Measurement of time–varying Multiple–Input Multiple–Output Channels

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ABSTRACT

We derive a criterion on the measurability / identifiability of Multiple–Input Multiple–Output (MIMO) channels based on the size of the so-called spreading support of its subchannels. Novel MIMO transmission techniques provide high-capacity communication channels in time-varying environments and exact knowledge of the transmission channel operator is of key importance when trying to transmit information at a rate close to channel capacity.

Keywords: Underspread operators, Multiple–Input Multiple–Output channels, spreading function, bandlimited Kohn–Nirenberg symbol

1. INTRODUCTION

The recovery of information from a signal that has traveled through a communications channel requires knowledge of — or at least some information on — the transmission channel at hand. In applications such as mobile telephony, neither the location of the subscriber nor the changing environment through which information is transmitted is known *a-priori*. To combat this problem, a pilot signal is send prior to information transmission with the hope that the corresponding channel output supplies the receiver with the measurements that are needed to invert the channel operator. The inverse of the channel operator allows the receiver to recover the information from the subsequently send information carrying signals.

In Single–Input Single–Output (SISO) channels, the channel input is considered to be a single variable function, which, after being transmitted, is distorted by the unknown transmission channel operator before arriving at the receiver (see [5, 13] and references within). In [14], the existence of pilot signals which identify linear SISO channel operators was shown to depend on the size of the spreading support of the channel operator. That is, it was shown that a channel operator is

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identifiable by the channel output corresponding to an appropriately chosen input signal if the *a-priori* known spreading support has area (Jordan content) less than one, while a channel operator cannot be identified by a single input/output pair if the area of the spreading support is larger than one (and nothing else is known of the channel operator). Loosely speaking, the size of the spreading support of an operator represents the amount of time–frequency dispersion that the channel inflicts on the transmission signal. Too much time–frequency dispersion cannot be resolved by a single channel output. Fortunately, channel operators with spreading support area much smaller than one, often called slowly time–varying or underspread operators, are the norm in mobile communications. The results in [14] described above were conjectured in the 1960s by Kailath [8] and Bello [1]. See [9] and [14] for some historical background on the channel identification problem for slowly time–varying channels and for further applications of identification theorems for underspread operators.

Multiple transmit and receive antenna methods have been developed to obtain high capacity wireless channels (see [5, 12, 13, 18] and references within). Methods which achieve high capacities often rely on the precise knowledge of the channel at the receiver and/or the transmitter (see [5], pp 298).

In such MIMO channel setups, N signals are transmitted by N antennas simultaneously. On the receiver side, M antennas record channel output signals that represent the superposition of the N input signals, each individually distorted depending on the path the signal has travelled from its transmitting antenna to the receiving antenna. Consequently, a linear MIMO channel operator can be modelled by a matrix of $N{\cdot}M$ SISO channel operators. It maps a vector of N transmission signals to M channel output signals.

In this paper, we extend the SISO results from [14] to linear MIMO channels. That is, we show that MIMO channel operators permit identification by one vector of input signals if at each of the receiving antennas the following condition holds: the sum of the areas of the N spreading supports of the subchannels leading to the receiving antenna is less than one. Conversely, we show that if the sum of the N spreading areas of the subchannels leading to one of the receiving antennas is larger than one, then identification is not possible.

For simplicity, we assume throughout this paper that the $N{\cdot}M$ subchannels within a MIMO channel are independent of each other. That is, information obtained on one of the $N{\cdot}M$ subchannels does not carry any information on another subchannel in the MIMO setup. The realistic assumption that the vicinity of the transmit antennas and the vicinity of the receive antennas lead to a dependent channel ensemble should allow for a relaxation of the measurability criterion given here.

Modern methods in time–frequency analysis, such as those involving Feichtinger's algebra and modulation spaces, have been used in [9, 15, 14] to streamline the analysis of operators with compactly supported spreading functions. Using these methods comes at the price of necessitating non–standard terminology when formulating results. Here, we bypass these methods in order to state results in terms of the better known Hilbert–Schmidt operators and tempered distributions. Further, the approach chosen here leads to a generalization of the results in [14] in the SISO case as well.

Section 2 is devoted to preliminaries and notation. We state our main result as Theorem 3.2 in Section 3. The result is then proven in Section 4 and Section 5

2. PRELIMINARIES AND NOTATION

The space of complex valued Lebesgue integrable functions on d-dimensional Euclidean space $\mathbb{R}^d$ is denoted by $L^1(\mathbb{R}^d)$. The *Fourier transform* $\widehat{f}$ of $f \in L^1(\mathbb{R}^d)$ is the continuous function

$$\widehat{f}(\gamma) = \int f(x)\, e^{-2\pi i \gamma \cdot x}\, dx\,, \quad \gamma \in \widehat{\mathbb{R}}^d,$$

where $\widehat{\mathbb{R}}^d$ is the dual group of $\mathbb{R}^d$, which, aside of notation, is identical to $\mathbb{R}^d$.

The space of square integrable functions $L^2(\mathbb{R}^d)$ consists of those Lebesgue measurable functions which satisfy

$$\|f\|_{L^2} = \left(\int |f(x)|^2\, dx \right)^{\frac{1}{2}} < \infty.$$

$L^2(\mathbb{R}^d)$ is a Hilbert space with inner product

$$\langle f, g \rangle = \int f(x)\overline{g(x)}\, dx, \quad f, g \in L^2(\mathbb{R}^d).$$

In case of vector valued functions $\boldsymbol{f} = (f_1, \ldots, f_N) \in L^2(\mathbb{R}^d)^N$ we set accordingly

$$\|\boldsymbol{f}\|_{L^2} = \sqrt{\sum_{n=1}^{N} \|f_n\|_{L^2}^2}.$$

For $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ we have $\|\widehat{f}\|_{L^2} = \|f\|_{L^2}$. In fact, the Fourier transform on $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ extends to a unitary operator on the Hilbert space $L^2(\mathbb{R}^d)$.

The set of *Schwartz class functions* $\mathcal{S}(\mathbb{R}) \subseteq L^2(\mathbb{R})$ on $\mathbb{R}$ consists of all infinitely differentiable functions which satisfy

$$p_{k,l}(f) = \sup_{x \in \mathbb{R}} \left| x^l f^{(k)}(x) \right| < \infty, \quad k, l \in \mathbb{N},$$

where $f^{(k)}$ denotes the k-th derivative of f. $\mathcal{S}(\mathbb{R})$ is a Frechét space whose metric is defined using the seminorms $p_{k,l}$, $k,l \in \mathbb{N}$. Hence, $f_n \to f$ in $\mathcal{S}(\mathbb{R})$ if and only if $p_{k,l}(f_n - f) \to 0$ for all $k,l \in \mathbb{N}$. The elements in the dual space $\mathcal{S}'(\mathbb{R})$ of bounded functionals on $\mathcal{S}(\mathbb{R})$ are called *tempered distributions*.

The usefulness of $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ in harmonic analysis stems in part from the fact that the Fourier transform defines a bijective isomorphism on $\mathcal{S}(\mathbb{R})$. Using duality, we can extend the Fourier transform on $\mathcal{S}(\mathbb{R})$ to the space $\mathcal{S}'(\mathbb{R})$ of tempered distributions. Since $\mathcal{S}'(\mathbb{R})$ contains constant functions, *Dirac's delta* $\delta : f \mapsto f(0)$, and *Shah distributions* $⧢_a = \sum_{n\in\mathbb{Z}} \delta_{an}$, where $\delta_{na} = T_{na}\delta$ and $a > 0$, it is justified to write $\widehat{⧢}_a = \frac{1}{a} ⧢_{\frac{1}{a}}$.

