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Linear independence of Gabor systems in finite dimensional vector spaces

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ABSTRACT. *We discuss the linear independence of systems of m vectors in n–dimensional complex vector spaces where the m vectors are time–frequency shifts of one generating vector. Such systems are called Gabor systems. When n is prime, we show that there exists an open, dense subset with full–measure of such generating vectors with the property that any subset of n vectors in the corresponding full Gabor system of n^2 vectors is linearly independent. We derive consequences relevant to coding, operator identification and time–frequency analysis in general.*

1. Introduction

The goal of this paper is to show that there exist Gabor frames for $\mathbb{C}^n$ consisting of n^2 vectors in $\mathbb{C}^n$ with the property that any n vectors in this frame are linearly independent. In other terminology, we say that the vectors in such a Gabor frame are in *linear general position* or possess the *Haar property* (cf. [4]).

This result, given as Theorem 1 in Section 2, has implications for operator identification (e.g., [2], [10]), for the structure of the discrete short time Fourier transform, and for the robust coding of signals transmitted over lossy channels. Section 3 summarizes these implications, and Section 4 contains the proof of a slightly more general form of Theorem 1.

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2. Notation and Terminology

Definition 1. Set $\omega = e^{2\pi i/n}$. The translation operator T is the unitary operator on $\mathbb{C}^n$ given by $Tx = T(x_0, \ldots, x_{n-1}) = (x_{n-1}, x_0, x_1, \ldots, x_{n-2})$, and the modulation operator M is the unitary operator defined by $Mx = M(x_0, \ldots, x_{n-1}) = (\omega^0 x_0, \omega^1 x_1, \ldots, \omega^{n-1} x_{n-1})$. We set $\pi(\lambda) = M^l T^k$ for $\lambda = (l, k)$. Given a vector $f \in \mathbb{C}^n$ the *finite Gabor system with window* f is the collection $\{\pi(\lambda)f\}_{\lambda \in \mathbb{Z}_n \times \mathbb{Z}_n}$, where $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$.

Given $n \in \mathbb{N}$, the *discrete Fourier matrix* W_n is defined by $W_n = (\omega^{pq})_{p,q=0}^{n-1}$ and given a vector $f \in \mathbb{C}^n$ the *Discrete Fourier Transform (DFT)* of f, denoted $\widehat{f}$, is the vector $\widehat{f} = \frac{1}{n} W_n^* f$.

Let $f = (f_0, f_1, \ldots, f_{n-1}) \in \mathbb{C}^n$ be given. For $k = 0, 1, \ldots, n-1$, let D_k be the diagonal matrix

$$D_k = \begin{pmatrix} f_k &&&&&& \\ & f_{k+1} &&&&& \\ && \ddots &&&& \\ &&& f_{n-1} &&& \\ &&&& f_0 && \\ &&&&& \ddots & \\ &&&&&& f_{k-1} \end{pmatrix}.$$

Define the $n \times n^2$ *full Gabor system matrix* A by

$$A = (D_0 \cdot W_n \,|\, D_1 \cdot W_n \,|\, \cdots \,|\, D_{n-1} \cdot W_n). \tag{2.1}$$

We will sometimes write $A = A(f)$ to emphasize that A is a matrix–valued function of $f \in \mathbb{C}^n$. It is clear that the columns of A are the vectors $\pi(\lambda)f$, for $\lambda \in \mathbb{Z}_n \times \mathbb{Z}_n$.

Definition 2. A family $\mathcal{F}$ of $m \geq n$ vectors in $\mathbb{C}^n$ has the *Haar property* (cf. [4]) if any subset $\mathcal{F}' \subset \mathcal{F}$ with $|\mathcal{F}'| = n$ is linearly independent.

The following is immediate.

Proposition 1. *The Gabor system* $\{\pi(\lambda)f\}_{\lambda \in \mathbb{Z}_n \times \mathbb{Z}_n}$ *has the Haar property if and only if every minor*[1] *of* A *of order* n *is nonzero.*

The main result of this paper is the following.

Theorem 1. *If* n *is prime then there is a dense open set* E *of full measure*[2] *in* $\mathbb{C}^n$ *such that for every* $f \in E$, *the Gabor system* $\{\pi(\lambda)f\}_{\lambda \in \mathbb{Z}_n \times \mathbb{Z}_n}$ *has the Haar property.*

[1] A *minor* of order k of an $n \times m$ matrix M is the determinant of a $k \times k$ submatrix of M obtained by deleting $n-k$ rows and $m-k$ columns of M, see Definition 9.

[2] E is of *full measure* if the Lebesgue measure of $\mathbb{C}^n \setminus E$ is 0.

3. Implications of Theorem 2.4

3.1 Operator identification and the short–time Fourier transform

Definition 3. A linear space of operators (matrices) $\mathcal{M} \subseteq \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n) \equiv Mat(n \times m)$ is called *identifiable with identifier* $f \in \mathbb{C}^m$ if the linear map $\varphi_f : \mathcal{M} \longrightarrow \mathbb{C}^n,\ M \mapsto Mf$ is injective, i.e., if $Mf \neq 0$ for all $M \in \mathcal{M} \setminus \{0\}$. If there exists an identifier for $\mathcal{M}$, then we call $\mathcal{M}$ *identifiable*. In other words, $\mathcal{M}$ is identifiable if there exists a vector $f \in \mathbb{C}^m$ such that for all $M \in \mathcal{M}$, $Mf = 0$ implies $M = 0$.

Example 1.

