An Amalgam Balian-Low theorem for symplectic lattices of rational density

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***Abstract*—A *Gabor space* is a space generated by a discrete set of time-frequency shifted copies of a single window function. Starting from the question of whether a Gabor space contains additional time-frequency shifts of the window function we establish a new Balian-Low type result. This result extends (for example) the well established Amalgam Balian-Low Theorem in the one dimensional case. The Gabor spaces considered in this note are generated by symplectic lattices of rational density.** [1]

I. Introduction

If the translation operator $T_u : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ is given by $T_u f(x) = f(x-u)$ and the modulation operator $M_\eta : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ is given by $M_\eta f(x) = e^{2\pi i \eta \cdot x} f(x)$, the Balian-Low Theorem establishes the fact that an orthonormal basis of Gabor type $\{T_k M_\ell \varphi\}_{k,\ell\in\mathbb{Z}^d}$ of $L^2(\mathbb{R}^d)$ consists of functions that are either poorly localized in time or in frequency (or in both). In this way, it The Balian-Low Theorem formulates a central shortcoming of time-frequency structured bases of $L^2(\mathbb{R}^d)$. The easiest example of such an orthonormal basis of Gabor type is given by the $L^2(\mathbb{R})$ function $\varphi(x) = \chi(x) = 1$ for $x \in [-1/2, 1/2]$ and 0 else. While $\chi(x)$ is compactly supported and therefore ideally localized in time, its Fourier transform

$$\widehat{\chi}(\xi) = \int \chi(x)\, e^{2\pi i x \xi}\, dx = \int_{-1/2}^{1/2} e^{2\pi i x\xi}\, dx = \frac{\sin(\pi\xi)}{\pi\xi}$$

decays poorly, for example, we have

$$\int \xi^2 |\widehat{\chi}(\xi)|^2\, d\omega = \int |\sin(\xi)|^2\, d\omega = \infty.$$

To formulate a fairly general version of the Balian-Low Theorem, we denote by Λ discrete subgroups of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, define the time-frequency shift operator $\pi(\lambda) = \pi(u,\eta)$ on $L^2(\mathbb{R}^d)$ by

$$(\pi(u,\eta)\varphi)(x) = (M_\eta T_u \varphi)(x) = e^{2\pi i x\cdot\eta}\varphi(x-u),$$

and denote Gabor systems generated by an L^2 function φ and a lattice Λ by $(\varphi, \Lambda) = \{\pi(\lambda)\varphi\}_{\lambda\in\Lambda}$.

Recall that a Riesz basis for $L^2(\mathbb{R}^d)$ is a system of functions $\{\varphi_j\}_{j\in J}$ that spans $L^2(\mathbb{R}^d)$ and that satisfies

$$A\|\{c_j\}\|^2_{\ell^2(J)} \le \Big\| \sum_{j\in J} c_j\varphi_j \Big\|_{L^2(\mathbb{R}^d)} \le B\|\{c_j\}\|^2_{\ell^2(J)} \qquad (1)$$

for some $0 < A \le B < \infty$. If $\{\varphi_j\}_{j\in J}$ satisfies (1) but does not span $L^2(\mathbb{R}^d)$, then we refer to $\{\varphi_j\}_{j\in J}$ as Riesz sequence, or as Riesz basis for its closed linear span. Clearly, an orthonormal basis is a Riesz basis with $A = B = 1$ and every orthonormal set forms a Riesz sequence.

Theorem 1 (Balian-Low). *If $(\varphi, \alpha\mathbb{Z}^d \times \frac{1}{\alpha}\mathbb{Z}^d)$ is a Riesz basis for $L^2(\mathbb{R}^d)$, then the uncertainty product is not finite, that is, for all $a, b \in \mathbb{R}^d$, we have*

$$\Big(\int \|x-a\|^2|\varphi(x)|^2 dx\Big)\Big(\int \|\omega-b\|^2|\widehat{\varphi}(\omega)|^2 d\omega\Big) = \infty. \quad (2)$$

Balian [Bal81] and Low [Low85] independently derived this result for $d = 1$ and for $(\varphi, \alpha\mathbb{Z} \times \frac{1}{\alpha}\mathbb{Z})$ being an orthonormal basis, but both of their proofs contained a gap, which was later filled by Coifman et. al [Dau90]. In that paper, the result was extended to Riesz bases. For general references on the Balian-Low Theorem as well as remarks on its history we refer the reader to [BHW95], [Hei07], [HP06].

