Irregular and multi-channel sampling in operator Paley-Wiener spaces

Y. M. Hong (1) and G. Pfander (2)

(1) Department of Mathematical Sciences, KAIST, Daejeon, South Korea.
(2) School of Engineering and Science, Jacobs university Bremen, 28759 Bremen, Germany.
phongym@kaist.ac.kr, g.pfander@jacobs-university.de

Abstract:

The classical sampling theorem states that a band-limited function can be reconstructed by its values taken at sufficiently dense grid. Recently, sampling theorem for functions has been generalized to so-called operator sampling, namely, operator identification in view of sampling theory. We generalize the uniform operator sampling to irregular version which shows that the sampling set is not necessarily uniformly distributed. We also develop multi-channel operator sampling for overspread operators.

1. Notations

The operator class considered here is the class of Hilbert-Schmidt operators. Hilbert-Schmidt operator H is defined by a bounded linear operator on $L^2(\mathbb{R}^d)$ which can be represented as an integral operator

$$\begin{aligned} Hf(x) &:= \int \kappa_H(x,t) f(t)dt \\ &= \int \kappa_H(x, x-t) f(x-t)dt \quad \text{a.e.} \end{aligned}$$

with kernel $\kappa_H \in L^2(\mathbb{R}^{2d})$. Then H is equivalently expressed by

$$\begin{aligned} Hf(x) &= \int \sigma_H(x,\xi)\hat{f}(\xi)\, e^{2\pi i x\cdot\xi} d\xi \\ &= \int\int \eta_H(t,\nu) M_\nu T_t f(x) dt d\nu \\ &= \int h_H(t,x) f(x-t)dt \quad \text{a.e.} \end{aligned}$$

where σ_H, η_H and h_H are Kohn-Nirenberg symbol, spreading function and time-varying impulse response of H, respectively, and T_t and M_ν are translation and modulation operators, respectively, that is, $T_t f(x) = f(x-t)$ and $M_\nu f(x) = e^{2\pi i x\nu} f(x)$.
The linear space of Hilbert-Schmidt operators $HS(L^2(\mathbb{R}^d))$ is a Hilbert space when it is given the Hilbert space structure of $L^2(\mathbb{R}^d)$ with

$$\langle H_1, H_2\rangle_{HS} := \langle \kappa_{H_1}, \kappa_{H_2}\rangle_{L^2}.$$

Note that

$$\|H\|_{HS} = \|\kappa_H\|_{L^2} = \|h_H\|_{L^2} = \|\sigma_H\|_{L^2} = \|\eta_H\|_{L^2}.$$

An operator class $\mathcal{H} \subseteq HS(L^2(\mathbb{R}^d))$ is identifiable if all $H \in \mathcal{H}$ extend to a domain containing some so–called identifier $f \in S_0'(\mathbb{R})$ with

$$A\|H\|_{HS} \leq \|Hf\|_{L^2} \leq B\|H\|_{HS} \;\text{ for all } H \in \mathcal{H}. \tag{1..1}$$

If we can choose f in (1..1) to be a tempered distribution supported on a discrete set, then we say that $\mathcal{H}$ permits operator sampling.
For any compact set $S \subseteq \mathbb{R}^{2d}$, we define the operator Paley-Wiener space by

$$OPW(S) := \{H \in HS(L^2(\mathbb{R}^d)) \,:\, \operatorname{supp}\eta_H \subseteq S\}.$$

Note that any $H \in OPW(S)$ with S compact extends to a bounded linear operator $H : S_0'(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ [1], [3]. Then our goal is to find the lower bound A in (1..1), as an upper bound always exists given by $B = \|f\|_{S_0'}$. In what follows, we use the notation $A(f) \asymp B(f)$ if there exist c and $C > 0$ independent of the object f in a given class such that

$$cA(f) \leq B(f) \leq CA(f).$$

2. Uniform sampling of Hilbert-Schmidt operators

We recall a uniform sampling for operators in the operator Paley-Wiener space [4].

Theorem 2..1 *For $\Omega, T, T' > 0$ and $0 < \Omega T' \leq \Omega T \leq 1$, choose $\varphi \in PW([-(\frac{1}{T}-\frac{\Omega}{2}), \frac{1}{T}-\frac{\Omega}{2}])$ with $\hat{\varphi} = 1$ on $[-\frac{\Omega}{2}, \frac{\Omega}{2}]$ and $r \in L^\infty(\mathbb{R})$ with $\operatorname{supp} r \subset [-T+T', T]$ and $r = 1$ on $[0, T']$. Then any $H \in OPW([0,T']\times[-\frac{\Omega}{2}, \frac{\Omega}{2}])$ permits operator sampling as*

$$\|H\|_{HS} = \sqrt{T}\|H\sum_{n\in\mathbb{Z}} \delta_{nT}\|_{L^2}$$

and operator reconstruction is possible by means of

$$h_H(t,x) = r(t)T\sum_{n\in\mathbb{Z}}(H\sum_{k\in\mathbb{Z}}\delta_{kT})(t+nT)\varphi(x-t-nT).$$

We remark that the results in [2] and [3] are based on the uniform sampling and rely on the properties of the Zak transform. Here, we consider orthonormal basis expansions based on Fourier series for a generalization to irregular sampling.