Similarly to the Fourier transform, the *time shift operator* T_t, $t \in \mathbb{R}^d$, given by $T_t f(x) = f(x-t)$ and the *modulation operator* M_ω, $\omega \in \widehat{\mathbb{R}}^d$, $M_\omega f(x) = e^{2\pi i \omega\cdot x} f(x)$ are unitary operators on $L^2(\mathbb{R}^d)$ and bijective isomorphism on $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ (equipped with the weak-$*$ topology). Note that M_ω is also called *frequency shift operator* since $\widehat{M_\omega f} = T_\omega \widehat{f}$. Further, we refer to $\pi(\lambda) = \pi(t,\nu) = T_t M_\nu$ for $\lambda = (t,\nu) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ as *time–frequency shift operator*.

The set $HS(L^2(\mathbb{R}))$ of *Hilbert–Schmidt operators* on $L^2(\mathbb{R})$ consists of those linear operators on $L^2(\mathbb{R})$ which satisfy

$$Hf(x) = \int \kappa_H(x,y) f(y)\, dy, \quad f \in \mathcal{S}(\mathbb{R}), \tag{1}$$

for $\kappa_H \in L^2(\mathbb{R}^2)$ [3, 4]. In fact, the density of $\mathcal{S}(\mathbb{R})$ in $L^2(\mathbb{R})$ together with $\langle Hf, g\rangle = \langle \kappa_H, g \otimes \overline{f}\rangle$ implies that (1) extends to a bounded operator on $L^2(\mathbb{R})$. Note further, that $HS(L^2(\mathbb{R}))$ is a Hilbert space with inner product $\langle H_1, H_2\rangle_{HS} = \langle \kappa_{H_1}, \kappa_{H_2}\rangle$ and corresponding norm. Hilbert–Schmidt operators are compact operators on $L^2(\mathbb{R})$. Note that some Hilbert–Schmidt operators can be extended to act on larger subsets of $\mathcal{S}'(\mathbb{R})$ than $L^2(\mathbb{R})$, a fact that will use later in this paper.

Every Hilbert–Schmidt operator can be expressed as a superposition of time and frequency shift operators. In fact, for H with $\kappa_H \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, we set

$$\eta_H(t,\nu) = \int \kappa_H(x, x-t) e^{-2\pi i \nu x}\, dx, \quad a.e.\ \nu \in \widehat{R}.$$

It is easy to see that in this case

$$\|\eta_H\|_{L^2} = \|\kappa_H\|_{L^2} = \|H\|_{HS}, \tag{2}$$

implying that the *spreading function* $\eta_H \in L^2(\mathbb{R}\times\widehat{\mathbb{R}})$ can be defined for any Hilbert–Schmidt operator H, and thereby extending (2) to all Hilbert–Schmidt operators[1]. As mentioned above, we

[1]The spreading function of an Hilbert–Schmidt operator, or, more general, of a pseudodifferential operator, is the

have

$$H = \iint \eta_H(t,\nu) T_t M_\nu \, d\nu dt = \int \eta_H(\lambda)\pi(\lambda)\, d\lambda \tag{3}$$

where the operator valued integral in (3) is understood weakly, that is, H is defined via

$$\langle Hf, g\rangle = \iint \eta_H(t,\nu) \int e^{2\pi i\nu(x-t)} f(x-t)\overline{g(x)} dx\, dt d\nu = \langle \eta_H, V_f g\rangle, \tag{4}$$

where the *short–time Fourier transform* $V_f g$ of $g \in L^2(\mathbb{R})$ with respect to $f \in L^2(\mathbb{R})$ is given by

$$V_f g(t,\nu) = \int g(x) e^{-2\pi i\nu(x-t)} \overline{f(x-t)}\, dx$$

and satisfies $V_f g \in L^2(\mathbb{R}\times\widehat{\mathbb{R}})$ [7].

To avoid double indices, we shall write at times $\eta(H)$ in place of η_H and, similarly, $\kappa(H)$ in place of κ_H.

We denote by $HS(L^2(\mathbb{R}))^{M\times N}$ the space of N-input, M-output MIMO channels whose $N{\cdot}M$ subchannels are Hilbert–Schmidt operators on $L^2(\mathbb{R})$ [6]. The operator space $HS(L^2(\mathbb{R}))^{M\times N}$ is equipped with norm

$$\|\boldsymbol{H}\|_{HS} = \sqrt{\sum_{m=1}^{M}\sum_{n=1}^{N} \|H_{mn}\|_{HS}^2}, \quad \boldsymbol{H} = \begin{pmatrix} H_{11} & \cdots & H_{1N} \\ \vdots & & \vdots \\ H_{M1} & \cdots & H_{MN} \end{pmatrix} \in HS(L^2(\mathbb{R}))^{M\times N}.$$ [2]

Further, the spreading function $\boldsymbol{\eta_H} = \boldsymbol{\eta}(\boldsymbol{H})$ of $\boldsymbol{H} = \begin{pmatrix} H_{11} & \cdots & H_{1N} \\ \vdots & & \vdots \\ H_{M1} & \cdots & H_{MN} \end{pmatrix} \in HS(L^2(\mathbb{R}))^{M\times N}$ and the *spreading support* of $\boldsymbol{H}$ are defined componentwise, that is, we have

$$\boldsymbol{\eta}(\boldsymbol{H}) = \begin{pmatrix} \eta(H_{11}) & \cdots & \eta(H_{1N}) \\ \vdots & & \vdots \\ \eta(H_{M1}) & \cdots & \eta(H_{MN}) \end{pmatrix} \in L^2(\mathbb{R}\times\widehat{\mathbb{R}})^{M\times N},$$

and

$$\text{supp }\boldsymbol{\eta}(\boldsymbol{H}) = \begin{pmatrix} \text{supp }\eta(H_{11}) & \cdots & \text{supp }\eta(H_{1N}) \\ \vdots & & \vdots \\ \text{supp }\eta(H_{M1}) & \cdots & \text{supp }\eta(H_{MN}) \end{pmatrix} \subseteq (\mathbb{R}\times\widehat{\mathbb{R}})^{M\times N}.$$

Our identifiability result for MIMO channels considers operator classes of the form

$$\mathcal{H}_{\boldsymbol{S}} = \Big\{\boldsymbol{H} \in HS(L^2(\mathbb{R}))^{M\times N} : \text{ supp }\boldsymbol{\eta}(\boldsymbol{H}) \subseteq \boldsymbol{S}\Big\}, \quad \boldsymbol{S} \subseteq (\mathbb{R}\times\widehat{\mathbb{R}})^{M\times N}.$$

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symplectic Fouriertransform of the operators Kohn–Nirenberg symbol. Consequently, the theory of pseudodifferential operators with compactly supported spreading functions coincides with the theory of pseudodifferential operators with bandlimited Kohn–Nirenberg symbols.

[2] It is easy to see that $HS(L^2(\mathbb{R}))^{N\times N} = HS(L^2(\mathbb{R})^N)$.

To avoid pathological cases, we shall only consider $\mathcal{H}_{\boldsymbol{S}}$ where $\boldsymbol{S}$ is the cartesian products of so called Jordan domains.

Definition 2.1. *A* Jordan domain $M \subseteq \mathbb{R}\times\widehat{\mathbb{R}}$ *is a bounded set whose boundary is a Lebesgue zero set.*

Clearly, our restriction to Jordan domains is not relevant to applications such as those in communications engineering. The following useful characterization of Jordan domains is well known. It is discussed in detail in [10].

