 \end{pmatrix} \mathbb{Z}^2.$$

A Gabor molecule extension of Theorem 1 implies that $\{H_\lambda f\}_{\lambda\in\Lambda}$ being a Riesz basis sequence for $L^2(\mathbb{R})$ necessitates $D(\Lambda') \leq 1$ [2, 4, 6]. A computation shows that $D(\Lambda') > 1$ follows from $D_{(2)}(\Lambda) > \sqrt{2}$. □

5. SUFFICIENT CRITERIA FOR IDENTIFICATION

Recall that for a given Hilbert–Schmidt operator H_0, a given lattice $\Lambda \subseteq \mathbb{R}^4$, and H_λ, $\lambda \in \Lambda$, given by (4), we have $\mathcal{H}_\Lambda = \overline{\operatorname{span}\{H_\lambda\}_{\lambda\in\Lambda}}$. To establish identifiability of $\mathcal{H}_\Lambda$, we search for an identifier f such that any choice of coefficients $\{c_\lambda\} \in \ell^2(\Lambda)$, can be computed from Hf. Equivalently, we require that $\{c_\lambda\}$ can be computed from the values of the inner products $v_\mu = \langle Hf, \pi(\mu)\gamma\rangle$, $\mu \in \mathcal{M}$, which are the Gabor coefficients of Hf with respect to a Gabor frame $\{\pi(\mu)\gamma\}_{\mu\in\mathcal{M}}$ for $L^2(\mathbb{R})$. To succeed with this, we need to solve the system of equations

$$v_\mu = \langle Hf, \pi(\mu)\gamma\rangle = \sum_{\lambda\in\Lambda} c_\lambda \langle H_\lambda f, \pi(\mu)\gamma\rangle = \sum_{\lambda\in\Lambda} c_\lambda A_{\mu;\lambda}. \tag{5}$$

If there exists f such that the map $A\colon Y \to \ell^2(\mathbb{Z}^2), Y \subset \ell^2(\mathbb{Z}^4)$ is invertible, then $\mathcal{H}_\Lambda$ is identifiable. On the other hand, if for every f belonging to a particular space of distributions (for example, the modulation space $M^\infty(\mathbb{R})$), the map A is not invertible, then $\mathcal{H}_\Lambda$ is not identifiable with identifiers from this space.

Designing f for identification can again be carried out on the coefficient level. In fact, with $\{\pi(\widetilde{\mu})\widetilde{\gamma}\}_{\widetilde{\mu}\in\widetilde{\mathcal{M}}}$ being an appropriately chosen Gabor frame for $L^2(\mathbb{R})$ (or, for example, an l^∞ frame for $M^\infty(\mathbb{R})$ [1]), we seek a coefficient sequence $\{d_{\widetilde{\mu}}\}$ so that the biinfinite matrix with entries

$$A_{\mu;\lambda} = \sum_{\widetilde{\mu}\in\widetilde{\mathcal{M}}} d_{\widetilde{\mu}} \langle H_\lambda \pi(\widetilde{\mu})\widetilde{\gamma}, \pi(\mu)\gamma\rangle.$$

Results obtained with this approach such as the explanatory given below [3].

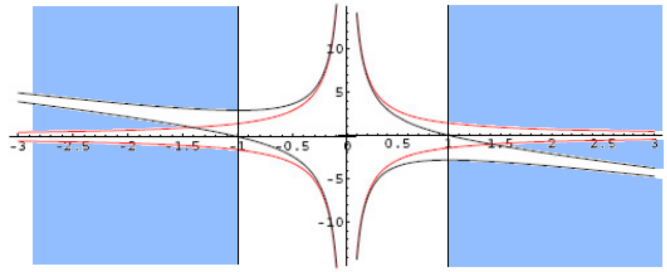

Fig. 1. The set (α, β) fulfilling the conditions in Proposition 7 lies in the shaded region.

Proposition 7. *Let H_0 be given by $\kappa_0(x,\omega) = e^{-\pi(x^2+\omega^2)}$, that is, $\eta_0(t,\nu) = \frac{1}{\sqrt{2}} e^{-\pi i\sqrt{2}t\nu} e^{-\frac{\pi}{2}(t^2+\nu^2)}$, and let $\Lambda = \left(\begin{smallmatrix} \alpha & 0 & 0 & \\ 0 & \beta & \alpha & 0 \end{smallmatrix}\right)^T \mathbb{Z}^2$. If α, β are such that $|\alpha(\beta+\alpha\sqrt{2})| \geq \sqrt{2}$, $|\alpha\beta| > \sqrt{2}$, $|\alpha| > 1$, then the operator family $\mathcal{H}_\Lambda$ is identifiable.*

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