Theorem 1 is a simple generalization of $d = 1$ to the multivariate setting. Consideration of a more general lattice Λ in place of $\alpha\mathbb{Z}^d \times \frac{1}{\alpha}\mathbb{Z}^d$ complicates things, and only partial answers are known to date. In [GHHK02], [BCM03], the higher dimensional Balian-Low Theorem was generalized to so-called symplectic lattices as considered herein and discussed in Section III. In addition, [GHHK02] supplies a weak Balian-Low theorem for a generic lattice Λ in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

A popular alternative to express the missing (joint) time and frequency localization of a Gabor Riesz basis is used in the Amalgam Balian-Low Theorem as proven by Benedetto et al [BHW95] in case $d = 1$ and in general by Ascensi et al [AFK14].

[1] In this paper, we describe results from a companion paper, preprint [CMP15], and extend these to the multivariate setting. Our results are compared to other Balian-Low type results in the multivariate setting.

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Let us recall the definition of the *Feichtinger algebra* $S_0(\mathbb{R}^d)$ as the set of functions with integrable short time Fourier transform Vf, where Vf is defined pointwise as

$$Vf(t,\nu) = \int f(x)\; e^{-\|x-t\|_2^2}\; e^{2\pi i x\cdot\nu}\,dx.$$

Note that the membership criterion of φ being in $S_0(\mathbb{R}^d)$, indicates good decay in both, time and frequency.

The *Amalgam Balian-Low Theorem* for Riesz bases in $L^2(\mathbb{R}^d)$ reads as follows.

Theorem 2 (Amalgam Balian-Low)**.** *If Λ is a subgroup of $\mathbb{R}^{2d}$ with $(\varphi,\Lambda)=\{\pi(\lambda)\varphi\}_{\lambda\in\Lambda}$ is a Riesz basis for $L^2(\mathbb{R}^d)$, then φ is not included in the Feichtinger algebra $S_0(\mathbb{R}^d)$.*

In this paper, we establish an Amalgam Balian-Low result for subspaces of $L^2(\mathbb{R}^d)$. Note that, for example, a Gaussian $g(x) = e^{-\|x\|_2^2} \in S_0(\mathbb{R}^d)$ has finite, in fact, minimal, uncertainty product (2), and for many sets Λ with density less than one, for example, $\Lambda = (1+\epsilon)\mathbb{Z}^{2d}$, $\epsilon > 0$, we have that $(g,\Lambda) = \{\pi(\lambda)g\}_{\lambda\in\Lambda}$ is a Riesz basis for its closed linear span $\mathcal{G}(g,\Lambda) = \overline{\operatorname{span}(g,\Lambda)}$ [Hei07], [PR13]. So subspaces of $L^2(\mathbb{R}^d)$ may very well have Riesz bases which are well localized in time and in frequency.

In order to still capture the Balian-Low phenomenon in the case of subspaces, we ask the question whether a so-called Gabor space $\mathcal{G}(g,\Lambda)$ is closed under time-frequency shifts. That is, we ask whether for $\mu \in \mathbb{R}^d\times\widehat{\mathbb{R}}^d\setminus\Lambda$ we may have $\pi(\mu)\varphi\in\mathcal{G}(\varphi,\Lambda)$? Using the fact that $\pi(\mu)\varphi\in\mathcal{G}(\varphi,\Lambda)$ would imply $\pi(\widetilde{\mu})\varphi \in \mathcal{G}(\varphi,\Lambda)$ for any $\widetilde{\mu}\in\widetilde{\Lambda}$ with $\widetilde{\Lambda}$ being the subgroup of $\mathbb{R}^d\times\widehat{\mathbb{R}}^d$ generated by μ and Λ, we ask whether it is possible that $\mathcal{G}(\varphi,\Lambda)=\mathcal{G}(\varphi,\widetilde{\Lambda})$ with Λ being a proper subgroup of Λ'.