Lemma 2.2. *If M is a Jordan domain, then its Lebesgue measure $\mu(M)$ satisfies*

$$\begin{aligned}\mu(M) &= \sup\{\mu(U) : U \subseteq M \text{ and } U \in \mathcal{U}_{KL} \text{ for some } K, L \in \mathbb{N},\ L \text{ prime }\}\\ &= \inf\{\mu(U) : U \supseteq M \text{ and } U \in \mathcal{U}_{KL} \text{ for some } K, L \in \mathbb{N},\ L \text{ prime }\}.\end{aligned}$$

where for $K, L \in \mathbb{N}$ we set $R_{KL} = [0, \frac{1}{K}] \times [0, \frac{K}{L}]$ and

$$\mathcal{U}_{KL} = \left\{ \bigcup_{j=1}^{J} \Big(R_{KL} + (\tfrac{m_j}{K}, \tfrac{n_j K}{L}) \Big) :\ m_j, n_j \in \mathbb{Z}, J \in \mathbb{N} \right\}.$$

3. STATEMENT OF RESULTS

The domain of Hilbert–Schmidt operators with compactly supported spreading function can be extended to include classes of tempered distributions (see Theorem 4.2 in [15]). For example, using (4), it is easy to see that any Hilbert–Schmidt operator with compactly supported spreading function maps $\sqcup\!\sqcup\!\sqcup_a$, $a \in \mathbb{R}^+$, to a function in $L^2(\mathbb{R})$. In fact, a simple computation in [9] shows that for $S = [-\frac{1}{2}, \frac{1}{2}]\times[-\frac{1}{2}, \frac{1}{2}] \subseteq \mathbb{R}\times\widehat{\mathbb{R}}$ we have

$$\|H \sqcup\!\sqcup\!\sqcup_1\|_{L^2(\mathbb{R})} = \|H\|_{HS}, \quad H \in \mathcal{H}_S.$$

Definition 3.1. *An operator class $\mathcal{H} \subseteq HS(L^2(\mathbb{R}))^{M\times N}$ is identifiable if there exists $\boldsymbol{f} \in S'(\mathbb{R})^N$ and positive A, B, with*

$$A\,\|\boldsymbol{H}\|_{HS} \le \|\boldsymbol{H} f\|_{L^2} \le B\,\|\boldsymbol{H}\|_{HS} \quad \text{for } \boldsymbol{H} \in \mathcal{H}.$$

In short, an operator class $\mathcal{H}$ is identifiable if there is $\boldsymbol{f}$ with the property that the induced map

$$\Phi_{\boldsymbol{f}} : \mathcal{H} \longrightarrow L^2(\mathbb{R})^N,\ \boldsymbol{H} \mapsto \boldsymbol{H}\boldsymbol{f}$$

is bounded and stable, that is, bounded above and below.

THEOREM 3.2. *Let* $\boldsymbol{S} = (S_{mn}) \subseteq (\mathbb{R}\times\widehat{\mathbb{R}})^{M\times N}$ *be the cartesian product of Jordan domains in* $\mathbb{R}\times\widehat{\mathbb{R}}$ *and let*

$$\mathcal{H}_{\boldsymbol{S}} = \Big\{\boldsymbol{H} \in HS(L^2(\mathbb{R}))^{M\times N} :\ \operatorname{supp} \boldsymbol{\eta}(\boldsymbol{H}) \subseteq \boldsymbol{S}\Big\}.$$

1. *If* $\sum_{n=1}^{N} \mu(S_{mn}) < 1$ *for all* $m \in \{1, \dots, M\}$*, then* $\mathcal{H}_{\boldsymbol{S}}$ *is identifiable.*
2. *If* $\sum_{n=1}^{N} \mu(S_{mn}) > 1$ *for some* $m \in \{1, \dots, M\}$*, then* $\mathcal{H}_{\boldsymbol{S}}$ *is not identifiable.*

4. PROOF OF THEOREM 3.2, PART *1*

Theorem 4.1 reduces Theorem 3.2, part *1*, for SISO channels ($M = N = 1$) to a question on the linear independence of columns of the following matrices: for any L–periodic sequence $c = \{c_k\}_{k\in Z}$ we set $\boldsymbol{A}(c) = [\boldsymbol{A}_0(c)\ \ \boldsymbol{A}_1(c)\ \ \cdots\ \ \boldsymbol{A}_{K-1}(c)] \in \mathbb{C}^{KL\times L}$ with $\boldsymbol{A}_k(c) = (c_{p+k}\, e^{2\pi i q(p+k)/L})_{p,q=0}^{L-1} \in \mathbb{C}^{L\times L}$.

THEOREM 4.1. *Let* $c = \{c_k\}_{k\in Z}$ *be a sequence with period* L *and* $f = \sum_k c_k\, \delta_{\frac{k}{K}} \in \mathcal{S}'(\mathbb{R})$*. Further, set*

$$U = \bigcup_{j=1}^{J} \Big(R_{KL} + (\tfrac{m_j}{K}, \tfrac{n_j K}{L})\Big), \qquad m_j, n_j \in \mathbb{Z}, J \in \mathbb{N},$$

where $R_{KL} = [0, \frac{1}{K}] \times [0, \frac{K}{L}]$.

Then f *identifies* $\mathcal{H}_U$ *if and only if the columns in* $\boldsymbol{A}(c)$ *with column indices in* $\{m_j L + n_j\}_j$ *are linearly independent.*

Clearly, this result is only applicable if the cardinality $|J|$ of J satisfies $|J| \leq L$ since $\boldsymbol{A}(c)$ has at most L linear independent columns. This requirement is equivalent to $\mu(U) \leq |J| \frac{1}{K} \frac{K}{L} \leq 1$.

If L is prime, then $|J| \leq L$ is also sufficient for the existence of an identifier for a SISO channel[10]:

THEOREM 4.2. *If* L *is prime then there exists* $c \in \mathbb{C}^L$ *such that any set of* L *columns of* $\boldsymbol{A}(c)$ *is linearly independent.*

Proof of Theorem 3.2, Part 1.

We choose $\boldsymbol{S} = (S_{mn}) \subseteq (\mathbb{R}\times\widehat{\mathbb{R}})^{M\times N}$ which satisfies $\sum_{n=1}^{N} \mu(S_{mn}) < 1$. Since all S_{mn} are assumed to be Jordan domains, there exists $K, L \in \mathbb{N}$, L prime, so that for each S_{mn} exists $U_{mn} \in \mathcal{U}_{KL}$ with $S_{mn} \subseteq U_{mn}$ and $\sum_{n=1}^{N} \mu(U_{mn}) < 1$ for $m = 1, \dots, M$.

Clearly, $\mathcal{H}_{\boldsymbol{S}} \subseteq \mathcal{H}_{\boldsymbol{U}}$with $\boldsymbol{U} = (U_{mn}) \subseteq (\mathbb{R}\times\widehat{\mathbb{R}})^{M\times N}$ implies that the identifiability of $\mathcal{H}_{\boldsymbol{S}}$ follows from the identifiability of $\mathcal{H}_{\boldsymbol{U}}$ which we shall prove now.

All U_{mn} are bounded, hence, we can choose $W > 0$ so that

$$U_{mn} \subseteq B_W^\infty(0) = \Big\{\|(t,\nu)\|_\infty = \max\{|t|,|\nu|\} \leq W\Big\} \quad \text{for } m = 1,\ldots,M,\ n = 1,\ldots,N.$$

For L and K chosen above, Theorem 4.2 allows us to choose an L–periodic sequence c so that any set of L columns from $\boldsymbol{A}(c)$ is linearly independent. We set

$$f_n = \pi(0,(n{-}1)2W)\sum_{k\in\mathbb{Z}} c_{k\mathrm{mod}L}\delta_{\frac{k}{K}} \quad \text{for } n = 1,\ldots,N\,,$$

and claim that $\boldsymbol{f} = (f_1,\ldots,f_N)^T$ identifies $\mathcal{H}_{\boldsymbol{U}}$.

