Operator Identification and Sampling

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Abstract:

Time–invariant communication channels are usually modelled as convolution with a fixed impulse–response function. As the name suggests, such a channel is completely determined by its action on a unit impulse. Time–varying communication channels are modelled as pseudodifferential operators or superpositions of time and frequency shifts. The function or distribution weighting those time and frequency shifts is referred to as the spreading function of the operator. We consider the question of whether such operators are identifiable, that is, whether they are completely determined by their action on a single function or distribution. It turns out that the answer is dependent on the size of the support of the spreading function, and that when the operators are identifiable, the input can be chosen as a distribution supported on an appropriately chosen grid. These results provide a sampling theory for operators that can be thought of as a generalization of the classical sampling formula for bandlimited functions.

1. Channel Models and Identification

A communications channel is said to be *measurable* or *identifiable* if its characteristics can be determined by its action on a single fixed input signal. A general model for linear (time-varying) communication channels is as operators of the form

$$Hf(x) = \int h_H(t,x)\, f(x-t)\, dt.$$

The function $h_H(t,x)$ is referred to as the *impulse response* of the channel and is interpreted as the response of the channel at time x to a unit impulse at time $x-t$, that is, originating t time units earlier. If $h_H(t,x) = h_H(t)$ then the characteristics of the channel are time-invariant and in this case the channel is modelled as a convolution operator. Such channels are identifiable since $h_H(t)$ can be recovered as the response of the channel to the input signal $\delta_0(t)$, the unit-impulse at $t=0$.

There are two representations of H that will be convenient for our purposes.

1. Letting $\eta_H(t,\nu) = \int h_H(t,x)\, e^{-2\pi i\nu(x-t)}\, dx$ gives

$$\begin{aligned} Hf(x) &= \iint \eta_H(t,\nu)\, e^{2\pi i\nu(x-t)}\, f(x-t)\, d\nu\, dt \\ &= \iint \eta_H(t,\nu)\, T_t\, M_\nu f(x)\, d\nu\, dt. \end{aligned}$$

$\eta_H(t,\nu)$ is the *spreading function* of H. If $\operatorname{supp}\eta_H \subseteq [0,a]\times[-b/2,b/2]$ for some $a,\, b>0$ then a is called the *maximum time-delay* and b the *maximum Doppler spread* of the channel.

2. Letting $\sigma_H(x,\xi) = \int h_H(t,x)\, e^{2\pi i t\xi}\, dt$ gives

$$Hf(x) = \int \sigma_H(x,\xi)\widehat{f}(\xi)\, e^{2\pi i x\xi}\, d\xi.$$

$\sigma_H(x,\xi)$ is the *Kohn-Nirenberg* (KN) symbol of H and we have the relation

$$\eta_H(t,\nu) = \iint \sigma_H(x,\xi)\, e^{-2\pi i(\nu x-\xi t)}\, dx\, d\xi.$$

In other words, the spreading function η_H is the *symplectic Fourier transform* of the KN symbol of H.

In 1963, T. Kailath [3, 4, 5] asserted that for time-variant communication channels to be identifiable it is necessary and sufficient that the maximum time-delay, a, and Doppler shift, b, satisfy $ab\le 1$ and gave an argument for this assertion based on counting degrees of freedom. In the argument, Kailath looks at the response of the channel to a train of impulses separated by at least a time units, so that in this sense the channel is being "sampled" by a succession of evenly-spaced impulse responses. The condition $ab\le 1$ allows for the recovery of sufficiently many samples of $h_H(t,x)$ to determine it uniquely.

Kailath's conjecture was given a precise mathematical framework and proved in [6]. The framework is as follows. Choose normed linear spaces $D(\mathbf{R})$ and $Y(\mathbf{R})$ of functions or distributions on $\mathbf{R}$, and a normed linear space of bounded linear operators $\mathcal{H}\subset\mathcal{L}(D(\mathbf{R}),Y(\mathbf{R}))$. Each fixed element $g\in D(\mathbf{R})$ induces a map $\Phi_g:\mathcal{H}\longrightarrow Y(\mathbf{R}),\ H\mapsto Hg$. If for some $g\in D(\mathbf{R})$, Φ_g is bounded above and below, that is, there are constants $0<A\le B$ such that for all $H\in\mathcal{H}$,

$$A\|H\|_{\mathcal{H}} \le \|Hg\|_Y \le B\,\|H\|_{\mathcal{H}}$$

then we say that $\mathcal{H}$ is *identifiable with identifier* $g \in D(\mathbf{R})$.