Note that this question is discussed in the literature in case of shift invariant spaces at length, see for example [ACH+10], [ASW11], [AKTW12], [TW14]. Consideration of shift invariant spaces is a special case of our setup, namely, it corresponds to $\mu,\Lambda\subseteq\mathbb{R}^d\times\{0\}$.

The main result of this paper is the following. The terminology used is described at length in Section III below.

Theorem 3. *Let Λ be a symplectic lattice of density $(Q/P)^d$, $P,Q\in\mathbb{N}$. If (φ,Λ) is a Riesz basis for its closed linear span $\mathcal{G}(\varphi,\Lambda)$ with $\varphi\in S_0(\mathbb{R}^d)$, then $\pi(u,\eta)\varphi\notin\mathcal{G}(\varphi,\Lambda)$ for any $(u,\eta)\notin\Lambda$.*

For examples of generators in case $d=1$ we refer to our companion paper [CMP15].

Note that Theorem 3 generalizes the Amalgam Balian-Low Theorem stated above as Theorem 2 in the case of $d=1$. Indeed, if $\mathcal{G}(\varphi,\Lambda)=L^2(\mathbb{R})$, then it contains $\pi(u,\eta)\varphi\in\mathcal{G}(\varphi,\Lambda)$ for all $(u,\eta)\in\mathbb{R}\times\widehat{\mathbb{R}}$. Moreover, (φ,Λ) being a Riesz basis for $L^2(\mathbb{R})$ implies that Λ has density one which is rational. For $d>1$, Theorem 2 covers general lattices of density 1, while Theorem 3 covers symplectic lattices of density less than or equal to 1.

Balian-Low type results are relevant, for example, in communications applications. For instance, in orthogonal frequency division multiplexing, short, OFDM, an information carrying sequence $\{c_{k,\ell}\}_{k\in\mathbb{Z},\ell\in L}$ is transmitted in the form of the signal

$$F\{c_{k,\ell}\} = \sum_{k\in\mathbb{Z}}\sum_{\ell\in L} c_{k,\ell}T_{k\alpha}M_{\ell\beta}\varphi.$$

In this model, we assume infinite transmission length but a limited frequency band $[-\Omega,\Omega]$, the frequency band corresponds to the

$$L=\{\ell : [\ell\beta-\Omega_0,\ell\beta+\Omega_0]\subseteq[-\Omega,\Omega]\},$$

where $\operatorname{supp}\widehat{\varphi}\subseteq[-\Omega_0,\Omega_0]$. To enable recovery of the information in $F\{c_{k,\ell}\}$ under the assumption that the channel can be inverted, we require that F is boundedly invertible. This is achieved by asserting that $(\varphi,\alpha\mathbb{Z}\times\beta\mathbb{Z})$ is a Riesz basis for its closed linear span. In practice, we must utilize a compactly supported functions φ, hence, we can only assume $\operatorname{supp}\widehat{\varphi}\subseteq[-\Omega_0,\Omega_0]$ to hold in an approximative sense. This can be achieved by choosing Ω_0 of reasonable size if $\varphi\in S_0(\mathbb{R})$ or if φ is a Schwartz class function.

Theorem 3 then implies that $\pi(u,\eta)\varphi\notin\mathcal{G}(\varphi,\alpha\mathbb{Z}\times\beta\mathbb{Z})$ whenever $(u,\eta)\notin\alpha\mathbb{Z}\times\beta\mathbb{Z}$, a property that has advantages and disadvantages. For example, channels generally introduce time-shifts in the channel, Theorem 3 shows that we cannot choose $\varphi\in S_0(\mathbb{R})$ so that the transmission space $\mathcal{G}(\varphi,\alpha\mathbb{Z}\times\beta\mathbb{Z})$ is invariant under time shifts $T_u=\pi(u,0)$ for $(u,0)\notin\alpha\mathbb{Z}\times\beta\mathbb{Z}$.

A. Related work

In addition to the results described above, we would like to point to the following related results.

This paper is motivated by recent results in the setting of principle shift-invariant spaces, that is, spaces that are the closed linear span $S(\varphi)$ of a system $\{T_k\varphi\}_{k\in\mathbb{Z}}\subseteq L^2(\mathbb{R})$. For example, Aldroubi et al showed the following Balian-Low type phenomenon [ASW11].