1. $\mathcal{M} = \mathrm{Mat}(3 \times 3)$, $\mathcal{M}$ not identifiable since $\dim \mathcal{M} = 9 \geq 3 = \dim \mathbb{C}^3$, and, hence, $\mathcal{M}$ cannot be mapped injectively by a linear map to $\mathbb{C}^3$. This reduces to the obvious statement that for every $f \in \mathbb{C}^3$ there is a nonzero matrix $M \in \mathcal{M}$ such that $Mf = 0$.
2. $\mathcal{M} = \left\{ \left(\begin{smallmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right),\ a, b, c \in \mathbb{C} \right\}$, $\mathcal{M}$ not identifiable since $\dim \mathcal{M} = 3 \geq 1 \geq \dim \mathcal{M}f \quad \forall f \in \mathbb{C}^3$, where $\mathcal{M}f = \{Mf,\ M \in \mathcal{M}\} = \mathrm{range}\, \varphi_f$. This is equivalent to the statement that for every $f \in \mathbb{C}^3$ there is a nonzero $M \in \mathcal{M}$ such that $Mf = 0$. This is accomplished by simply choosing $\left(\begin{smallmatrix} a \\ b \\ c \end{smallmatrix} \right)$ to be a nonzero vector orthogonal to f.
3. $\mathcal{M} = \left\{ \left(\begin{smallmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{smallmatrix} \right),\ a, b, c \in \mathbb{C} \right\}$, $\mathcal{M}$ identifiable since $\left(\begin{smallmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{smallmatrix} \right) \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix} \right) = \left(\begin{smallmatrix} a \\ b \\ c \end{smallmatrix} \right)$

Definition 4. The *spreading function* of a matrix $H \in \mathrm{Mat}(n \times n)$ given by $H = (h_{i,j})_{i,j=0}^{n-1}$, denoted $\eta_H \in \mathbb{C}^{n^2}$, is defined by

$$\eta_H(l, k) = \frac{1}{n} \sum_{m=0}^{n-1} h_{m,m-k}\, \omega^{-ml}$$

for k, $l = 0, \ldots, n-1$, where here and in the following, indices are taken modulo n.

Lemma 1. *The family of operators* $\{M^l T^k\}_{(l,k) \in \mathbb{Z}_n \times \mathbb{Z}_n} \subseteq Mat(n \times n)$ *is a basis for* $Mat(n \times n)$*. In particular,*

$$H = \sum_{l=0}^{n-1} \sum_{k=0}^{n-1} \eta_H(l, k) M^l T^k = \sum_{\lambda \in \mathbb{Z}_n \times \mathbb{Z}_n} \eta_H(\lambda) \pi(\lambda),$$

where $\eta_H \in \mathbb{C}^{n^2}$ *is the spreading function of* H*.*

Proof. Note first that for each k, $\eta_H(\cdot, k)$ is the discrete Fourier transform of the vector $(h_{p,p-k})_{p\in\mathbb{Z}_n}$ and the identity

$$h_{p,p-k} = \sum_{l=0}^{n-1} \eta_H(l,k)\,\omega^{pl}$$

holds for each p, $k \in \mathbb{Z}_n$. Given $x \in \mathbb{C}^n$,

$$\begin{aligned}(Hx)_p &= \sum_{k=0}^{n-1} h_{p,k}\,x_k \\ &= \sum_{k=0}^{n-1} h_{p,p-k}\,x_{p-k} \\ &= \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} \eta_H(l,k)\,\omega^{pl}\,x_{p-k} \\ &= \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} \eta_H(l,k)\,(M^l T^k\,x)_p\end{aligned}$$

and the result follows. □

Definition 5. For $\Lambda \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$ define $\mathcal{H}_\Lambda = \operatorname{span}\{\pi(\lambda),\ \lambda\in\Lambda\} = \{H \in Mat(n\times n) : \operatorname{supp}\eta_H \subseteq \Lambda\}$.

Lemma 2. *Let $f \in \mathbb{C}^n$. The family $\{\pi(\lambda)f\}_{\lambda\in\Lambda}$ is linearly independent if and only if f identifies $\mathcal{H}_\Lambda$.*

Proof. The vector f fails to identify $\mathcal{H}_\Lambda$ if and only if there is an element $H \in \mathcal{H}_\Lambda \setminus \{0\}$ such that $Hf = \sum_{\lambda\in\Lambda} \eta_H(\lambda)\pi(\lambda)f = 0$. But by Lemma 1 $H \neq 0$ if and only if $\eta_H \neq 0$. Hence f fails to identify H_Λ if and only if $\sum_{\lambda\in\Lambda} \eta_H(\lambda)\pi(\lambda)f = 0$ for some $\eta_H \neq 0$, that is, if and only if $\{\pi(\lambda)f\}_{\lambda\in\Lambda}$ fails to be linearly independent. □

Definition 6. We define the short time Fourier transform V_f with respect to window $f \in \mathbb{C}^n$ on $\mathbb{C}^n$ by setting for $g \in \mathbb{C}^n$, $V_f g(\lambda) = \langle g, \pi(\lambda)f\rangle, \lambda \in \mathbb{Z}_n\times\mathbb{Z}_n$.