Taking $D = S_0'$, $Y = L^2$, and $\mathcal{H}_S = \{H \in HS(L^2)\colon \eta_H \in S_0(\mathbf{R}\times\widehat{\mathbf{R}}),\ \operatorname{supp}\eta_H \subseteq S\}$ where $S \subseteq \mathbf{R}\times\widehat{\mathbf{R}}$, $HS(L^2)$ is the class of Hilbert-Schmidt operators, and S_0 is the Feichtinger algebra (defined below), the following was proved in [6].

Theorem 1. If $S = [0,a]\times[-b/2,b/2]$ then $\mathcal{H}_S$ is identifiable if and only if $ab \le 1$. In this case an identifier is given by $g = \sum_n \delta_{na}$.

2. Distributional Spreading Functions and Operator Sampling

The requirement that $\eta_H \in S_0$ excludes some very natural operators from consideration in this formalism, for example the identity operator ($\eta_H(t,\nu) = \delta_0(t)\delta_0(\nu)$), convolution operators ($\eta_H(t,\nu) = h(t)\delta_0(\nu)$ giving $Hf = f * h$), and multiplication operators, ($\eta_H(t,\nu) = \delta_0(t)\widehat{m}(\nu)$ giving $Hf = m\cdot f$).

A more natural setting for operator identification is the *modulation spaces* (see [2] for a full treatment of the subject). For convenience we give the definitions below for modulation spaces on $\mathbf{R}$, but all definitions and results can be extended to $\mathbf{R}^d$. For $\varphi \in \mathcal{S}(\mathbf{R})$ define for $f \in \mathcal{S}'(\mathbf{R})$ the *short-time Fourier transform (STFT)* of f by

$$\begin{aligned} V_\varphi f(t,\nu) &= \langle f, T_t M_\nu \varphi\rangle \\ &= \int f(x)\, e^{-2\pi i\nu(x-t)}\, \overline{\varphi(x-t)}\, dx. \end{aligned}$$

For $1 \le p,\ q \le \infty$ define the modulation space $M^{p,q}(\mathbf{R})$ by

$$M^{p,q}(\mathbf{R}) = \{f \in \mathcal{S}'(\mathbf{R})\colon V_\varphi f \in L^{p,q}(\mathbf{R})\},$$

that is, for which

$$\|V_\varphi\|_{L^{p,q}} = \left(\int\left(\int |V_\varphi f(t,\nu)|^p\, dt\right)^{q/p}\right)^{1/q}$$

is finite. The usual modifications are made if p or $q = \infty$. $M^{p,q}$ is a Banach space with respect to the norm $\|f\|_{M^{p,q}} = \|V_\varphi f\|_{L^{p,q}}$ and different nonzero choices of $\varphi \in \mathcal{S}$ define equivalent norms. The space $M^{1,1}$ is the Feichtinger algebra denoted S_0 and $M^{\infty,\infty}$ is its dual S_0'. The space S_0' contains the Dirac impulses $\delta_x : f \mapsto f(x)$ for $x \in \mathbf{R}$ as well as distributions of the form $g = \sum_j c_j \delta_{x_j}$, $x_j \in \mathbf{R}$ and $\{c_j\} \subseteq \mathbf{C}$ a bounded sequence.

In our next step toward operator sampling we observe that it is possible to take $D = S_0'$, $Y = S_0'$, and $\mathcal{H}_S = \{H \in \mathcal{L}(D,Y)\colon \eta_H \in S_0',\ \operatorname{supp}\eta_H \subseteq S\}$ in the operator identification formalism. Indeed the following theorem was shown in [10].

Theorem 2. The operator class $\mathcal{H}_S$ (defined above) is identifiable if $S = [0,a]\times[-b/2,b/2]$ and $ab < 1$, and is not identifiable if $ab > 1$.

3. A Theory of Operator Sampling

In discussing identifiability of operators in various settings, we have been content to show that an operator is completely determined by its actions on a fixed input in terms of a norm inequality. The next step is to find an explicit reconstruction formula for the impulse response of the channel operator directly from its response to the identifier. Such formulas illustrate a connection between operator identification and classical sampling theory and lead to a definition of *operator sampling*.

If, in the operator identification formalism described earlier, an operator class $\mathcal{H}$ is identified by a distribution of the form $g = \sum_j c_j \delta_{x_j}$, then we call $\{x_j\}$ a *set of sampling* for $\mathcal{H}$ and g a *sampling function* for the operator class $\mathcal{H}$. In the results obtained so far, operator sampling is possible only for operators with compactly supported spreading function, and in order to interpret Theorem 1 in this context we make the following definition.