Theorem 4. *If $\{T_k\varphi\}$ is a Riesz basis for its closed linear span, then*

1) *$T_{\frac{1}{N}}\varphi\in S(\varphi)$ for some $N\in\mathbb{N}\setminus\{1\}$ implies $\int|x|^{1+\epsilon}|\varphi(x)|^2\,dx=\infty$ for all $\epsilon>0$, and*
2) *$T_u\varphi\in S(\varphi)$ for all $u\in\mathbb{R}$ implies $\varphi\notin L^1(\mathbb{R})$.*

Gabardo and Han gave Balian-Low type results for Gabor spaces, as considered in this paper. In [GH04], they prove the following.

Theorem 5. *Let $(\varphi,\alpha\mathbb{Z}\times\beta\mathbb{Z})$ be a frame for $\mathcal{G}(\varphi,\alpha\mathbb{Z}\times\beta\mathbb{Z})$. If*

1) *$\alpha\beta\in\mathbb{N}\setminus\{1\}$ and $(\varphi,\alpha\mathbb{Z}\times\beta\mathbb{Z})$ is not Riesz, or*
2) *$(\alpha\beta)^{-1}\in\mathbb{N}\setminus\{1\}$ and $\mathcal{G}(\varphi,\alpha\mathbb{Z}\times\beta\mathbb{Z})\neq L^2(\mathbb{R})$,*

then (2) *holds.*

Clearly, the assumption that $(\varphi,\alpha\mathbb{Z}\times\beta\mathbb{Z})$ is a frame for $\mathcal{G}(\varphi,\alpha\mathbb{Z}\times\beta\mathbb{Z})$ is weaker than the condition considered in this paper, namely, that $(\varphi,\alpha\mathbb{Z}\times\beta\mathbb{Z})$ is a Riesz basis.

Theorems 3 and 5 are indeed unrelated. It is worth noting that both cases considered in Theorem 5 are rather unusual:

a generic Gabor system of density $1/N$, $N \geq 2$, forms a Riesz sequence and if the density exceeds two, then one would expect $\mathcal{G}(\varphi, \alpha\mathbb{Z} \times \beta\mathbb{Z}) = L^2(\mathbb{R})$.

Another subspace Balian-Low Theorem not discussed here in detail is Theorem 8 in [GHHK02]

Balian-Low type phenomenons remain an active research area. In fact, very recently Nitzan and Olsen [NO13] proved a strengthening of the $d = 1$ Balian-Low Theorem.

For general Balian-Low type results, we refer the reader to [BHW92], [BHW95], [BHW98], [BW94], [DLL95], [FG97], [Jan08].

II. The Zak transform

The proof of Theorem 3 hinges on utilizing well known properties of the Zak transform. The Zak transform is an operator mapping $L^2(\mathbb{R}^d)$ to $L^2_{\text{loc}}(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, densely defined on $L^2(\mathbb{R}^d)$ by

$$Zf(x, \omega) = \sum_{k \in \mathbb{Z}^d} f(x + k)\, e^{-2\pi i k \cdot \omega}.$$

The Zak transform is quasiperiodic, that is, for $n, m \in \mathbb{Z}^d$, we have

$$Zf(x + n, \omega) = e^{2\pi i n \cdot \omega} Zf(x, \omega)$$

and

$$Zf(x, \omega + m) = Zf(x, \omega).$$

Moreover, we have

$$\|Zf\|_{L^2([0,1]^d \times [0,1]^d)} = \|f\|_{L^2(\mathbb{R}^d)},$$

and, indeed, Z is a unitary map onto the space of quasiperiodic functions on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ equipped with the $L^2([0,1]^d \times [0,1]^d)$ norm.

The Zak transform of a time-frequency shifted function satisfies

$$\big(Z\pi(u, \eta) f\big)(x, \omega) = e^{2\pi i \eta \cdot x} Zf(x - u, \omega - \eta),$$

which, together with quasiperiodicity leads to

$$\big(Z\pi(k, \ell) f\big)(x, \omega) = e^{2\pi i (\ell \cdot x + k \cdot \omega)} Zf(x, \omega),$$

for $k, \ell \in \mathbb{Z}^d$.