3. Irregular sampling in Operator Paley-Wiener spaces

Considering a frame expansion rather than orthonormal basis expansion, we obtain more general irregular sampling theorems for Hilbert-Schmidt operators.
Given an operator class $\mathcal{H}$, a sequence $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ in $\mathbb{R}$ is said to be a set of sampling for $\mathcal{H}$, if for some $\{c_k\}_{k\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$, we have $\sum_{k\in\mathbb{Z}} c_k\delta_{\lambda_k} \in S_0'(\mathbb{R})$ and $\sum_{k\in\mathbb{Z}} c_k\delta_{\lambda_k}$ identifies $\mathcal{H}$.
In the following we let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$, $\lambda_{k+1} > \lambda_k$ for all $k \in \mathbb{Z}$.

Theorem 3..1 *If Λ is uniformly discrete, then a necessary condition for Λ being a set of sampling for $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$ is $D^-(\Lambda) \geq \Omega$ and a sufficient condition is $D^-(\Lambda) > \Omega$ and $\lambda_{k+1} - \lambda_k \geq T$.*

Theorem 3..2 *Let Λ be a set of sampling for $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$ and let $\{e^{-2\pi i\lambda_k\xi}\}_{k\in\mathbb{Z}}$ be a Riesz basis for $L^2[-\frac{\Omega}{2},\frac{\Omega}{2}]$, then $\lambda_{k+1} - \lambda_k \geq T$.*

Example 3..3 *Let $\Lambda_r = \{\lambda_k\}_{k\in\mathbb{Z}}$ be given by $\lambda_k = k$ for $k \neq 0$ and $\lambda_0 = r$, $r \in \mathbb{R}$. The set Λ_r is a set of sampling for $PW([-\frac{1}{2},\frac{1}{2}])$ if and only if $r \notin \mathbb{Z}\setminus\{0\}$. To see this, note that as $\{e^{2\pi ik\xi}\}_{k\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\frac{1}{2},\frac{1}{2}]$, so is $\{e^{2\pi ik\xi}\}_{k\neq 0} \cup \{e^{2\pi ir\xi}\}$ if $r \notin \mathbb{Z}\setminus\{0\}$. By Theorem 3..2, $\Lambda_r = \{\lambda_k\}_{k\in\mathbb{Z}}$ is a set of sampling for $OPW([0,1]\times[-\frac{1}{2},\frac{1}{2}])$ if and only if $r = 0$.*

Note that the condition $\lambda_{k+1} - \lambda_k \geq T$ for $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$ in Theorem 3..2 is not necessary if $\{e^{-2\pi i\lambda_k\xi}\}_{k\in\mathbb{Z}}$ is not a Riesz basis but a frame for $L^2[-\frac{\Omega}{2},\frac{\Omega}{2}]$.

4. Multi-channel operator sampling

It has been shown that underspread operators can be identifiable by single channel output [2, 3]. In this section we develop multi-channel sampling for overspread operators. In other words, overspread operators can be measured by multiple channel outputs.

Theorem 4..1 *For $H \in OPW(S)$ where $S \subseteq [0,N]\times[-\frac{M}{2},\frac{M}{2}]$ for some $M, N \in \mathbb{N}$, H is recovered from the MN identifiers $\{\sum_{n\in\mathbb{Z}} e^{2\pi ijn/MN}\delta_{\frac{n}{M}}\}_{j=0}^{MN-1}$, that is,*

$$\|H\|_{HS}^2 = \frac{1}{M^2N}\sum_{j=0}^{MN-1}\|H(\sum_{n\in\mathbb{Z}} e^{2\pi ijn/MN}\delta_{\frac{n}{M}})\|^2.$$

Alternatively, if we consider periodic nonuniform samples for delta-trains, then we obtain the following theorem.

Theorem 4..2 *For $H \in OPW(S)$ where $S \subseteq [0,N]\times[-\frac{M}{2},\frac{M}{2}]$ for some $M, N \in \mathbb{N}$, H is recovered from MN identifiers $\{\sum_{n\in\mathbb{Z}}\delta_{nN+\alpha_j}\}_{j=1}^{MN}$ by*

$$\|H\|_{HS}^2 \asymp \sum_{j=1}^{MN}\|H(\sum_{n\in\mathbb{Z}}\delta_{nN+\alpha_j})\|^2$$

where $0 \leq \alpha_j < N$ for all $1 \leq j \leq MN$ and $\alpha_i \neq \alpha_j$ for $i \neq j$.

References:

[1] E. Cordero and K. Gröchenig. Time-frequency analysis of localization operators. *J. Funct. Anal.*, 205:107–131, 2003.

[2] W. Kozek and G. Pfander. Identification of operators with bandlimited symbols. *SIAM J. Math. Anal.*, 37(3):867–888, 2005.

[3] G. E. Pfander and D. F. Walnut. Operator identification and feichtinger's algebra. *Sampl. Theory Signal Image Process.*, 5:183–200, 2006.

[4] G. E. Pfander and D. F. Walnut. On the sampling of functions and operators with an application to multiple–input multiple–output channel identification. *Proc. SPIE Vol. 6701, Wavelets XII; Dimitri Van De Ville, Vivek K. Goyal, Manos Papadakis, Eds*, pages 67010T–1 –67010T–14, 2007.