To see this, note that the choice of W implies that $T_{(0,(n-1)2W)}U_{mn} \cap T_{(0,(n'-1)2W)}U_{mn'} = \emptyset$ for all $n \neq n'$ and $m = 1,\ldots,M$. For $U_m = \bigcup_{n=1}^N T_{(0,(n-1)2W)}U_{mn}$, $m = 1,\ldots,M$, we have $\mu(U_m) = \sum_{n=1}^N \mu(U_{mn}) < 1$, and, by Theorem 4.1, f_1 identifies $\mathcal{H}_{U_m} \subseteq HS(\mathbb{R})$ for $m = 1,\ldots,M$, that is, there exists $A, B > 0$ such that for all $H \in \mathcal{H}_{U_m}$, $m = 1,\ldots,M$ we have

$$A\|H\|_{HS} = A\|\eta_H\|_{L^2} \leq \|Hf_1\|_{L^2} \leq B\|H\|_{HS}. \tag{5}$$

For $\boldsymbol{H} \in \mathcal{H}_{\boldsymbol{U}}$ we set $\boldsymbol{g} = (g_1,\ldots,g_M) = \boldsymbol{H}\boldsymbol{f}$ and compute for $m = 1,\ldots,M$,

$$\begin{aligned} g_m &= \sum_{n=1}^N H_{mn}\ f_n = \sum_{n=1}^N H_{mn}\circ\pi(0,(n{-}1)2W)\ f_1 \\ &= \sum_{n=1}^N \int \eta\big(H_{mn}\circ\pi(0,(n{-}1)2W)\big)(\lambda)\ \pi(\lambda)f_1\,d\lambda \\ &= \int \left(\sum_{n=1}^N \eta\big(H_{mn}\big)(\lambda-(0,(n{-}1)2W))\right)\ \pi(\lambda)f_1\,d\lambda. \end{aligned}$$

Since $\operatorname{supp} T_{(0,(n-1)2W)}\eta\big(H_{mn}\big) \subseteq T_{(0,(n-1)2W)}U_{mn} \subseteq U_m$ and

$$\mu\Big(\ \operatorname{supp} T_{(0,(n-1)2W)}\eta\big(H_{mn}\big) \cap \operatorname{supp} T_{(0,(n'-1)2W)}\eta\big(H_{mn'}\big)\ \Big) = 0$$

for all $n \neq n'$ and all $m = 1,\ldots,M$, we can apply (5) to obtain

$$\|g_m\|_{L^2}^2 \ \geq\ A^2\left\|\sum_{n=1}^N T_{(0,(n-1)2W)}\eta\big(H_{mn}\big)\right\|_{L^2}^2 = A^2\sum_{n=1}^N \big\|\eta\big(H_{mn}\big)\big\|_{L^2}^2$$

and

$$\|\boldsymbol{g}\|_{L^2}^2 \ =\ \sum_{m=1}^M \|g_m\|_{L^2}^2 \geq A^2\sum_{m=1}^M\sum_{n=1}^N \big\|\eta\big(H_{mn}\big)\big\|_{L^2}^2 = A^2\,\|\boldsymbol{H}\|_{HS}^2$$

The upper bound involving B follows in the same manner. □

5. PROOF OF THEOREM 3.2, PART *2*

We shall now show that the condition $\sum_{n=1}^{N} \mu(S_{mn}) \leq 1$, $m = 1 \dots M$, is necessary for the identifiability of $\mathcal{H}_{\boldsymbol{S}}$, $\boldsymbol{S} = (S_{mn})$.

Without loss of generality, we assume a Multiple–Input Single–Output (MISO) scenario, that is, we consider $M = m = 1$ and write $S_n = S_{1n}$ and $H_n = H_{1n}$. In fact, if there there exists $\boldsymbol{S}$ in the MIMO case with $\sum_{n=1}^{N} \mu(S_{m_0 n}) > 1$ for $m_0 \in \{1, \dots, M\}$ and $\mathcal{H}_{\boldsymbol{S}}$ identifiable, then defining $\boldsymbol{S}'$ by $S'_n = S_{m_0 n}$ would lead to a contradiction of Theorem 3.2, part *2*, in the MISO case.

The proof of Theorem 3.2, part *2*, is organized as the corresponding proof in [15]. The crux is to show that operators in the class $\mathcal{H}_{\boldsymbol{S}}$ with $\sum_{n=1}^{N} \mu(S_n) > 1$ carry to many, in time and frequency tightly packed, degrees of freedom, that is, too much information to be embedded in a stable manner in a single output signal.

To see this, we shall fix $\boldsymbol{S}$ with $\sum_{n=1}^{N} \mu(S_n) > 1$. For this $\boldsymbol{S}$, we construct a bounded and stable synthesis (information embedding) map $E : l_0(\mathbb{Z}^2) \longrightarrow \mathcal{H}_S$ where $l_0(\mathbb{Z}^2)$ is equipped with the $l^2(\mathbb{Z}^2)$-norm, and a bounded and stable analysis (information recovery) operator $C : L^2(\mathbb{R}) \longrightarrow l^2(\mathbb{Z}^2)$ with the property that **all** compositions

$$C \circ \Phi_{\boldsymbol{f}} \circ E : \; l_0(\mathbb{Z}^2) \longrightarrow l^2(\mathbb{Z}^2), \quad \boldsymbol{f} \in \mathcal{S}'(\mathbb{R})^N,$$

are not stable. The stability of E and C implies that the boxed-in operators $\Phi_{\boldsymbol{f}} : \mathcal{H}_S \longrightarrow L^2(\mathbb{R})$, $\boldsymbol{f} \in \mathcal{S}'(\mathbb{R})^N$, must not be stable, showing that $\mathcal{H}_{\boldsymbol{S}}$ is not identifiable if $\sum_{n=1}^{N} \mu(S_n) > 1$.

Before proving Theorem 3.2, part *2*, we state three lemmas, some of whose proofs can be found in [15]. Lemma 5.1 concerns the conjugation of Hilbert–Schmidt operators by time–frequency shifts. In Lemma 5.2 we construct a prototype operator which is later used to construct a Riesz bases for its closed linear span in $\mathcal{H}_S$, that is, a family of Hilbert–Schmidt operators $\{H_{k,l}\}_{k,l \in \mathbb{Z}}$ for which the map

$$\begin{array}{rccc} E : & l^2(\mathbb{Z}^2) & \rightarrow & HS(L^2(\mathbb{R})) \\ & \{c_{kl}\}_{k,l \in \mathbb{Z}} & \mapsto & \sum_{k,l \in \mathbb{Z}} c_{k,l} H_{k,l} \end{array}$$

is well defined, bounded, and stable. Lemma 5.3 generalizes the fact that $m \times n$ matrices with $m < n$ have a nontrivial kernel and, therefore, are not stable, to operators acting on $l^2(\mathbb{Z}^2)$. In fact, the bi-infinite matrices $M = (m_{j',j})_{j',j \in \mathbb{Z}^2}$ considered in Lemma 5.3 are not dominated by its diagonal $m_{j,j}$ — which would correspond to square matrices — but by a slanted diagonal $m_{j,\lambda j}$, $j \in \mathbb{Z}^2$, with $\lambda > 1$.