Last but not least, we mention that $S_0(\mathbb{R}^d)$ is invariant under the Fourier transform, so $\varphi \in S_0(\mathbb{R})$ if and only if $\widehat{\varphi} \in S_0(\mathbb{R}^d)$, and $\varphi \in S_0(\mathbb{R}^d)$ implies that φ and $Z\varphi$ are continuous functions.

III. Symplectic lattices and metaplectic operators

Symplectic geometry is a popular tool in time-frequency analysis as many results for rectangular lattices can be extended to symplectic lattices using a unitary equivalence arguments [Grö01]. Further examples that illustrate the role of symplectic geometry in time-frequency analysis can be found in [KP06], [Pfa13].

A symplectic matrix is defined similarly to unitary matrices in Euclidean space. A matrix $A \in \mathbb{R}^{2d \times 2d}$ with $\det A = 1$ is an element of the symplectic group if it preserves the standard symplectic form, that is, $[A(x,\xi)^T, A(x',\xi')^T] = [(x,\xi),(x',\xi')] = [x'\xi - x\xi']$. Using the Stone - von Neumann Theorem we can establish for such A the existence of a unitary operator $U = U(A)$ on $L^2(\mathbb{R}^d)$, a so-called metaplectic operator, with

$$\pi(A(x,\xi)) = U(A)\pi(x,\xi)U(A)^*, \quad (x,\xi) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

A lattice Λ is a discrete subgroup of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$; Λ is referred to as full rank if $\Lambda = A(\mathbb{Z}^d \times Z^d)$ for some $A \in \mathbb{R}^{2d \times 2d}$ full rank. The density of a full rank lattice is unambiguously defined by $D(\Lambda) = 1/|\det(A)|$.

The lattice $\Lambda = \alpha A(\mathbb{Z}^d \times Z^d) \subseteq \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ is called symplectic if the generating matrix A can be chosen to be symplectic. The density of a symplectic lattice $\Lambda = \alpha A(\mathbb{Z}^d \times Z^d)$ is

$$D(\Lambda) = 1/\det(\alpha A) = \frac{1}{\alpha^{2d}\det(A)} = \frac{1}{\alpha^{2d}},$$

hence, the symplectic lattices considered in Theorem 3 satisfy $\alpha = \sqrt{P/Q}$ with $Q, P \in \mathbb{N}$. In particular, the lattice

$$\frac{1}{Q}\mathbb{Z}^d \times P\mathbb{Z}^d = \sqrt{\frac{P}{Q}}\,\big(\frac{1}{\sqrt{PQ}}\,\mathbb{Z}^d \times \sqrt{PQ}\,\mathbb{Z}^d\big) = \sqrt{\frac{P}{Q}}\,B(\mathbb{Z}^d \times \mathbb{Z}^d)$$

is of this type.

For fixed $P, Q \in \mathbb{N}$, $\alpha = \sqrt{P/Q}$, symplectic A and $B = \text{diag}(1/\sqrt{PQ}, \ldots, 1/\sqrt{PQ}, \sqrt{PQ}, \ldots, \sqrt{PQ})$ defined implicitly above, we observe that
$(\varphi, \alpha A(\mathbb{Z}^d \times \mathbb{Z}^d))$ is a Riesz sequence if and only if $(U(A)^*\varphi, \alpha(\mathbb{Z}^d \times \mathbb{Z}^d))$ is a Riesz sequence if and only if $(U(B)U(A)^*\varphi, (\frac{1}{Q}\mathbb{Z}^d \times P\mathbb{Z}^d))$ is a Riesz sequence.

Similarly, $\pi(u,\eta)\varphi \in \mathcal{G}(\varphi, \Lambda)$ with $(u,\eta) \notin \Lambda$ if and only if $\pi(\widetilde{u},\widetilde{\eta})U(B)U(A)^*\varphi \in \mathcal{G}(U(B)U(A)^*\varphi, \alpha(\mathbb{Z}^d \times \mathbb{Z}^d))$ with $(\widetilde{u},\widetilde{\eta}) = BA^{-1}(u,\eta)^T \notin \frac{1}{Q}\mathbb{Z}^d \times P\mathbb{Z}^d$.