Lemma 3. *If f identifies $\mathcal{H}_\Lambda$ with $|\Lambda| = n$, then $V_f g|_\Lambda \neq 0$ for all $g \neq 0$.*

Proof. If f identifies $\mathcal{H}_\Lambda$ with $|\Lambda| = n$, then by Lemma 2, $\{\pi(\lambda)f\}_{\lambda\in\Lambda}$ is a basis for $\mathbb{C}^n$. Hence $V_f g(\lambda) \neq 0$ for at least one $\lambda \in \Lambda$ whenever $g \neq 0$. □

Theorem 2. *For $f \in \mathbb{C}^n \setminus \{0\}$, the following are equivalent:*

1. *$\{\pi(\lambda)f\}_{\lambda\in\mathbb{Z}_n\times\mathbb{Z}_n}$ has the Haar property.*
2. *$\mathcal{H}_\Lambda$ is identifiable by f if and only if $|\Lambda| \leq n$*

3. *For all $g \in \mathbb{C}^n$, $g \neq 0$, the vector $V_f g \in \mathbb{C}^{n^2}$ has at most $n-1$ components which equal 0.*

4. *$V_f g(\lambda)$ is completely determined by its values on any set Λ with $|\Lambda| = n$.*

Proof. **1.** $\Longleftrightarrow$ **2.** This follows immediately from Lemma 2.

2. $\Longrightarrow$ **3.** If **3** does not hold then there is a Λ with $|\Lambda| = n$ such that $V_f g|_\Lambda = 0$ and $g \neq 0$. By Lemma 3 f does not identify $\mathcal{H}_\Lambda$ and hence **2** fails to hold.

3. $\Longrightarrow$ **1.** If **1** does not hold then there is a set $\Lambda \in \mathbb{Z}_n \times \mathbb{Z}_n$ such that $|\Lambda| = n$ and $\{\pi(\lambda)f\}_{\lambda\in\Lambda}$ is not linearly independent. Let g be a nonzero vector perpendicular to span $\{\pi(\lambda)f\}_{\lambda\in\Lambda}$. Then for this g,

$$V_f g(\lambda) = \langle g, \pi(\lambda)f\rangle = 0$$

for all $\lambda \in \Lambda$. Since $|\Lambda| = n$, **3** does not hold.

1. $\Longrightarrow$ **4.** If **1** holds then for any Λ with $|\Lambda| = n$, $\{\pi(\lambda)f\}_{\lambda\in\Lambda}$ is a basis for $\mathbb{C}^n$. If $V_f g(\lambda) = \langle g, \pi(\lambda)f\rangle = 0$ for all $\lambda \in \Lambda$ then $g = 0$ and $V_f g$ is identically zero.

4. $\Longrightarrow$ **3.** If **3** does not hold then there is a $g \neq 0$ such that $V_f g$ has at least n components which vanish. If **4** also holds then $g = 0$, a contradiction. □

Corollary 1. *If $\{\pi(\lambda)f\}_{\lambda\in\mathbb{Z}_n\times\mathbb{Z}_n}$ has the Haar property, then $f_i \neq 0$ and $\widehat{f}_i \neq 0$ for all $i \in \mathbb{Z}_n$.*

Proof. If $f_{i_0} = 0$ for $i_0 \in \mathbb{Z}_n$, choose $g = (1, 0, 0, 0, \dots, 0)$ and observe that we have $V_f g(i_0, k) = 0$ for $k \in \mathbb{Z}_n$. Hence, Theorem 2.**3** is not satisfied and $\{\pi(\lambda)f\}_{\lambda\in\mathbb{Z}_n\times\mathbb{Z}_n}$ does not have the Haar property.

To see that $\widehat{f}_i \neq 0$ for all $i \in \mathbb{Z}_n$, note that by a straightforward calculation $(\pi(l,k)f)^\wedge = \omega^{kl}\,\pi(k,-l)\widehat{f}$ where, as before, the indices are taken modulo n. This means that

$$V_f g(l,k) = \langle g, \pi(l,k)f\rangle = n\,\omega^{kl}\,\langle \widehat{g}, \pi(k,-l)\widehat{f}\rangle = n\,\omega^{kl}\,V_{\widehat{f}}(\widehat{g})(k,-l).$$

Now assuming that $\widehat{f}_{i_0} = 0$ for $i_0 \in \mathbb{Z}_n$, and choosing $\widehat{g} = (1, 0, 0, 0, \dots, 0)$ we have that $V_f g(-k, i_0) = 0$ for $k \in \mathbb{Z}_n$ and the result follows as before. □

Corollary 2. *For n prime, $\mathcal{H}_\Lambda$ is identifiable if and only if $|\Lambda| \leq n$.*

Proof. This follows immediately from Theorem 1 and Theorem 2. □

3.2 Uniform tight finite frames and channels with erasures

Definition 7. A *frame* in a Hilbert space is a set of vectors $\{x_k\}_{k\in K}$ with the property that there exist constants c_1, $c_2 > 0$, called the *frame bounds* such that for all x in the Hilbert space

$$c_1\,\|x\|^2 \le \sum_{k\in K} |\langle x, x_k\rangle|^2 \le c_2\,\|x\|^2. \tag{3.1}$$

A frame is *tight* if $c_1 = c_2$ and is *uniform* if $\|x_j\| = \|x_k\|$ for all j and k.

It is obvious that, in an n–dimensional Hilbert space, any collection of $m \ge n$ vectors spanning the space is a (finite) frame for the space.

If our Hilbert space is $\mathbb{C}^n$ then it is convenient to represent a finite frame for $\mathbb{C}^n$, $\{x_k\}_{k=1}^m$, as an $m \times n$ matrix F whose rows are the complex conjugates of the m vectors $\{x_k\}_{k=1}^m$. In this case the frame coefficients of a vector x are given by the vector Fx, and the sum in (3.1) reduces to $\langle x, F^*Fx\rangle$ and the inequality (3.1) can be written as $c_1\,I \le F^*F \le c_2\,I$. The frame is tight if and only if F^*F is a multiple of the identity matrix.