Lemma 5.1. *For $P \in HS(\mathbb{R})$ with spreading function $\eta_P \in L^2(\mathbb{R}\times\widehat{\mathbb{R}})$ set $\widetilde{P} = M_\omega T_{p-r} P T_r M_{\xi-\omega} \in HS$. Then $\eta_{\widetilde{P}} = e^{2\pi i\omega p} M_{(\omega,r)}\, T_{(p,\xi)}\, \eta_P$ and $\widetilde{P} \in HS(\mathbb{R})$.*

Lemma 5.2. *Fix $\lambda > 1$ with $1 < \lambda^4 < \mu(S)$ and choose even functions $\eta_1, \eta_2 \in \mathcal{S}(\mathbb{R})$ with values in $[0,1]$ and*

$$\eta_1(t) = \begin{cases} 1 & \text{for } |t| \le \frac{1}{2\lambda K} \\ 0 & \text{for } |t| \ge \frac{1}{2K} \end{cases} \quad \text{and} \quad \eta_2(\nu) = \begin{cases} 1 & \text{for } |\nu| \le \frac{K}{2\lambda L} \\ 0 & \text{for } |\nu| \ge \frac{K}{2L} \end{cases}.$$

The operator $P \in \mathcal{H}_{R_{KL}}$ defined by $\eta_P = \eta_1 \otimes \eta_2$ has the properties:

a) The operator family

$$\left\{ M_{\lambda K k}\, T_{\frac{1}{K}m - \frac{\lambda L}{K}l}\, P\, T_{\frac{\lambda L}{K}l}\, M_{\frac{K}{L}n - \lambda K k} \right\}_{k,l,m,n\in\mathbb{Z}} \tag{6}$$

is a Riesz basis for its closed linear span in the Hilbert space of Hilbert–Schmidt operators $HS(\mathbb{R})$.

b) For $f \in \mathcal{S}'(\mathbb{R})$, there exists $C_f, L_f \in \mathbb{N}$ and $d_1, d_2 : \mathbb{R} \to \mathbb{R}_0^+$ which decay rapidly at infinity with

$$|PT_y M_\omega f(x)| \le C_f\, d_1(x)(1 + \|(y,\omega)\|_\infty)^{L_f}, \quad x \in \mathbb{R},$$

and

$$|\widehat{PT_y M_\omega} f(\xi)| \le C_f\, d_2(\xi)(1 + \|(y,\omega)\|_\infty)^{L_f}, \quad \xi \in \widehat{\mathbb{R}}.$$

Proof. (a) See [9].

(b) For $f \in \mathcal{S}(\mathbb{R})$, we compute

$$Pf(x) = \iint \eta_1(t)\eta_2(\nu) e^{2\pi i\nu(x-t)} f(x-t)\, d\nu\, dt = \int \eta_1(t)\check{\eta}_2(x-t) f(x-t)\, dt = \eta_1 * (\check{\eta}_2 f),$$

and, therefore, $\widehat{Pf}(\xi) = \widehat{\eta_1}(\xi) \cdot \eta_2 * \widehat{f}(\xi)$. The rapid decay and smoothness of $\widehat{\eta_1}$ together with the fact that $\operatorname{supp} \eta_2$ compact and η_2 smooth implies that $\widehat{Pf}$ and, therefore, Pf is well defined for $f \in \mathcal{S}'(\mathbb{R})$. In fact, we can conclude that $\widehat{Pf}$, and, therefore, $Pf \in \mathcal{S}(\mathbb{R})$ for $f \in \mathcal{S}'(\mathbb{R})$.

Further, we obtain for $f \in \mathcal{S}(\mathbb{R})$ and $\xi \in \widehat{\mathbb{R}}$ that

$$\begin{aligned} |(PT_{-y}M_{-\omega}f)\widehat{\ }(\xi)| &= \left| \widehat{\eta_1}(\xi) \int \eta_2(\xi-\nu)\, M_{-y}T_\omega \widehat{f}(\nu) d\nu \right| = |\widehat{\eta_1}(\xi)| \left| \langle M_{-y}T_\omega \widehat{f},\, T_\xi \eta_2 \rangle \right| \\ &= |\widehat{\eta_1}(\xi)| \left| \langle \widehat{f},\, M_y T_{\xi-\omega}\eta_2 \rangle \right| = |\widehat{\eta_1}(\xi)| \left| V_{\eta_2}\widehat{f}(\xi-\omega, y) \right|. \end{aligned}$$

The weak-$*$ density of $\mathcal{S}(\mathbb{R})$ in $\mathcal{S}'(\mathbb{R})$ extends the equality above to $f \in \mathcal{S}'(\mathbb{R})$. Theorem 11.2.3 in [7] provides us now with $C'_f, L'_f \in \mathbb{N}$ and

$$\begin{aligned} |(PT_{-y}M_{-\omega}f)\widehat{\ }(\xi)| &= |\widehat{\eta_1}(\xi)| \left| V_{\eta_2}\widehat{f}(\xi-\omega, y)\right| \leq C'_f |\widehat{\eta_1}(\xi)|(1+|y|+|\xi-\omega|)^{L'_f} \\ &\leq C'_f |\widehat{\eta_1}(\xi)|\,(1+|y|+|\xi|+|\omega|)^{L'_f} \\ &\leq C'_f |\widehat{\eta_1}(\xi)|\,(1+|(\xi)|)^{L'_f}(1+|y|+|\omega|)^{L'_f} \leq d_2(\xi)(1+\|(y,\omega)\|_\infty)^{L'_f}, \end{aligned}$$

where $d_2 = C'_f\, 2^{L'_f}\, |\widehat{\eta_1}(\xi)|\,(1+|\xi|)^{L'_f}$ is rapidly decaying.

Similarly, we conclude that for $f \in \mathcal{S}(\mathbb{R})$ and $x \in \mathbb{R}$ we have

$$\begin{aligned} |PT_{-y}M_{-\omega}f(x)| &= \left| \int \eta_1(s-x)\check{\eta}_2(s)\, T_{-y}M_{-\omega}f(s)\, ds \right| \\ &= |\langle M_\omega T_y(\check{\eta}_2 T_x \eta_1),\ f\rangle| = |V_{\check{\eta}_2 T_x\eta_1} f(y,\omega)| \end{aligned}$$

Within the proof of Theorem 11.2.3 in [7], the existence of $C_f, L_f \in \mathbb{N}$ are given with $C_f \geq C'_f$, $L_f \geq L'_f$, and

$$|PT_{-y}M_{-\omega}f(x)| = |V_{\check{\eta}_2 T_x\eta_1} f(y,\omega)| \leq C_f \max_{m,n\leq L_f} \sup_{t\in\mathbb{R}} |t^n \frac{\partial^n}{\partial t^n} \check{\eta}_2 T_x \eta_1(t)|\ (1+\|(y,\omega)\|_\infty)^{L_f}.$$

Note that since $\check{\eta}_2, \eta_1 \in \mathcal{S}(\mathbb{R})$, each $\sup_{t\in\mathbb{R}}\ |t^n \frac{\partial^n}{\partial t^n}\check{\eta}_2 T_x\eta_1(t)|$, $m,n \leq L_f$, decays faster than any polynomial. This implies that also $d_1(x) = \max_{m,n\leq L_f}\ \sup_{t\in\mathbb{R}}\ |t^n \frac{\partial^n}{\partial t^n}\check{\eta}_2 T_x\eta_1(t)|$ also decays faster than any polynomial. □

LEMMA 5.3. *Given $M = (m_{j',j}) : l^2(\mathbb{Z}^2) \to l^2(\mathbb{Z}^2)$. If there exists a polynomial p of degree $L \in \mathbb{N}$ and a monotonically decreasing function $w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $w(x) = o\left(x^{-(L+2)}\right)$ satisfying*

$$|m_{j',j}| < w(\|\lambda j' - j\|_\infty)\ p(\|j\|_\infty), \quad \|\lambda j' - j\|_\infty > K_0$$

for some constants $\lambda > 1$ and $K_0 > 0$, then M is not stable. The proof of Lemma 5.3 is included in the appendix.