Given a Jordan domain $S \subseteq \mathbf{R}^2$, define the *operator Paley-Wiener space* $OPW^2(S)$ by

$$OPW^2(S) = \{H \in HS(L^2)\colon \operatorname{supp}\eta_H \subseteq S\}.$$

OPW^2 is a Banach space with respect to the Hilbert-Schmidt norm $\|H\|_{OPW^2} = \|\eta_H\|_{L^2}$. Then Theorem 1 can be extended as follows ([8]).

Theorem 3. Let $\Omega,\ T,\ T' > 0$ with $T' < T$ and $\Omega T < 1$. Then $OPW^2([0,T']\times[-\Omega/2,\Omega/2])$ is identifiable with identifier $g = \sum_n \delta_{nT}$ and moreover we have the formula

$$h_H(t,x) = r(t)\sum_{k\in\mathbf{Z}} (Hg)(t+kT)\varphi(x-t-kT)$$

unconditionally in $L^2(\mathbf{R}^2)$, where $r \in \mathcal{S}(\mathbf{R})$ is such that $r = 1$ on $[0,T']$ and vanishes outside a sufficiently small neighborhood of $[0,T']$, and where $\varphi \in \mathcal{S}(\mathbf{R})$ is such that $\widehat{\varphi} = 1$ on $[-\Omega/2,\Omega/2]$ and vanishes outside a sufficiently small neighborhood of $[-\Omega/2,\Omega/2]$.

In the more general modulation space setting we can define the operator Paley-Wiener space $OPW^{p,q}(S)$ by

$$\begin{aligned} OPW^{p,q}(S) &= \{H \in \mathcal{L}(S_0,S_0') \\ &\quad\colon \operatorname{supp}\eta_H \subseteq S,\ \ \sigma_H \in M^{pq,11}\} \end{aligned}$$

where $\sigma_H(x,\xi) \in M^{pq,11}$ means that the two-dimensional STFT of σ_H satisfies

$$\int\left(\int\left(\int |V_{\varphi\otimes\varphi}\sigma_H(t_1,t_2,\nu_1,\nu_2)|^p dt_1\right)^{q/p} dt_2\right)^{1/p} d\nu_1 d\nu_2$$

is finite. Here

$$V_{\varphi\otimes\varphi}(t_1,t_2,\nu_1,\nu_2) = \langle f, T_{t_1}M_{\nu_1}\varphi\otimes T_{t_2}M_{\nu_2}\varphi\rangle.$$

$OPW^{p,q}$ is a Banach space with respect to the norm $\|H\|_{OPW^{p,q}} = \|\sigma_H\|_{M^{pq,11}}$. In this case, Theorem 3 generalizes as follows ([8]).

Theorem 4. Let $1 \le p,\ q \le \infty$, $\Omega,\ T,\ T' > 0$ with $T' < T$ and $\Omega T < 1$. Then $OPW^{p,q}([0,T']\times[-\Omega/2,\Omega/2])$ is identifiable with identifier $g = \sum_n \delta_{nT}$ and moreover we have the formula

$$h_H(t,x) = r(t)\sum_{k\in\mathbf{Z}} (Hg)(t+kT)\varphi(x-t-kT)$$

unconditionally in $M^{1p,q1}(\mathbf{R}^2)$ and in the weak-* sense if p or $q = \infty$, where r and φ are as in Theorem 3.

Example 1. If we take H to be ordinary convolution by $h_H(t)$, this means that $h_H(t,x)$ depends only on t, that is, $h_H(t,x) = h_H(t)$. In this case H can be identified in principle by $g = \delta_0$, the unit impulse, since $Hg(x) = h_H(x)$. Translating this into our operator sampling formalism results in something slightly different.

Assume that $h \in M^{1,q}$ is supported in the interval $[0,T']$ and that $T > T'$, and $\Omega > 0$ are chosen so that $\Omega T < 1$. In this case, $\eta_H(t,\nu) = h(t)\,\delta_0(\nu)$ and $\sigma_H(x,\xi) = \widehat{h}(\xi)$. Therefore $\sigma_H \in M^{\infty q,11}$ and $H \in OPW^{\infty,q}([0,T'] \times \{0\})$.