Proposition 2. *For any $f \ne 0$, the collection $\{\pi(\lambda)f\}_{\lambda\in\mathbb{Z}_n\times\mathbb{Z}_n}$ is a uniform tight finite frame for $\mathbb{C}^n$ with frame bound $c_1 = c_2 = n^2\,\|f\|^2$.*

Proof. Let $F = A^*$ where A is given by (2.1). Then the rows of F are the complex conjugates of the elements of the Gabor system $\{\pi(\lambda)f\}_{\lambda\in\mathbb{Z}_n\times\mathbb{Z}_n}$. Specifically

$$F = \begin{pmatrix} W_n^*\,D_0^* \\ W_n^*\,D_1^* \\ \cdots \\ W_n^*\,D_{n-1}^* \end{pmatrix}$$

so that

$$\begin{aligned} F^*F &= D_0\,W_n\,W_n^*\,D_0^* + D_1\,W_n\,W_n^*\,D_1^* + \cdots + D_{n-1}\,W_n\,W_n^*\,D_{n-1}^* \\ &= n\,(D_0\,D_0^* + D_1\,D_1^* + \cdots + D_{n-1}\,D_{n-1}^*) \\ &= \left(n^2 \sum_{k=0}^{n-1} |f_k|^2\right) I \end{aligned}$$

and $\{\pi(\lambda)f\}_{\lambda\in\mathbb{Z}_n\times\mathbb{Z}_n}$ is a tight frame for $\mathbb{C}^n$. Moreover it is clear that

$$\|\pi(\lambda)f\| = \left(\sum_{k=0}^{n-1} |f_k|^2\right)^{1/2}$$

for each $\lambda \in \mathbb{Z}_n \times \mathbb{Z}_n$ so that the frame is also uniform. □

The basic problem we are interested in is the transmission of information in the form of a vector $x \in \mathbb{C}^n$ over a channel in such a way that recovery of the information at the receiver is robust to errors introduced by the channel. In the particular model of interest we first transform the signal x by forming $y = Fx \in \mathbb{C}^m$. This vector is then quantized in some fashion yielding $\widehat{y} = Q(y)$. In other words, we transmit not x but the quantized frame coefficients of x. Each such quantized coefficient is considered a *packet* of data sent over the channel. It is assumed that the channel distorts the transmitted vector by erasing packets at random. Robustness to this sort of distortion means maximizing the number of packets that can be erased while still allowing reconstruction of the signal as accurately as possible from the remaining packets. For more details see [3, 6, 7, 8, 14] and the references cited therein.

Definition 8. ([6]) A frame $\mathcal{F} = \{x_k\}_{k=1}^m$ in $\mathbb{C}^n$ is *maximally robust to erasures* if the removal of any $l \leq m - n$ vectors from $\mathcal{F}$ leaves a frame.

If the rows of F form a frame that is maximally robust to erasures, then if no more than $m - n$ packets are erased by our theoretical channel then the error in the reconstructed signal $\widehat{x}$ recovered from the received packets will be due entirely to quantization error in the coefficients $\widehat{y}$. Indeed if the quantization error is modelled as zero-mean uncorrelated noise, the mean square error of the reconstructed signal is minimized if and only if the frame is uniform and tight (Theorem 3.1, [6]).

The above discussion is summarized in following theorem.

Theorem 3. *The following are equivalent.*

1. $\{\pi(\lambda)f\}_{\lambda \in \mathbb{Z}_n \times \mathbb{Z}_n}$ *has the Haar property.*
2. $\{\pi(\lambda)f\}_{\lambda \in \mathbb{Z}_n \times \mathbb{Z}_n}$ *is maximally robust to erasures.*
3. *The* $n^2 \times n$ *matrix* F *whose rows are the complex conjugates of the vectors in the Gabor system* $\{\pi(\lambda)f\}_{\lambda \in \mathbb{Z}_n \times \mathbb{Z}_n}$ *has the property that every minor of order* n *is nonzero.*

4. Minors of Full Gabor System Matrices

The goal of this section is to prove Theorem 4 which, in light of Theorem 1, is a generalization of Theorem 1.

Theorem 4. *If* n *is prime then there exists a dense open set* E *of full measure*[3] *in* $\mathbb{C}^n$ *such that if* $f \in E$ *then every minor of* $A = A(f)$ *is nonzero.*

Before getting to the proof, we specify in the following two subsections some notation and basic results from the theory of determinants.

4.1 Basic results on determinants

The following is adapted from [11], Chapter 2 and [1], Chapter 6.

[3] In fact E is the complement of the union of the zero sets of finitely many homogeneous polynomials in n complex variables.

Definition 9. The *determinant* of an $n \times n$ matrix $A = (a_{i,j})_{i,j=1}^n$, denoted $\det(A)$, is defined to be

$$\det(A) = \sum_j (-1)^{t(j)} \, a_{1j_1} a_{2,j_2} \cdots a_{nj_n} \tag{4.1}$$

where $j = (j_1, j_2, \ldots, j_n)$ runs through all permutations of $\{1, 2, \ldots, n\}$ and $t(j)$ is the parity of j, that is, the number of interchanges of pairs of elements required to transform j into $(1, 2, \ldots, n)$.

For each permutation $j = (j_1, j_2, \ldots, j_n)$, the set of matrix elements $\{a_{1j_1}, a_{2,j_2}, \ldots, a_{nj_n}\}$ is referred to as a *diagonal* of A. Then $\det(A)$ is the sum over all diagonals of A of the products of those diagonals weighted by ± 1.