Now all pieces are in place to prove necessity of the condition $\displaystyle\sum_{n=1}^{N} \mu(S_n) \leq 1$ for the identifiability of $\mathcal{H}_{\boldsymbol{S}}$, $\boldsymbol{S} = (S_{mn})$.

Proof of Theorem 3.2, part 2.

Fix $\boldsymbol{S} = (S_n)$ with $\sum_{n=1}^N \mu(S_n) > 1$. Without restriction of generality, we shall assume that $S_n \in \mathcal{U}_{KL}$ for some $K, L \in \mathbb{N}$ and all $n = 1, \ldots, N$, and that $S_n \cap S_{n'} = \emptyset$ for $n \neq n'$. Hence, there exists $\mathcal{J} = \{0, 1, 2, ..., J-1\} \subseteq \mathbb{N}$ so that $S = \bigcup_{n=1}^N S_n = \bigcup_{j\in\mathcal{J}} \left(R_{KL} + (\frac{m_j}{K}, \frac{n_j K}{L})\right)$, $(m_j, n_j) \neq (m_{j'}, n_{j'})$ for $j \neq j'$. We have $\mu(R_{KL}) = \frac{1}{L}$, and, since $\mu(S) = \sum_{n=1}^N \mu(S_n) > 1$, we have $J > L$.

Fix $\boldsymbol{f} = (f_1, \ldots, f_n) \in \mathcal{S}'(\mathbb{R})^N$. Choose λ, η_1, η_2, P , $C_{\boldsymbol{f}} = \max_n C_{f_n}$, $L_{\boldsymbol{f}} = \max_n L_{f_n}$, and to $C_{\boldsymbol{f}}$ and $L_{\boldsymbol{f}}$ corresponding d_1 and d_2 according to Lemma 5.2. For $n = 1, \ldots, N$ define

$$J_n = \Big\{ j \in \{0, \ldots, J-1\} : \; R_{KL} + (\tfrac{m_j}{K}, \tfrac{n_j K}{L}) \subseteq S_n \Big\} .$$

The synthesis operator $E : l_0(\mathbb{Z}^2) \to \mathcal{H}_S$ mentioned above is given by

$$E : \sigma_{k,l''} = \sigma_{k,lJ+j} \mapsto \sum_{k,l \in Z} \sum_{j=0}^{J-1} \sigma_{k,lJ+j} \; \iota(j) M_{\lambda K k} \, T_{\frac{1}{K} m_j + \frac{\lambda L}{K} l} \, P \, T_{-\frac{\lambda L}{K} l} \, M_{\frac{K}{L} n_j - \lambda K k},$$

where

$$\iota(j) : HS(\mathbb{R}) \longrightarrow HS(\mathbb{R})^N, \quad H \mapsto H \cdot (\mathbf{1}_{J_1}(j), \ldots, \mathbf{1}_{J_M}(j)) = (0, \ldots 0, \overset{n^{th} \text{ position if } j \in J_n}{H}, 0, \ldots, 0) \; .$$

Since

$$\Big\{ M_{\lambda K k} \, T_{\frac{1}{K} m - \frac{\lambda L}{K} l} \, P \, T_{\frac{\lambda L}{K} l} \, M_{\frac{K}{L} n - \lambda K k} \Big\}_{k,l,m,n \in \mathbb{Z}}$$

is a Riesz basis for its closed linear span in $\mathcal{H}_S \subseteq HS(\mathbb{R})$, we have that

$$\Big\{ \iota(j) M_{\lambda K k} \, T_{\frac{1}{K} m_j + \frac{\lambda L}{K} l} \, P \, T_{-\frac{\lambda L}{K} l} \, M_{\frac{K}{L} n_j - \lambda K k} \Big\}_{k,l \in \mathbb{Z}, j \in \mathcal{J}}$$

is a Riesz basis for its closed linear span in $HS(\mathbb{R})^N$. We conclude that E is bounded and stable.

To construct a stable analysis operator C, we choose the Gaussian $g_0 : \mathbb{R} \to \mathbb{R}^+, \quad x \mapsto e^{-\pi x^2}$, and note that Lyubarski [11] and Seip and Wallsten [16, 17] have shown that $\{M_{ka'} T_{lb'} g_0\}$ is a frame whenever $a'b' < 1$.[3] Since $\lambda^2 K \frac{\lambda^2 L}{KJ} = \lambda^4 \frac{L}{J} = \frac{\lambda^4}{\mu(S)} < 1$, this implies that the analysis map given by

$$C : L^2(\mathbb{R}) \to l^2(\mathbb{Z}^2), \quad f \mapsto \Big\{ \langle f, M_{\lambda^2 K \, k} T_{\frac{\lambda^2 L}{KJ} \, l} g_0 \rangle \Big\}_{k,l}$$

is bounded and stable.

For simplicity of notation, set $\alpha = K$ and $\beta = \frac{L}{KJ}$. Let us now consider the composition

$$\begin{array}{ccccccc} l_0(\mathbb{Z}^2) & \overset{E}{\to} & \mathcal{H}_{\boldsymbol{S}} & \overset{\Phi_{\boldsymbol{f}}}{\to} & L^2(\mathbb{R}) & \overset{C}{\to} & l^2(\mathbb{Z}^2) \\ \{\sigma_{k,l''}\} & \mapsto & E\{\sigma_{k,l''}\} & \mapsto & E\{\sigma_{k,l''}\} \, \boldsymbol{f} & \mapsto & \{ \langle E\{\sigma_{k,l''}\} \, \boldsymbol{f}, \; M_{\lambda^2 \alpha k'} T_{\lambda^2 \beta l'} \, g_0 \rangle \}_{k',l'} \; . \end{array}$$

We set $f_j = f_n$ whenever $j \in J_n$ and note that the bi–infinite matrix

$$M = \Big(m_{k',l',k,l''} \Big) = \Big(m_{k',l',\,k\,,lJ+j} \Big) = \Big(\langle \; M_{\lambda \alpha k} \, T_{\frac{m_j}{\alpha} + \lambda \beta l J} \, P \, T_{-\lambda \beta l J} \, M_{\frac{n_j}{\beta J} - \lambda \alpha k} \, f_j \; , \; M_{\lambda^2 \alpha k'} T_{\lambda^2 \beta l'} \, g_0 \, \rangle \Big) \, ,$$

[3] For background on frame theory see [2, 7].