Note that metaplectic operators are isometries on $S_0(\mathbb{R}^d)$, hence, to establish Theorem 3, it satisfies to consider the case $\Lambda = \frac{1}{Q}\mathbb{Z}^d \times P\mathbb{Z}^d$ with P, Q relatively prime natural numbers.

IV. Proof of Theorem 3

Section III allows us to assume without loss of generality that $\Lambda = \frac{1}{Q}\mathbb{Z}^d \times P\mathbb{Z}^d$ with $P, Q \in \mathbb{N}$.

The proof is by contradiction. We assume that $(\varphi, \frac{1}{Q}\mathbb{Z}^d \times P\mathbb{Z}^d)$ is a Riesz basis for the Gabor space $\mathcal{G}(\varphi, \frac{1}{Q}\mathbb{Z}^d \times P\mathbb{Z}^d)$ with $\varphi \in S_0(\mathbb{R}^d)$ and $\pi(u,\eta)\varphi \in \mathcal{G}(\varphi, \frac{1}{Q}\mathbb{Z}^d \times P\mathbb{Z}^d)$ where $(u,\eta) \notin \frac{1}{Q}\mathbb{Z}^d \times P\mathbb{Z}^d$.

We shall now show that each component (u,η) can be replaced by a rational number, while preserving non-inclusion in $\frac{1}{Q}\mathbb{Z}^d \times P\mathbb{Z}^d$. This will allow us to assume, without loss of generality, that there exists $R \in \mathbb{N}$ with $R(u,\eta) \in \frac{1}{Q}\mathbb{Z}^d \times P\mathbb{Z}^d$.

Indeed, in case $u_1 \notin \mathbb{Q}$, we have that $\mathbb{R} \times \{u_2\} \times \ldots \times \{u_d\} \times \{\eta_1\} \times \{\eta_2\} \times \ldots \times \{\eta_d\}$ is in the closure of the subgroup generated by (u,η) and Λ. Hence, we can replace u_1 with a rational number which is not in $\frac{1}{Q}\mathbb{Z}^d \times P\mathbb{Z}^d$. Successively, we can replace all irrational components of u and η by rationals.

A. The case $Q = 1$

Set $N = Ru \in \mathbb{Z}^d$ and $M = R\eta \in \mathbb{Z}^d$ where $R \in \mathbb{N}$ is chosen so that $(Ru, R\eta) \in \mathbb{Z}^d \times P\mathbb{Z}^d$, $N \cdot \eta/2$ is an integer and P divides M.

Observe that $\pi(u,\eta)\varphi \in \mathcal{G}(\varphi, \mathbb{Z}^d \times P\mathbb{Z}^d)$ together with $(\varphi, \mathbb{Z}^d \times P\mathbb{Z}^d)$ being a Riesz basis of $\mathcal{G}(\varphi, \mathbb{Z}^d \times P\mathbb{Z}^d)$ implies that there exists a sequence $c = (c_{k,\ell}) \in \ell^2(\mathbb{Z}^{2d})$ with

$$\begin{aligned} e^{2\pi i\eta\cdot x} Z\varphi(x-u, \omega-\eta) &= \sum_{k,\ell\in\mathbb{Z}} c_{k,\ell}\, e^{2\pi i(P\ell\cdot x + k\cdot\omega)} Z\varphi(x,\omega) \\ &= h(x,\omega) Z\varphi(x,\omega), \end{aligned}$$

where h is a locally L^2 function which is $1/P$ periodic in each x_i and 1 periodic in each ω_i. Using the quasiperiodicity of the Zak transform, $N \cdot \eta/2$ is an integer, and

$$Z\varphi(x,\omega) = e^{-2\pi i\eta\cdot(x+u)} h(x+u, \omega+\eta) Z\varphi(x+u, \omega+\eta),$$

we can compute the relationship

$$Z\varphi(x,\omega) = \; e^{2\pi i(N\cdot\omega - M\cdot x)}\, Z\varphi(x,\omega) \prod_{r=1}^{R} h(x+ru, \omega+r\eta)$$

which establishes

$$\prod_{r=1}^{R} h(x+ru, \omega+r\eta) = e^{2\pi i(M\cdot x - N\cdot\omega)}. \tag{3}$$

Note that (3) holds a-priori only on $\operatorname{supp} Z\varphi$. This relationship can be extended to hold on $\mathbb{R}\times\widehat{\mathbb{R}}$ using the fact that $(\varphi, \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d)$ is a Riesz sequence for its closed linear span and that $Z\varphi$ is continuous, implying that h is continuous. See [CMP15] for more detailed arguments.