If $g = \sum_n \delta_{nT}$ then Hg is simply the T–periodized impulse response $h(t)$, and it follows that

$$\begin{aligned} & r(t) \sum_{k\in\mathbf{Z}} (Hg)(t+kT)\varphi(x-t-kT) \\ = \;& r(t)\, h(t) \sum_{k\in\mathbf{Z}} \varphi(x-t-kT) \;=\; h(t) \end{aligned}$$

since $r(t) = 1$ on $[0,T']$ and vanishes outside a neighborhood of $[0,T']$ and since $\sum_k \varphi(x-t-kT) = 1$ by the Poisson Summation Formula and in consideration of the support constraints on $\widehat{\varphi}$. Indeed the theorem says that the sum $\sum_k \varphi(x-t-kT)$ converges to 1 in the $M^{\infty,1}$ norm and in particular uniformly on compact sets.

Example 2. If we take H to be multiplication by some fixed function $m \in M^{p,1}$ with $\operatorname{supp} \widehat{m} \subseteq [-\Omega/2,\Omega/2]$ then $\eta_H(t,\nu) = \delta_0(t)\widehat{m}(\nu)$, $h(t,x) = \delta_0(t)\, m(x-t)$, and $\sigma_H(x,\xi) = m(x)$. Therefore $\sigma_H \in M^{p\infty,11}$ and $H \in OPW^{p,\infty}(\{0\} \times [-\Omega/2,\Omega/2])$.

If $g = \sum_n \delta_{nT}$, with $T > 0$ chosen small enough that $\Omega T < 1$, then $Hg = \sum_n m(nT)\,\delta_{nT}$, and it follows from Theorem 4 that

$$\begin{aligned} & \delta_0(t)\, m(x-t) \\ = \;& r(t) \sum_{k\in\mathbf{Z}} (Hg)(t+kT)\varphi(x-t-kT) \\ = \;& r(t) \sum_{k\in\mathbf{Z}} \sum_{n\in\mathbf{Z}} m(nT)\, \delta_{(n-k)T}(t)\varphi(x-t-kT) \\ = \;& \sum_{n\in\mathbf{Z}} m(nT)\, \varphi(x-nT) \end{aligned}$$

by support considerations on the function $r(t)$. Therefore we have the summation formula

$$m(x) = \sum_n m(nT)\, \varphi(x-nT)$$

where the sum converges unconditionally in $M^{p,1}$ if $1 \le p < \infty$ and weak-* if $p = \infty$, and moreover there are constants $0 < A \le B$ such that for all such f,

$$A\|f\|_{M^{p,1}} \le \|\{f(nT)\}\|_{\ell^p} \le B\|f\|_{M^{p,1}}.$$

Taking $p = 2$, this recovers the classical sampling formula when the sampling is above the Nyquist rate.

4. Spreading functions with nonrectangular support and Bello's conjecture

In 1969, P. A. Bello [1] argued that what is important for channel identification is not the product ab of the maximum time-delay and Doppler shift of the channel but the area of the support of the spreading function. It is notable that Kailath also asserted something along these lines. This means that a time-variant channel whose spreading function has essentially arbitrary support is identifiable as long as the area of that support is smaller than one.

Using ideas from [6], Bello's conjecture was proved in [9].

Theorem 5. $\mathcal{H}_S$ is identifiable if $\mathrm{vol}^+(S) < 1$, and not identifiable if $\mathrm{vol}^-(S) > 1$. Here $\mathrm{vol}^+(S)$ is the outer Jordan content and $\mathrm{vol}^-(S)$ the inner Jordan content of S. In this case, the channel is identified by $g = \sum_n c_n\, \delta_{n/L}$ where $L \in \mathbf{N}$ and the L–periodic sequence $\{c_n\}$ is chosen based on the geometry of S.

We next present a generalization of Theorem 4 to this case. Before stating the result, a few preliminaries are required.

Definition 1. Given $L \in \mathbf{N}$, let $\omega = e^{-2\pi i/L}$ and define the *translation operator* T on $(x_0,\,\ldots,\,x_{L-1}) \in \mathbf{C}^L$ by

$$Tx = (x_{L-1}, x_0,\, x_1,\, \ldots, x_{L-2}),$$

and the *modulation operator* M on $\mathbf{C}^L$ by

$$Mx = (\omega^0 x_0, \omega^1 x_1,\, \ldots,\, \omega^{L-1}x_{L-1}).$$

Given a vector $c \in \mathbf{C}^L$ the *finite Gabor system with window* c is the collection $\{T^q M^p c\}_{q,p=0}^{L-1}$.

Note that the discrete Gabor system defined above consists of L^2 vectors in $\mathbf{C}^L$ so is necessarily overcomplete.