$l'' = lJ + j$, represents the operator $C \circ \Phi_{\boldsymbol{f}} \circ E$ with respect to the canonical basis of $l^2(\mathbb{Z}^2)$, since

$$\begin{aligned}\Big(C \circ \Phi_{\boldsymbol{f}} \circ E\ \{\sigma_{k,lJ+j}\}\Big)_{k',l'} &= \langle \sum_{k,l}\sum_{j=0}^{J-1} \sigma_{k,lJ+j}\ M_{\lambda\alpha k}\, T_{\frac{m_j}{\alpha}+\lambda\beta lJ}\, P\, T_{-\lambda\beta lJ}\, M_{\frac{n_j}{\beta J}-\lambda\alpha k}\, f_j\ ,\ M_{\lambda^2\alpha k'} T_{\lambda^2\beta l'} g_0 \rangle \\ &= \sum_{k,l}\sum_{j=0}^{J-1} \langle\, M_{\lambda\alpha k}\, T_{\frac{m_j}{\alpha}+\lambda\beta lJ}\, P\, T_{-\lambda\beta lJ}\, M_{\frac{n_j}{\beta J}-\lambda\alpha k}\, f_j\ ,\ M_{\lambda^2\alpha k'} T_{\lambda^2\beta l'} g_0 \,\rangle \sigma_{k,lJ+j} \\ &= \sum_{k,l}\sum_{j=0}^{J-1} m_{k',l',k,lJ+j}\ \sigma_{k,lJ+j}\,.\end{aligned}$$

In order to use Lemma 5.3 to show that M, and, therefore, $C \circ \Phi_{\boldsymbol{f}} \circ E$ is not stable, we have to obtain bounds on the matrix entries of M. Lemma 5.2, part b, together with the rapidly decaying function

$$\widetilde{d}_1 = C_{\boldsymbol{f}} \sum_{j=0}^{J-1} T_{\frac{m_j}{\alpha}-\lambda\beta j} d_1,$$

will provide us with these bounds. In fact, for $k, l, k', l' \in \mathbb{Z}$, we have

$$\begin{aligned}|m_{k',l',k,l''}| &= |m_{k',l',k,lJ+j}| \\ &= \Big|\langle\, M_{\lambda\alpha k}\, T_{\frac{m_j}{\alpha}+\lambda\beta lJ}\, P\, T_{-\lambda\beta lJ}\, M_{\frac{n_j}{\beta J}-\lambda\alpha k}\, f_j\ ,\ M_{\lambda^2\alpha k'} T_{\lambda^2\beta l'} g_0 \,\rangle\Big| \\ &\le \langle\, T_{\lambda\beta(lJ+j)} \big(T_{\frac{m_j}{\alpha}-\lambda\beta j} \Big| P\, T_{-\lambda\beta lJ}\, M_{\frac{n_j}{\beta J}-\lambda\alpha k}\, f_j \Big|\,\big)\ ,\ T_{\lambda^2\beta l'} g_0 \,\rangle \\ &\le \widetilde{d}_1 * g_0\, (\lambda\beta(\lambda l' - l''))\, (1 + \|(\lambda\beta lJ, \frac{n_j}{\beta J} - \lambda\alpha k)\|_\infty)^{L_{\boldsymbol{f}}},\end{aligned}$$

and

$$\begin{aligned}|m_{k',l',k,l''}| &= |m_{k',l',k,lJ+j}| \\ &= \Big|\langle\, T_{\lambda\alpha k}\, M_{-\frac{m_j}{\alpha}-\lambda\beta lJ} \big(P\, T_{-\lambda\beta lJ}\, M_{\frac{n_j}{\beta J}-\lambda\alpha k}\, f_j\,\big)^{\widehat{}},\ T_{\lambda^2\alpha k'} M_{-\lambda^2\beta l'} g_0 \,\rangle\Big| \\ &\le \langle\, T_{\lambda\alpha k} \Big|\big(P\, T_{-\lambda\beta lJ}\, M_{\frac{n_j}{\beta J}-\lambda\alpha k}\, f_j\,\big)^{\widehat{}}\Big|\ ,\ T_{\lambda^2\alpha k'} g_0 \,\rangle \\ &\le d_2 * g_0(\lambda\alpha(\lambda k' - k))(1 + \|(\lambda\beta lJ, \frac{n_j}{\beta J} - \lambda\alpha k)\|_\infty)^{L_{\boldsymbol{f}}}.\end{aligned}$$

In these calculations, we used that $g_0 \ge 0$, $\widehat{g_0} = g_0$, and $g_0(-x) = g_0(x)$, and the Parseval–Plancherel identity. Since $\widetilde{d}_1$, d_2, and g_0 decay rapidly, the same holds for $\widetilde{d}_1 * g_0$ and $d_2 * g_0$. We set

$$w(x) = \max\big\{\widetilde{d}_1 * g_0(\lambda\beta x),\ \widetilde{d}_1 * g_0(-\lambda\beta x),\ d_2 * g_0(\lambda\alpha x),\ d_2 * g_0(-\lambda\alpha x)\big\},$$

and choose a polynomial p of degree $L_{\boldsymbol{f}}$ which satisfies

$$(1 + \|(\lambda\beta lJ, \frac{n_j}{\beta J} - \lambda\alpha k)\|)^{L_f} \le p(\|(k,l)\|_\infty), \quad j = 1, \ldots, J,$$

and obtain $|m_{k',l',k,l}| \le w\big(\max\{|\lambda k' - k|,\ |\lambda l' - l|\}\big)\, p(\|(k,l)\|_\infty)$ with $w = o\,(x^{-n})$ for $n \in \mathbb{N}$. Lemma 5.3 implies that M is not stable, and therefore $C \circ \Phi_{\boldsymbol{f}} \circ E$ and thus $\Phi_{\boldsymbol{f}}$ are not stable. $\square$

6. APPENDIX

Proof of Lemma 5.3

Without loss of generality, we may assume $p(x) = (1+x)^L$. First, we show that if $w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $w(x) = o\left(x^{-(L+2)}\right)$ is monotonically decreasing, then

$$K_1^{2L} \sum_{K \geq K_1} K \sum_{k \geq K} k\, w(k)^2 \to 0 \text{ as } K_1 \to \infty. \tag{7}$$

This limit is proven using the Riemann integral criterium for sums. To this end, we pick $v \in C_0(\mathbb{R}^+)$ with $w(x) \leq v(x)\, x^{-(L+2)}$ and observe that

$$\begin{aligned}
\sum_{K \geq K_1+2} K \sum_{k \geq K} k\, w(k)^2 &\leq \sum_{K \geq K_1+1} K \sum_{k \geq K+1} k\, w(k)^2 \\
&\leq \int_{K_1}^{\infty} x \int_x^{\infty} y\, w(y)^2\, dy\, dx \\
&\leq \int_{K_1}^{\infty} x \int_x^{\infty} y\, v(y)^2 y^{-2L-4}\, dy\, dx \\
&\leq \int_{K_1}^{\infty} x \int_x^{\infty} v(y)^2 y^{-2L-3}\, dy\, dx \\
&\leq \frac{\|v|_{[K_1,\infty)}\|_\infty^2}{2L+2} \int_{K_1}^{\infty} x\, x^{-2L-2}\, dx \\
&\leq \frac{\|v|_{[K_1,\infty)}\|_\infty^2}{2L+2} \int_{K_1}^{\infty} x^{-2L-1}\, dx \\
&\leq \frac{\|v|_{[K_1,\infty)}\|_\infty^2}{2L(2L+2)} K_1^{-2L} = o(K_1^{-2L}).
\end{aligned}$$

Since $\|v|_{[K_1,\infty)}\|_\infty \to 0$ as $K_1 \to \infty$, (7) follows.