We complete the proof for $Q = 1$ by showing that a continuous function h as constructed above does not exist. This follows from Proposition 6 which is a simple generalization of Proposition 3 in [CMP15]; its proof is therefore omitted.

Proposition 6. *Let $R \in \mathbb{N}$ and $P, M \in \mathbb{Z}^d$ and $u \in \mathbb{R}^d$. If $h(x)$ is continuous on $\mathbb{R}^d$ and $1/P_i$ periodic in x_i, $i = 1, \ldots, d$, with*

$$e^{2\pi i M\cdot x} = \prod_{r=0}^{R-1} h(x+ru),$$

then RP_i divides M_i for $i = 1, \ldots, d$.

We apply Proposition 6 to $h(x,\omega)$ constructed above. The function h is continuous, satisfies (3), and is $1/P$ periodic in $x_1, \ldots, x_d$ and 1-periodic in $\omega_1, \ldots, \omega_d$. Therefore Proposition 6 is applicable with $M = R\eta$, $N = -Ru$, $P_1 = \ldots = P_d = P$, and $P_{d+1}, \ldots, P_{2d} = 1$. We conclude $M_i/RP \in \mathbb{Z}$, that is, $\eta = M/R \in P\mathbb{Z}^d$, and, similarly $u = -N/R \in \mathbb{Z}$, that is, $(u,\eta) \in \Lambda = \mathbb{Z}\times P\mathbb{Z}$, a contradiction.

B. The rational case $\frac{P}{Q} \notin \mathbb{N}$.

Similarly as before, we first fix $R \in \mathbb{N}$ with $(Ru, R\eta) \in \mathbb{Z}^d \times P\mathbb{Z}^d$, set $N = Ru$ and $M = R\eta$, where by increasing R we can further assume $N \cdot \eta/2$ is an integer and P divides each component of M. As above, we argue that $\pi(u,\eta)\varphi \in \mathcal{G}(\varphi, \frac{1}{Q}\mathbb{Z}^d \times P\mathbb{Z}^d)$ implies that for some sequence $c = (c_{k,\ell}) \in \ell^2(\mathbb{Z}^{2d})$ we have with $[Q] = \{0, 1, \ldots, Q-1\}$

$$\begin{aligned} &e^{2\pi i\eta\cdot x} Z\varphi(x-u, \omega-\eta) \\ &= \sum_{k,\ell\in\mathbb{Z}^d} c_{k,\ell}\, e^{2\pi i P\ell\cdot x} Z\varphi(x - \tfrac{k}{Q}, \omega) \\ &= \sum_{q\in[Q]^d} \sum_{k,\ell\in\mathbb{Z}^d} c_{q+kQ,\ell}\, e^{2\pi i(P\ell\cdot x + k\cdot\omega)} Z\varphi(x - \tfrac{q}{Q}, \omega) \\ &= \sum_{q\in[Q]^d} h_q(x,\omega)\, Z\varphi(x - \tfrac{q}{Q}, \omega), \end{aligned}$$

where

$$h_q(x,\omega) = \sum_{k,\ell\in\mathbb{Z}^d} c_{q+kQ,\ell}\, e^{2\pi i(P\ell\cdot x + k\cdot\omega)}$$

are locally L^2 functions, $1/P$ periodic in each component of x and 1 periodic in each component of ω.