Given an $m \times n$ matrix A and $1 \leq p \leq \min(m, n)$, the determinant of a $p \times p$ submatrix of A obtained by deleting $m - p$ rows and $n - p$ columns is called a *minor of order* p of A. If the p rows and columns retained have indices given by $i = (i_1, i_2, \ldots, i_p)$ with $i_1 < i_2 < \cdots < i_p$ and $j = (j_1, j_2, \ldots, j_p)$ with $j_1 < j_2 < \cdots < j_p$ respectively then the corresponding minor of order p is denoted

$$A \begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ j_1 & j_2 & \cdots & j_p \end{pmatrix} \equiv A \begin{pmatrix} i \\ j \end{pmatrix}$$

If A is square then the *complimentary minor* corresponding to $i = (i_1, i_2, \ldots, i_p)$ and $j = (j_1, j_2, \ldots, j_p)$, denoted

$$A \begin{pmatrix} i_1 & i_1 & \cdots & i_p \\ j_1 & j_2 & \cdots & j_p \end{pmatrix}^c \equiv A \begin{pmatrix} i \\ j \end{pmatrix}^c,$$

is defined to be the determinant of the $(n - p) \times (n - p)$ submatrix of A obtained by deleting the rows with indices in i and the columns with indices in j.

The *complimentary cofactor* corresponding to i and j is defined as

$$A^c \begin{pmatrix} i \\ j \end{pmatrix} = (-1)^{(|i|+|j|)} \, A \begin{pmatrix} i \\ j \end{pmatrix}^c$$

where $|i| = \sum_{k=1}^p i_k$ and $|j| = \sum_{k=1}^p j_k$.

Theorem 5. (Laplace Expansion) *Let A be an $n \times n$ matrix and let $j = (j_1, j_2, \ldots, j_p)$, $j_1 < j_2 < \cdots < j_p$, be a choice of p column indices, $1 \leq p \leq n$. Then*

$$\det(A) = \sum_i A \begin{pmatrix} i \\ j \end{pmatrix} A^c \begin{pmatrix} i \\ j \end{pmatrix} = \sum_i (-1)^{(|i|+|j|)} \, A \begin{pmatrix} i \\ j \end{pmatrix} A \begin{pmatrix} i \\ j \end{pmatrix}^c \tag{4.2}$$

where i runs through all $\binom{n}{p}$ choices of p row indices $i = (i_1, i_2, \ldots, i_p)$, $i_1 < i_2 < \cdots < i_p$.

Proof. The proof of this theorem is given in [11] and consists of showing that each term in the sum contributes exactly $p!(n-p)!$ terms in the sum (4.1) defining $\det(A)$, and that $\binom{n}{p}p!(n-p)! = n!$ thereby accounting for all terms in this sum. □

The proof of the following theorem is found in [1], Section 35.

Theorem 6. (Extended Laplace Expansion) *Let A be an $n \times n$ matrix and let s be a partition of the column indices of A, that is, $s = (s_1, s_2, \ldots, s_m)$ where s_k is a set of p_k column indices such that $s_k \cap s_{k'} = \emptyset$ when $k \neq k'$ and where $p_1 + \cdots + p_m = n$. Then*

$$\det(A) = \sum_t (-1)^{\sum_{k=1}^m |s_k|+|t_k|} A\begin{pmatrix} t_1 \\ s_1 \end{pmatrix} A\begin{pmatrix} t_2 \\ s_2 \end{pmatrix} \cdots A\begin{pmatrix} t_m \\ s_m \end{pmatrix} \tag{4.3}$$

where $t = (t_1, t_2, \ldots, t_m)$ runs through all

$$\binom{n}{p_1}\binom{n-p_1}{p_2}\cdots\binom{n-p_1-\cdots-p_{m-1}}{p_m}$$

partitions of the row indices into subsets of size $(p_1, p_2, \ldots, p_m)$.

Proof. This theorem follows from Theorem 5 and a straightforward induction argument based on the following considerations. Given s as in the statement of the theorem, (4.2) says that

$$\det(A) = \sum_{t_1} (-1)^{(|t_1|+|s_1|)} A\begin{pmatrix} t_1 \\ s_1 \end{pmatrix} A\begin{pmatrix} t_1 \\ s_1 \end{pmatrix}^c$$

where t_1 runs through all choices of p_1 row indices of A. Note now that $A\begin{pmatrix} t_1 \\ s_1 \end{pmatrix}^c$ is the determinant of an $(n-p_1) \times (n-p_1)$ matrix formed by deleting the rows of A with indices in t_1 and columns with indices in s_1. Hence applying (4.2) again,

$$A\begin{pmatrix} t_1 \\ s_1 \end{pmatrix}^c = \sum_{t_2} (-1)^{(|t_2|+|s_2|)} A\begin{pmatrix} t_2 \\ s_2 \end{pmatrix} A\begin{pmatrix} t_2 \\ s_2 \end{pmatrix}^c$$

where t_2 runs through all choices of p_2 row indices of the $(n-p_1) \times (n-p_1)$ submatrix of A formed by deleting the rows of A with indices in t_1 and columns with indices in s_1. Hence

$$\det(A) = \sum_{t_1}\sum_{t_2} (-1)^{(|t_1|+|s_1|+|t_2|+|s_2|)} A\begin{pmatrix} t_1 \\ s_1 \end{pmatrix} A\begin{pmatrix} t_2 \\ s_2 \end{pmatrix} A\begin{pmatrix} t_2 \\ s_2 \end{pmatrix}^c.$$

Continuing in this fashion we arrive finally at

$$\det(A) = \sum_{t_1}\sum_{t_2}\cdots\sum_{t_m}(-1)^{\sum_{k=1}^{m}(|t_k|+|s_k|)}\, A\binom{t_1}{s_1}\, A\binom{t_2}{s_2}\cdots A\binom{t_m}{s_m}$$

where we have made the observation that at the penultimate step we have the equality

$$A\binom{t_{m-1}}{s_{m-1}}^{c} = A\binom{t_m}{s_m}.$$

□

Note that by combining this result with the counting argument in the proof of Theorem 5, it follows that each term of the sum in (4.3) contributes exactly $p_1!p_2!\cdots p_m!$ terms to the sum in (4.1) which defines $\det(A)$ and that

$$\binom{n}{p_1}\binom{n-p_1}{p_2}\cdots\binom{n-p_1-\cdots-p_{m-1}}{p_m}p_1!p_2!\cdots p_m! = n!,$$

thereby accounting for all terms in this sum. Indeed the proof of Theorem 6 in [1] (Section 35) follows precisely these lines.

