Support size restrictions on time-frequency representations of functions on finite abelian groups

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We obtain uncertainty principles for finite abelian groups that relate the cardinality of the support of a function to the cardinality of the support of its short-time Fourier transform. These uncertainty principles are based on well-established uncertainty principles for the Fourier transform. In terms of applications, the uncertainty principle for the short-time Fourier transform implies the existence of a class of equal norm tight Gabor frames that are maximally robust to erasures.

1 Uncertainty principles

Let G be a finite abelian group with dual group $\widehat{G}$ consisting of the group homomorphisms $\xi : G \mapsto S^1$. The space of functions $\{f : G \longrightarrow \mathbb{C}\}$ will be denoted by $\mathbb{C}^G$, and the support size of a function is $\|f\|_0 := |\{x : f(x) \neq 0\}|$. The Fourier transform is defined as $\widehat{f}(\xi) = \sum_{x\in G} f(x)\,\overline{\xi(x)}$ for $f \in \mathbb{C}^G, \xi \in \widehat{G}$. The Euclidean norm on $\mathbb{C}^G$ will be denoted by $\|\cdot\|_2$.

A well-known result [2] states that $\|f\|_0 \cdot \|\widehat{f}\|_0 \geq |G|$ for $f \in \mathbb{C}^G\backslash\{0\}$. This inequality can be improved for groups of prime order, namely for $G = \mathbb{Z}_p$ with p prime, $\|f\|_0 + \|\widehat{f}\|_0 \geq |G| + 1$ holds for all $f \in \mathbb{C}^G\backslash\{0\}$ [3, 8]. We illustrate the possible pairs $(\|f\|_0, \|\hat{f}\|_0)$ for different groups in Figure 1.

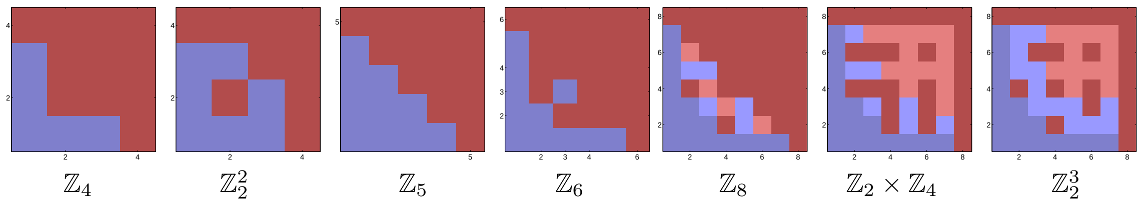

Fig. 1: Achieved combinations $(\|f\|_0, \|\widehat{f}\|_0)$ are represented by a red square, nonexistent ones by a blue square. Numerically verified combinations are represented in a lighter shade.

Let $g \in \mathbb{C}^G\backslash\{0\}$ be a window function. The short-time Fourier transform with respect to g is given by

$$V_g f(x,\xi) = \sum_{y\in G} f(y)\overline{g(y-x)\xi(y)}, \quad f \in \mathbb{C}^G, (x,\xi) \in G \times \widehat{G}.$$

The linear mapping $V_g : \mathbb{C}^G \longrightarrow \mathbb{C}^{G\times\widehat{G}}$ has a matrix representation that will be denoted by $A_{G,g}$. For groups G of prime order the fact that for a generic g, all minors of $A_{G,g}$ are non-zero allows us to establish the fact that the cardinality of the support of the short-time Fourier transform must be larger than $|G|^2 - |G| + 1$ [5, 6].

Theorem 1.1 *Let $G = \mathbb{Z}_p$, p prime. For almost every $g \in \mathbb{C}^G$, $\|f\|_0 + \|V_g f\|_0 \geq |G|^2 + 1$ for all $f \in \mathbb{C}^G \setminus \{0\}$. Moreover, for $1 \leq k \leq |G|$ and $1 \leq l \leq |G|^2$ with $k + l \geq |G|^2 + 1$ there exists f with $\|f\|_0 = k$ and $\|V_g f\|_0 = l$.*

The result stated in Theorem 1.1 can be improved further, namely we can choose a unimodular window function $g \in \mathbb{C}^{\mathbb{Z}_p}$, that is, a vector g all of whose entries have absolute value 1 [5].

Similar to [7], in order to establish lower bounds on $\|V_g f\|_0$ for a general group G, we define for $0 < k \leq |G|$,

$$\phi(G,k) := \max_{g\in\mathbb{C}^G\backslash\{0\}} \min\big\{\|V_g f\|_0 :\ f \in \mathbb{C}^G \text{ and } 0 < \|f\|_0 \leq k\big\}.$$