Similarly as in [KZZ04], [ZZ97], [ZZ93], [CMP15], we define the quasiperiodic, infinite length vector valued function

$$\mathcal{Z}_p^\circ\varphi(x,\omega) = Z\varphi(x - \tfrac{p}{Q}, \omega) = e^{-2\pi i s\cdot\omega} \mathcal{Z}_r^\circ(x,\omega)$$

for $p = Qs + r$, $r \in [Q]^d$, $s \in \mathbb{Z}^d$, to obtain the biinfinite matrix equation

$$\mathcal{Z}^\circ\varphi(x,\omega) = e^{-2\pi i\eta\cdot(x+u)} H(x+u, \omega+\eta)\, \mathcal{Z}^\circ\varphi(x+u, \omega+\eta)$$

where

$$H_{pq}(x,\omega) = e^{2\pi i\eta\frac{p}{Q}}\, h_{q-p}(x - \tfrac{p}{Q}, \omega) \text{ if } q-p \in [Q]^d \text{ and 0 else.}$$

As done in Section IV-A, we use our assumptions to show

$$\mathcal{Z}^\circ\varphi(x,\omega) = \; e^{2\pi i(N\cdot\omega - M\cdot x)} \prod_{r=1}^{R} H(x+ru, \omega+r\eta) \mathcal{Z}^\circ\varphi(x,\omega).$$

Using the fact that $H(x,\omega)$ is $1/P$ periodic in the components of x and that P divides all components of M, we have also for $p \in [Q]^d$

$$\begin{aligned} &\mathcal{Z}^\circ\varphi(x + \frac{p}{P}, \omega) \\ &= \; e^{2\pi i(N\cdot\omega - M\cdot x)} \prod_{r=1}^{R} H(x+ru, \omega+r\eta) \mathcal{Z}^\circ\varphi(x + \frac{p}{P}, \omega), \end{aligned}$$

and, with I denoting the identity operator,

$$e^{2\pi i(M\cdot x - N\cdot\omega)} I = \prod_{r=1}^{R} H(x+ru, \omega+r\eta) \tag{4}$$

on the space of quasiperiodic sequence in the span of $\mathcal{Z}^\circ\varphi(x + \frac{p}{P}, \omega)$, $p \in [Q]^d$. The following lemma implies that (4) is an identity of operators on the entire space of Q-quasiperiodic sequences for a.e. (x,ω) .

Lemma 7. *If $\varphi \in S_0(\mathbb{R}^d)$ and $(\varphi, \frac{1}{Q}\mathbb{Z}^d \times P\mathbb{Z}^d)$ is a Riesz basis for its closed linear span, then $\mathcal{Z}^\circ\varphi(x + \frac{p}{P}, \omega)$, $p \in [Q]^d$, spans the space of Q-quasiperiodic sequences for almost every $(x,\omega) \in \mathbb{R}^d \times \mathbb{R}^d$.*

We omit the proof of this lemma as it is similar to the proof of Lemma 5 in the companion paper[CMP15].

In order to use finite dimensional model we choose $\widetilde{H}(x,\omega) \in \mathbb{C}^{Q^d \times Q^d}$ so that for any $\mathcal{Z} \in \mathbb{C}^{Q^d}$ we have

$$\Big(H(x,\omega)\mathcal{Z}^\circ\Big)_p = \Big(\widetilde{H}(x,\omega)\mathcal{Z}\Big)_p, \quad p \in [Q]^d.$$

We obtain

$$e^{2\pi i(M\cdot x - N\cdot\omega)} I = \prod_{r=1}^{R} \widetilde{H}(x+ru, \omega + r\eta)$$

and with $h(x,\omega) = \det \widetilde{H}(x,\omega)$,

$$e^{2\pi i Q(M\cdot x - N\cdot\omega)} = \prod_{r=1}^{R} h(x+ru, \omega + r\eta).$$

As done in [CMP15], we can argue that $h(x,\omega)$ is continuous. As $h(x,\omega)$ is $1/P$ periodic in the components of x and 1 periodic in the components of ω, we can invoke Proposition 6 to realize that R divides each component of QN. Hence $RL = QN$ for some $L \in \mathbb{N}^d$ and $u = N/R = L/Q \in \frac{1}{Q}\mathbb{Z}^d$. Similarly, RP divides QM component wise and, hence, $\frac{Q}{P}\eta = \frac{QM}{RP} \in \mathbb{Z}^d$; since by assumption $(P,Q)=1$ we have $\eta \in P\mathbb{Z}^d$, contradicting $(u,\eta) \notin \frac{1}{Q}\mathbb{Z} \times P\mathbb{Z}$.

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