Now, we shall use (7) to show that $\inf_{x \in l_0(\mathbb{Z}^2)}\{\frac{\|Mx\|_{l^2}}{\|x\|_{l^2}}\} = 0$. To this end, fix $\epsilon > 0$ and note that (7) provides us with a $K_1 > K_0$ satisfying

$$(K_1+3)^{2L} \sum_{K \geq K_1} K \left(\sum_{k \geq K} k\, w(k)^2\right) \leq 2^{-6} \left(\frac{\lambda - 1}{\lambda}\right)^{2L} \epsilon^2 .$$

Set $N = \left\lceil \frac{\lambda(K_1+1)}{\lambda-1} \right\rceil$ and $\widetilde{N} = \lceil \frac{N}{\lambda} \rceil + K_1$. Then $N \leq \frac{\lambda(K_1+2)}{\lambda-1}$, and $N \geq \frac{\lambda(K_1+1)}{\lambda-1}$ implies $\lambda N \geq \lambda K_1 + \lambda + N$ and

$$N \geq K_1 + \frac{N}{\lambda} + 1 > K_1 + \left\lceil \frac{N}{\lambda} \right\rceil = \widetilde{N}.$$

Therefore, $(2\widetilde{N}+1)^2 < (2N+1)^2$ and the matrix

$$\widetilde{M} = (m_{j',j})_{\|j'\|_\infty \leq \widetilde{N}, \|j\| \leq N} : \mathbb{C}^{(2N+1)^2} \to \mathbb{C}^{(2\widetilde{N}+1)^2}$$

has a nontrivial kernel. We can therefore choose $\widetilde{x} \in \mathbb{C}^{(2N+1)}$ with $\|\widetilde{x}\|_2 = 1$ and $\widetilde{M}\widetilde{x} = 0$. Define $x \in l_0(\mathbb{Z}^2)$ according to $x_j = \widetilde{x}_j$ if $\|j\|_\infty \leq N$ and $x_j = 0$ otherwise, so by construction we have $\|x\|_{l^2} = 1$, and $(Mx)_{j'} = 0$ for $\|j'\|_\infty \leq \widetilde{N}$.

To estimate $(Mx)_{j'}$ for $\|j'\|_\infty > \widetilde{N}$, we fix $K > K_1$ and one of the $2^3\big(\lceil \frac{N}{\lambda} \rceil + K\big)$ indices $j' \in \mathbb{Z}^d$ with $\|j'\|_\infty = \lceil \frac{N}{\lambda} \rceil + K$. We have $\|\lambda j'\|_\infty \geq N + K\lambda$ and $\|\lambda j' - j\|_\infty \geq K\lambda \geq K$ for all $j \in \mathbb{Z}^d$ with $\|j\|_\infty \leq N$. Therefore

$$\begin{aligned}
|(Mx)_{j'}|^2 &= \Big| \sum_{\|j\|_\infty \leq N} m_{j',j} x_j \Big|^2 \\
&\leq \|x\|_2^2 \sum_{\|j\|_\infty \leq N} |m_{j',j}|^2 \\
&\leq \sum_{\|j\|_\infty \leq N} w(\|\lambda j' - j\|_\infty)^2 (1 + \|j\|_\infty)^{2L} \\
&\leq (N{+}1)^{2L} \sum_{\|j\|_\infty \leq N} w(\|\lambda j' - j\|_\infty)^2 \\
&\leq (N{+}1)^{2L} \sum_{\|j\|_\infty \geq K} w(\|j\|_\infty)^2 \\
&= (N{+}1)^{2L} 2^3 \sum_{k \geq K} k\, w(k)^2.
\end{aligned}$$

Finally, we compute

$$\begin{aligned}
\|Mx\|_{l^2}^2 &= \sum_{j' \in \mathbb{Z}^d} |(Mx)_{j'}|^2 \\
&= \sum_{\|j'\|_\infty \geq \lceil \frac{N}{\lambda} \rceil + K_1} |(Mx)_{j'}|^2 \\
&= 2^3 \sum_{\|j'\|_\infty \geq \lceil \frac{N}{\lambda} \rceil + K_1} (N{+}1)^{2L} \sum_{k \geq \|j'\|_\infty} k\, w(k)^2 \\
&\leq 2^6 (N{+}1)^{2L} \sum_{K \geq \lceil \frac{N}{\lambda} \rceil + K_1} K \sum_{k \geq K} k\, w(k)^2 \\
&\leq 2^6 \left(\frac{\lambda(K_1 + 2)}{\lambda - 1} + 1 \right)^{2L} \sum_{K \geq \lceil \frac{N}{\lambda} \rceil + K_1} K \sum_{k \geq K} k\, w(k)^2 \\
&\leq 2^6 \left(\frac{\lambda}{\lambda - 1} \right)^{2L} (K_1 + 3)^{2L} \sum_{K \geq \lceil \frac{N}{\lambda} \rceil + K_1} K \sum_{k \geq K} k\, w(k)^2 \leq \epsilon^2
\end{aligned}$$

and obtain $\|Mx\|_{l^2} \leq \epsilon$. Since ϵ was chosen arbitrarily and $\|x\|_{l^2} = 1$, we have $\inf_{x \in l_0(\mathbb{Z}^2)} \{ \frac{\|Mx\|_{l^2}}{\|x\|_{l^2}} \} = 0$ and M is not stable. $\square$

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References

[1] P.A. Bello. Measurement of random time-variant linear channels. *IEEE Trans. Comm.*, 15:469–475, 1969.

[2] O. Christensen. *An introduction to frames and Riesz bases.* Applied and Numerical Harmonic Analysis. Birkhäuser Boston Inc., Boston, MA, 2003.

[3] J. Dieudonné. *Treatise on analysis. Vol. II.* Translated from the French by I. G. MacDonald. Pure and Applied Mathematics, Vol. 10-II. Academic Press, New York, 1970.

[4] S.A. Gaal. *Linear Analysis and Representation Theory.* Springer–Verlag, New York, 1973.

[5] A. Goldsmith. *Wireless Communications.* Cambridge University Press, Cambridge, 2005.

[6] N. Grip and G.E. Pfander. A discrete model for the efficient analysis of time–varying narrowband communications channels.

[7] K. Gröchenig. *Foundations of Time-Frequency Analysis.* Applied and Numerical Harmonic Analysis. Birkhäuser, Boston, MA, 2001.

[8] T. Kailath. Measurements on time–variant communication channels. *IEEE Trans. Inform. Theory*, 8(5):229–236, Sept. 1962.

[9] W. Kozek and G.E. Pfander. Identification of operators with bandlimited symbols. *SIAM J. Math. Anal.*, 37(3):867–888, 2006.

[10] J. Lawrence, G.E. Pfander, and D. Walnut. Linear independence of Gabor systems in finite dimensional vector spaces. *J. Fourier Anal. Appl.*, 11(6):715–726, 2005.

[11] Y.I. Lyubarskii. Frames in the Bargmann space of entire functions. *Adv. Soviet Math.*, 429:107–113, 1992.

[12] H.S. Malvar. *Wireless Communications over MIMO Channels.* John Wiley & Sons, Chichester, 2006.

[13] A. Paulraj, R. Nabar, and D. Gore. *Introduction to Space-Time Wireless Communications.* Cambridge University Press, Cambridge, 2003.

[14] G.E. Pfander and D. Walnut. Measurement of time–variant channels. *IEEE Trans. Info. Theory*, 52(11):4808–4820, 2006.

[15] G.E. Pfander and D. Walnut. Operator identifcation and Feichtinger's algebra. *Sampl. Theory Signal Image Process.*, 5(2):151–168, 2006.

[16] K. Seip. Density theorems for sampling and interpolation in the Bargmann-Fock space. I. *J. Reine Angew. Math.*, 429:91–106, 1992.

[17] K. Seip and R. Wallstén. Density theorems for sampling and interpolation in the Bargmann-Fock space. II. *J. Reine Angew. Math.*, 429:107–113, 1992.

[18] E. Telatare. Capacity of multi-antenna Gaussian channels. *Eur. trans. telecommun.*, 10:585–959, 1999.

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