4.2 Generalized Vandermonde determinants.

The proof of Theorem 1 requires the following lemma, whose proof may be found in [5]. We will describe this proof below for completeness, with notation taken from [5].

Lemma 4. *If n is prime then every minor of the discrete Fourier matrix W_n is nonzero.*

The proof of Lemma 4 is based on the theory of *generalized Vandermonde determinants.*

Definition 10. Given an p–tuple $a = (a_0, a_1, \ldots, a_{p-1})$ of distinct non-negative integers and a point $x = (x_0, x_1, \ldots, x_{p-1}) \in \mathbb{C}^p$, define the *generalized Vandermonde determinant*, denoted

$$V_a(x) = V_a(x_0, x_1, \ldots, x_{p-1}),$$

to be the determinant of the matrix $(x_k^{a_l})_{k,l=0}^{p-1}$. With $s = (0, 1, \ldots, p-1)$, $V_s(x)$ is the *standard Vandermonde determinant.*

It is well–known that the standard Vandermonde determinant is given by

$$V_s(x) = V_s(x_0, x_1, \ldots, x_{p-1}) = \prod_{0\le k<l\le p}(x_k - x_l)$$

and hence does not vanish if and only if the x_k are distinct. Suppose that a minor of order p of the Fourier matrix W_n is given by $W_n\binom{i}{j}$ where $i = (i_0, i_1, \ldots, i_{p-1})$ and $j = (j_0, j_1, \ldots, j_{p-1})$. Then $W_n\binom{i}{j}$ is the generalized Vandermonde determinant $V_j(\omega^{i_0}, \omega^{i_1}, \ldots, \omega^{i_{p-1}})$ where $\omega = e^{2\pi i/n}$. If we consider $V_a(x)$ to be a polynomial in the p variables x_k, $k = 0, 1, \ldots, p-1$, then $V_a(x)/V_s(x)$ is a homogeneous polynomial in x with integer coefficients.[4] We will denote this polynomial by $P_a(x)$.

Fundamental to the theory of generalized Vandermonde determinants are the following results of Mitchell [13] (see [5] for elementary proofs and interesting consequences of these results).

Theorem 7. *Let $a = (a_0, \ldots, a_{p-1})$ with $0 \le a_0 < a_1 < \cdots < a_{p-1}$. Then all the coefficients of the polynomial $P_a(x)$ are nonnegative.*

Theorem 8. *Let $a = (a_0, \ldots, a_{p-1})$. Then the sum of the coefficients of $P_a(x)$ is*

$$V_s(a_0, a_1, \ldots, a_{p-1})/V_s(0, 1, \ldots, p-1).$$

In other words,

$$P_a(1, 1, \ldots, 1) = \frac{V_s(a_0, a_1, \ldots, a_{p-1})}{V_s(0, 1, \ldots, p-1)} = \prod_{0 \le k < l \le p} \frac{(a_k - a_l)}{(k - l)}.$$

We can now prove Lemma 4 as follows. Given a choice of p row and column indices of W_n, denoted respectively by $i = (i_0, i_1, \ldots, i_{p-1})$ and $j = (j_0, j_1, \ldots, j_{p-1})$, suppose that

$$W_n\binom{i}{j} = V_j(\omega^{i_0}, \omega^{i_1}, \ldots, \omega^{i_{p-1}}) = 0.$$

Now consider the polynomial in $z \in \mathbb{C}$ defined by

$$P(z) = \frac{V_j(z^{i_0}, z^{i_1}, \ldots, z^{i_{p-1}})}{V_s(z^{i_0}, z^{i_1}, \ldots, z^{i_{p-1}})} = \frac{V_j(z^{i_0}, z^{i_1}, \ldots, z^{i_{p-1}})}{\prod_{0 \le k < l \le p}(z^{i_k} - z^{i_l})}. \qquad (4.4)$$

Since n is prime and since the $0 \le i_k \le n-1$ are distinct integers the denominator of the last term in (4.4) is nonzero when $z = \omega$, and since its numerator is assumed to vanish, $P(\omega) = 0$. Moreover, $P(\omega) = 0$ implies

[4] It follows by direct calculation that for each $k < l$, $(x_k - x_l)$ divides $V_a(x)$. Since $\mathbb{Z}[x]$ is a unique factorization domain ([9], p. 164, Thm. 6.14) and since $(x_k - x_l)$ is irreducible, $V_a(x)/V_s(x)$ is a polynomial. That it is homogeneous follows from the fact that a polynomial P is homogeneous of degree k if and only if $P(ax) = a^k P(x)$.

that P is divisible by $z^{n-1} + z^{n-2} + \cdots + z + 1$ in $\mathbb{Z}[z]$.[5] Consequently $P(1)$ is an integer multiple of n.