Definition/Proposition 2. The Zak Transform is defined for $f \in \mathcal{S}(\mathbf{R})$ by $Zf(t,\nu) = \sum_n f(t-n)\, e^{2\pi i n\nu}$. $Zf(t,\nu)$ satisfies the quasi-periodicity relations $Zf(t+1,\nu) = e^{2\pi i\nu}\, Zf(t,\nu)$ and $Zf(t,\nu+1) = Zf(t,\nu)$. Z can be extended to a unitary operator from $L^2(\mathbf{R})$ onto $L^2([0,1]^2)$.

If the spreading function of H, $\eta_H(t,\nu)$, is supported in a bounded Jordan region $S \subseteq \mathbf{R} \times \widehat{\mathbf{R}}$ with $\mathrm{vol}^+(S) < 1$, then by appropriately shifting and scaling η_H we can assume without loss of generality that for some $L \in \mathbf{N}$, $S \subseteq [0,1] \times [0,L]$ and that S meets at most L of the L^2 rectangles $R_{q,m} = ([0,1/L] \times [0,1]) + (q/L, m)$, $0 \le q,\, m < L$ whose union is $[0,1] \times [0,L]$. We can further assume that S does not meet any of the rectangles $R_{q,m}$ on the "edge" of the larger rectangle, specifically it does not meet $R_{q,m}$ with $q = 0$, $m = 0$, $q = L-1$ or $m = L-1$.

The following Lemma connects the output $Hg(x)$ where $g = \sum_n c_n\, \delta_{n/L}$ to the spreading function $\eta_H(t,\nu)$. From this a reconstruction formula analogous to that in Theorem 4 can be derived.

Lemma 1. Given a period-L sequence (c_n) and $g = \sum_n c_n\, \delta_{n/L}$, then for (t,ν) in a sufficiently small neighborhood of $[0,1/L] \times [0,1]$,

$$\begin{aligned} & e^{-2\pi i\nu p/L}\,(Z \circ H)g(t+p/L,\nu) \\ = \;& \sum_{q=0}^{L-1} \sum_{m=0}^{L-1} (T^q\, M^m c)_p\, e^{-2\pi i\nu q/L}\, \eta_H(t+q/L,\nu+m). \end{aligned}$$

In other words, the spreading function can be realized as coefficients on the vectors of a finite Gabor system. The system is in general underdetermined since there are L

equations and L^2 unknowns. If, however, the support set S of the spreading function $\eta_H(t,\nu)$ satisfies $\mathrm{vol}^+(S) < 1$ and since S meets at most L of the rectangles $R_{q,m}$, there are at most L nonzero unknowns in the above linear system. If the resulting $L \times L$ matrix is invertible, then η_H can be determined uniquely from Hg. The vector c must be chosen so that this matrix is invertible. It is shown in [7] that if L is prime then such a c always exists.
We can prove the following theorem (cf. [8], [9]).

Theorem 6. Let $1 \le p,\, q \le \infty$. If $\mathrm{vol}^-(S) > 1$ then $OPW^{p,q}(S)$ is not identifiable. If $\mathrm{vol}^+(S) < 1$ then $OPW^{p,q}(S)$ is identifiable via operator sampling, and the identifier is of the form $g = \sum_n c_n \delta_{n/L}$ where $L \in \mathbf{N}$ and (c_n) is an appropriately chosen period-L sequence. Moreover, we have the formula

$$\begin{aligned} h_H(t,x) &= \sum_{j=0}^{L-1} r_j(t) \sum_{k\in\mathbf{Z}} b_{j,k}\,(Hg)(t-q_j/L+k/L) \\ &\qquad \times \varphi_j(x-t-q_j/L-k/L) \end{aligned}$$

unconditionally in $M^{1p,q1}$ and in the weak-* sense if $p = \infty$ or $q = \infty$. For $0 \le j < L$, the rectangles R_{q_j,m_j} are precisely those that meet S. Also for each $0 \le j < L$, $r_j(t)\widehat{\varphi}_j(\nu) = 1$ on R_{q_j,m_j} and vanishes outside a small neighborhood of R_{q_j,m_j}, and $b_{j,k}$ is a period-L sequence in k based on the inverse of the matrix derived from the discrete Gabor system that appears in Lemma 1.

5. Conclusion

This paper contains a brief overview of some recent results on the measurement and identification of communication channels and the relation of these results to sampling theory. These connections provide explicit reconstruction formulas for identification of operators modelling time-variant linear channels.

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