Proposition 1.2 *For $k \leq |G|$, let d_1 be the largest divisor of $|G|$ which is less than or equal to k and let d_2 be the smallest divisor of $|G|$ which is larger than or equal to k. Then*

$$\phi(G,k) \geq \frac{|G|^2}{d_1 d_2}(d_1 + d_2 - k).$$

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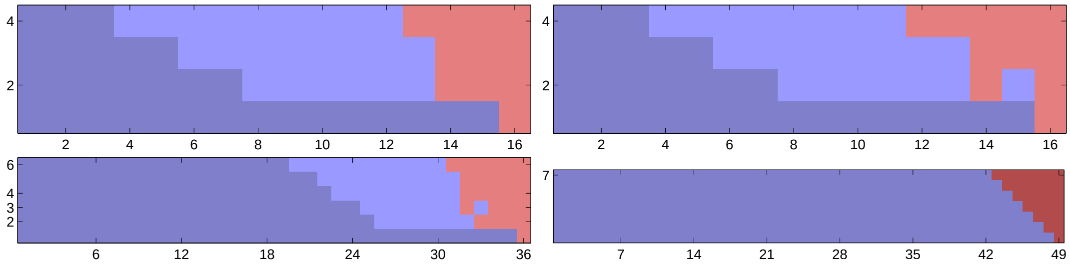

Fig. 2: Possible pairs $(\|f\|_0, \|V_g f\|_0)$ for the groups $\mathbb{Z}_4, \mathbb{Z}_2^2, \mathbb{Z}_6, \mathbb{Z}_7$ for a generic g. Achieved combinations $(\|f\|_0, \|V_g f\|_0)$ are represented by a red square, nonexistent ones by a blue square. Numerically verified combinations are represented in a lighter shade.

For $G = \mathbb{Z}_{pq}$ the bound can be improved, namely

$$\phi(G,k) \geq \begin{cases} p^2(q^2 - k + 1) & \text{if } k < q; \\ (p^2 - \frac{k}{q} + 1)(q^2 - q + 1) & \text{else.} \end{cases}$$

We illustrate the possible pairs $(\|f\|_0, \|V_g f\|_0)$ for a generic g for different groups in Figure 1. We note that for the cyclic groups $\mathbb{Z}_4$ and $\mathbb{Z}_6$ and for generic g, $\|V_g f\|_0 \geq |G|^2 - |G| + 1$ for all $f \in \mathbb{C}^G \setminus \{0\}$. While such a statement turns out to be false in the case of arbitrary abelian groups (for instance, $\mathbb{Z}_2^2$), we believe that for cyclic groups the inequality remains valid, namely that for G cyclic, $\big\{(\|f\|_0, \|V_g f\|_0),\ f \in \mathbb{C}^G \backslash \{0\}\big\} = \big\{(\|f\|_0, \|\widehat{f}\|_0 + |G|^2 - |G|),\ f \in \mathbb{C}^G \backslash \{0\}\big\}$. We note that checking this equality for $\mathbb{Z}_8$ turned out to be too expensive numerically.

2 Applications: Gabor frames and erasure channels

In generic communication systems, information (a vector $f \in \mathbb{C}^G$) is not sent directly, but must be coded in such a way that allows recovery of f at the receiver regardless of errors and disturbances introduced by the channel. We can choose a frame $\{\varphi_k\}_{k\in K}$ for $\mathbb{C}^G$ and send the coded coefficients $\{\langle f, \varphi_k\rangle\}_{k\in K}$ (see for example [1, 4] for definition and properties of frames in finite-dimensional vector spaces and Gabor frames). If none of the transmitted coefficients are lost, a dual frame $\{\widetilde{\varphi}_k\}$ of $\{\varphi_k\}$ can be used by the receiver to recover f via the inversion formula $f = \sum_k \langle f, \varphi_k\rangle \widetilde{\varphi}_k$.

In the case of an erasure channel, some coefficients are lost during the transmission. Suppose that only the coefficients $\{\langle f, \varphi_k\rangle\}_{k\in K'}$, $K' \subset K$ are received. The original vector f can still be recovered if and only if the subset $\{\varphi_k\}_{k\in K'}$ remains a frame for $\mathbb{C}^G$. Of course this requires $|K'| \geq |G| = \dim \mathbb{C}^G$.

Definition 2.1 A frame $\mathcal{F} = \{\varphi_k\}_{k\in K}$ in $\mathbb{C}^G$ is *maximally robust to erasures* if the removal of any $l \leq |K| - |G|$ vectors from $\mathcal{F}$ leaves a frame.

For any $g \in \mathbb{C}^G \backslash \{0\}$, the columns of the matrix $A_{G,g}$ form an equal norm tight Gabor frame for $\mathbb{C}^G$ [5].

Theorem 2.2 *For $g \in \mathbb{C}^G \backslash \{0\}$, the following are equivalent:*

1. *For all $f \in \mathbb{C}^G \setminus \{0\}$, $\|V_g f\|_0 \geq |G|^2 - |G| + 1$.*
2. *The Gabor system, consisting of the columns of the matrix $A_{G,g}$, is an equal norm tight frame which is maximally robust to erasures.*

For $|G|$ prime, Theorem 1.1 guarantees the validity of Statement 1 for a generic g and in particular, for some unimodular g. As Figure 1 shows, Statement 1 is true for $G = \mathbb{Z}_4, \mathbb{Z}_6$. It remains yet an open question to verify it for general cyclic groups and show the existence of such frames in the general case.

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