Now, by Theorem 8, it also holds that

$$P(1) = \prod_{0 \le k < l \le p} \frac{(j_k - j_l)}{(k - l)}.$$

However, since $0 \le j_k \le n - 1$ and are distinct, $P(1)$ cannot be a multiple of n since in that case n would be the product of integers strictly less than n contradicting the assertion that n is prime. Hence $W_n \begin{pmatrix} i \\ j \end{pmatrix}$ is not zero.

4.3 Proof of Theorem 4.1.

Fix $1 \le l \le n$ and let M be an $l \times l$ submatrix of A formed by deleting $n^2 - l$ columns and rows of A. Associate to M the n-tuple $(l_0, l_1, \ldots, l_{n-1})$ where l_k is the number of columns of the matrix $D_k \cdot W_n$ that appear in M (see (2.1)).

Note that $\det(M)$ is a homogeneous polynomial of degree l in the variables $f_0, f_1, \ldots, f_{n-1}$. It will be sufficient to show that this polynomial does not vanish identically. We will do this by finding at least one monomial in this polynomial that has a nonzero coefficient. We define this monomial below by means of a recursive algorithm.

Definition 11. The minor $\det(M)$ given by (4.1) is formally a sum of $l!$ monomials in $f_0, f_1, \ldots, f_{n-1}$. Of those monomials that formally appear in this sum we define p_M recursively as follows. If $l = 1$ then M is simply a multiple of a single variable f_j and we define $p_M = f_j$. For $l > 1$, let f_j be the variable of lowest index appearing in M. Choose any entry of M in which f_j appears, eliminate from M the row and column containing that entry, and call the remaining matrix M'. Define $p_M = f_j \, p_{M'}$.

Of course it now must be argued that this definition makes sense, that is, that the monomial p_M is uniquely determined by M. It is clear that what must be shown is that the choice of the term in M containing the variable f_j does not effect the variable of least index appearing in M'. So suppose that f_j is the variable of lowest index appearing in M. There are three possibilities. (i) the variable f_j appears in more than one row of M, (ii) the variable f_j appears in exactly one row and more than one column of M, and (iii) the variable f_j appears in exactly one row and one column of M.

[5] That P is divisible by $z^{n-1} + z^{n-2} + \cdots + z + 1$ in $\mathbb{Q}[z]$ follows from the fact that $I = \{f \in \mathbb{Q}[z] : f(\omega) = 0\}$ is an ideal in the principle ideal domain $\mathbb{Q}[z]$ generated by $z^{n-1} + z^{n-2} + \cdots + z + 1$ ([9], p. 123). That P also factors in $\mathbb{Z}[z]$ is an application of Gauss' Lemma, ([12], p. 181).

Consider case (i). Because M is a submatrix of A, and because of the structure of A given in (2.1), it follows that the variable f_j cannot appear twice in the same column of M. Hence no matter which term containing f_j is chosen, the variable f_j will still appear in the reduced matrix M', and will be the variable of least index appearing in M'.

Consider case (ii). Again by the structure of A, and since M is a submatrix of A, the columns in which f_j appears must come from the same submatrix $D_k \cdot W_n$ of A. Consequently the variables appearing in each such column are the same and appear in the same order in each column. Hence all the variables that are removed by eliminating one of the columns in which f_j appears still appear in the reduced matrix M'. Hence the term of lowest index in M' is unaffected.

Consider case (iii). In this case, there is no ambiguity about which row and column to eliminate so M' is uniquely determined.

Lemma 5. *The number of diagonals of M that correspond to the monomial p_M is $\prod_{k=0}^{n-1} l_k!$.*

Proof. For any submatrix M of A, we define $\mu(M)$ to be the number of diagonals of M whose product is a multiple of p_M. The proof proceeds by induction on l. If $l = 1$ then the result is obvious. Let M be given with its associated n-tuple $(l_0, l_1, \ldots, l_{n-1})$. We may assume without loss of generality that the variable of smallest index in p_M with a nonzero exponent is f_0. In this case, there is a row of M in which the variable f_0 appears l_j times for some index j. Choose one of these terms and delete the row and column in which it appears. Call the remaining matrix M'. The n-tuple associated with M' is $(l_0, \ldots, l_{j-1}, l_j - 1, l_{j+1}, \ldots, l_{n-1})$, and this n-tuple does not depend on which term was chosen from the given row to form M'. By Definition 11, $p_M = f_0\, p_{M'}$ and by the induction hypothesis

$$\mu(M') = l_0! \cdots l_{j-1}!\,(l_j - 1)!\, l_{j+1}! \cdots l_{n-1}!.$$

Since there are l_j ways to choose a term from the given row to produce M' we have that

$$\mu(M) = l_j\, \mu(M') = l_0! \cdots l_{j-1}!\, l_j (l_j - 1)!\, l_{j+1}! \cdots l_{n-1}! = \prod_{k=0}^{n-1} l_k!$$

which was to be proved. □

Proof of Theorem 4: Let $s = (s_1, s_2, \ldots, s_m)$ be the partition of the column indices of M defined as follows. Let $0 \le j_1 < j_2 < \cdots < j_m < n$ be such that $l_{j_k} > 0$ and let s_k be the set of those l_{j_k} column indices of M corresponding to columns chosen from the submatrix $D_{j_k} \cdot W_n$ of A. By (4.3), $\det(M)$ is given by the sum

$$\det(M) = \sum_t (-1)^{\sum_{k=1}^m |s_k|+|t_k|} M\binom{t_1}{s_1} M\binom{t_2}{s_2} \cdots M\binom{t_m}{s_m} \tag{4.5}$$

where the sum runs over all partitions of the row indices of M for which t_k is the same size as s_k. Note that each term in this sum is a multiple of a monomial of degree l in the variables $f_0, f_1, \ldots, f_{n-1}$.

We will now choose a term in this sum that is a nonzero multiple of the monomial p_M. That is, we will choose a particular $t = (t_1, t_2, \ldots, t_m)$ with the property that the product

$$M\binom{t_1}{s_1} M\binom{t_2}{s_2} \cdots M\binom{t_m}{s_m} \tag{4.6}$$

is a nonzero multiple of p_M. Define this partition as follows. Choose any diagonal of M whose product is formally a multiple of p_M. Define t_k to be the set of row indices of M such that the term in the chosen diagonal in that row is in one of the columns whose index is in s_k. Then for each k, the submatrix of M formed by choosing the columns indexed by s_k and the rows indexed by t_k has the property that the same variable f_j appears in each row of the submatrix. Hence the minor $M\binom{t_k}{s_k}$ is a product of these variables and of a minor of the Fourier matrix W_n. Moreover, the product of all the variables that appear in each of the submatrices is precisely p_M. Finally we conclude that the quantity in (4.6) is p_M multiplied by a coefficient which is the product of m minors of W_n. By Lemma 4 this coefficient is nonzero.

Finally we assert that the term in the sum (4.5) described above is the only one that is a multiple of p_M. To see why this is true note that the product (4.6) represents $\prod_{k=0}^{n-1} l_k!$ terms in the sum for $\det(M)$ given by (4.1). However, by Lemma 5 this is precisely the number of terms in the sum (4.1) for $\det(M)$ in which the monomial p_M formally appears. Hence the coefficient of p_M in $\det(M)$ is nonzero and the polynomial $\det(M)$ is not identically zero.

Let $\mathcal{Z}(M)$ be the set of zeros of $\det(M)$ and define $E^c = \cup_M \mathcal{Z}(M)$ where M runs through all $l \times l$ submatrices of A, $1 \leq l \leq n$. Since this is a finite union and since each set $\mathcal{Z}(M)$ has measure zero, is closed, and has empty interior, E is an open dense subset of $\mathbb{C}^n$ of full measure. Clearly if $f \in E$ then every minor of the matrix $A(f)$ is nonzero.

Example 2. The Matrix

$$M = \left(\begin{array}{cc|ccc|c} \omega^0 f_2 & \omega^0 f_2 & \omega^0 \underline{f_3} & \omega^0 f_3 & \omega^0 f_3 & \omega^0 f_6 \\ \omega^1 f_3 & \omega^2 f_3 & \omega^0 f_4 & \omega^2 f_4 & \omega^5 f_4 & \omega^3 \underline{f_0} \\ \omega^2 f_4 & \omega^4 f_4 & \omega^0 f_5 & \omega^4 \underline{f_5} & \omega^3 f_5 & \omega^6 f_1 \\ \omega^3 f_5 & \omega^6 f_5 & \omega^0 f_6 & \omega^6 f_6 & \omega^1 \underline{f_6} & \omega^2 f_2 \\ \omega^5 \underline{f_0} & \omega^3 f_0 & \omega^0 f_1 & \omega^3 f_1 & \omega^4 f_1 & \omega^1 f_4 \\ \omega^6 f_1 & \omega^5 \underline{f_1} & \omega^0 f_2 & \omega^5 f_2 & \omega^2 f_2 & \omega^4 f_5 \end{array} \right)$$

gives an exemplary submatrix M of $A(f)$ in the case $n = 7$.

M is obtained by removing the row of $A(f)$ with index 4, and all columns with indices not in $\{15, 16, 21, 23, 26, 45\}$. Recall that rows and columns are numbered starting with 0 in this paper. Underlined are the appearances of f_j which contribute to the construction of p_M which is given by $p_M(f) = f_0^2 f_1 f_3 f_6$.

The 7-tuple assigned to M in the proof of Theorem 4 is

$$(l_0, l_1, \ldots, l_{n-1}) = (0, 0, 2, 3, 0, 0, 1),$$

and the partition s is given by

$$s = (s_1, s_2, s_3) = (\{15, 16\}, \{21, 23, 26\}, \{45\}).$$

We conclude that $t_1 = \{4, 5\}$, $t_2 = \{0, 2, 3\}$, and $t_3 = \{1\}$.

We have

$$\begin{aligned} & M\begin{pmatrix} t_1 \\ s_1 \end{pmatrix} M\begin{pmatrix} t_2 \\ s_2 \end{pmatrix} M\begin{pmatrix} t_3 \\ s_3 \end{pmatrix} \\ = \; & \det \begin{pmatrix} \omega^5 \underline{f_0} & \omega^3 f_0 \\ \omega^6 f_1 & \omega^5 \underline{f_1} \end{pmatrix} \det \begin{pmatrix} \omega^0 \underline{f_3} & \omega^0 f_3 & \omega^0 f_3 \\ \omega^0 f_5 & \omega^4 \underline{f_5} & \omega^3 f_5 \\ \omega^0 f_6 & \omega^6 f_6 & \omega^1 \underline{f_6} \end{pmatrix} \cdot \det \left(\omega^3 \underline{f_0} \right) \\ = \; & f_0^2 f_1 f_3 f_6 \cdot \det \begin{pmatrix} \omega^5 & \omega^3 \\ \omega^6 & \omega^5 \end{pmatrix} \det \begin{pmatrix} \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^4 & \omega^3 \\ \omega^0 & \omega^6 & \omega^1 \end{pmatrix} \cdot \omega^3 \end{aligned}$$

and this coefficient of $f_0^2 f_1 f_3 f_6$ is nonzero by Lemma 4. The counting argument in the proof of Theorem 4 shows that $p_M(f)$ appears with this coefficient in $\det M$.

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