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Summary

The uncertainty principle for functions on finite Abelian groups provides us with lower bounds on the cardinality of the support of Fourier transforms of functions of small support. We discuss novel results in this realm and generalize these to obtain results relating the support sizes of functions and their *short–time Fourier transforms*. We then apply these results to construct a class of equal norm tight Gabor frames that are maximally robust to erasures. We discuss consequences of our findings to the theory of recovering and storing signals with sparse time–frequency representations.

Contents

1 Introduction

Uncertainty principles establish restrictions on how well localized the Fourier transform of a well localized function can be and vice versa [DS89, Grö03, FS97]. In the case of a function defined on a finite Abelian group, localization is generally expressed through the cardinality of the support of the function. Due to its relevance for compressed sensing and, in particular, for the recovery of lossy signals under the assumption of restricted spectral content [CRT06], the uncertainty principle for functions on finite Abelian groups has recently drawn renewed interest.

In this realm, a classical result on uncertainty states that the product of the number of nonzero entries in a vector representing a nontrivial function on an Abelian group and the number of nonzero entries in its Fourier transform is not smaller than the order of the group [DS89, MÖP04]. This result can be improved for any nontrivial Abelian group [Mes06]. For example, for groups of prime order, the sum of the number of nonzero entries in a vector and the number of nonzero entries in its Fourier transform exceeds the order of the group [Tao05].

The objective of this technical report is to establish corresponding results for joint time–frequency representations, that is, to obtain restrictions on the minimal cardinality of the support of joint time–frequency representations of functions defined on finite Abelian groups. The central results in this paper are published in [KPR].

As first example, let us consider the simplest time–frequency representation of a function, namely the one that is given by the tensor product of a function and its Fourier transform. In this case, the classical uncertainty principle for nontrivial functions on finite Abelian groups states that the cardinality of the support of this tensor is at least the order of the group.

In this paper though our focus lies on time–frequency representations given by short–time Fourier transforms. It is easy to see that, again, the cardinality of the support of any short–time Fourier transform of a nontrivial function defined on a finite Abelian group is bounded below by the order of the group. As seen below, we can improve this bound by using the subgroup structure of the groups and/or by allowing only well-chosen window functions. For example, we establish in Theorem 4.5 that for any group of prime order and for almost every window function on the group, the sum of the cardinality of the support of the analyzed function and the cardinality of the support of its short–time Fourier transform exceeds the square of the order of the group.

In addition to the above, we give applications of our results to the theory of so-called Gabor frames and the theory of sparse signal recovery. For instance, the results on the cardinality of the support of short–time Fourier transforms can be translated into criteria for the recovery of encoded signals from a channel with

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erasures.

The paper is organized as follows. In Section 2 we give a brief but self-contained account of the Fourier transformation and of the short–time Fourier transformation for functions defined on finite Abelian groups. Section 3 discusses uncertainty principles which relate the cardinality of the support of functions to the cardinality of the support of their Fourier transforms. We start Section 3 with a classical result which is based on standard norm estimates [DS89]. In Section 3.1 we state results based on the minors of Fourier transform matrices and which apply only to functions defined on cyclic groups of prime order [Tao05]. Finite Abelian groups of any order are analyzed in Section 3.2. There, the underlying subgroup structure of finite Abelian groups is used to obtain improvements to the classical uncertainty result discussed above [Mes06]. In addition, we provide numerical evidence on the achieved support set pairs for the Fourier transformation on groups of order less than or equal to 16.

Section 4 is devoted to uncertainty inherent in the short–time Fourier transformation. Following the organization of Section 3, a discussion of general results is followed by results for functions defined on cyclic groups of prime order in Section 4.1. Other finite Abelian groups are covered in Section 4.2. We conclude our discussion of the cardinality of the support set of short–time Fourier transforms in Section 4.3 with a question on the possible cardinalities of the support of short–time Fourier transforms with respect to an optimally chosen window function. In fact, one of the major difficulties to obtain uncertainty principles for the short–time Fourier transform is its dependence on the chosen window function. Our results are complemented by numerical experiments.

In Section 5 we give applications of the results of Section 4 to communications engineering. In Section 5.1 we discuss the identification/measurement problem for time-varying operators/channels. Also we consider channel coding for the transmission of information through channels with erasures. In addition, we show the existence of a large class of Gabor type. In Section 5.2 we briefly discuss connections of our work to the recovery of signals which have a sparse representation in a given dictionary.

2 Background and Notation

For any finite set A we set $\mathbb{C}^A = \{f : A \longrightarrow \mathbb{C}\}$. For $|A| = |B| = n$, $\mathbb{C}^A \cong \mathbb{C}^B \cong \mathbb{C}^n$ as vector spaces, where $|A|$ denotes the cardinality of the set A. Further, for $A \subseteq B$, we write $A^c = B \backslash A$ and we define the embedding operator $i_A : \mathbb{C}^A \longrightarrow \mathbb{C}^B$ where $i_A f(x) = f(x)$ for $x \in A$ and $i_A f(x) = 0$ for $x \in A^c$. Correspondingly, we define the restriction operator $r_A : \mathbb{C}^B \longrightarrow \mathbb{C}^A$. Similarly, every map $S : A \longrightarrow B$ induces a map $\widetilde{S} : \mathbb{C}^B \longrightarrow \mathbb{C}^A$, $\left(\widetilde{S} f\right)(a) = f(S(a))$. If S is bijective, then $\widetilde{S}$ is bijective as well. For $M \in \mathbb{C}^{m \times n}$ and $A \subseteq \{0, 1, \ldots, n-1\}$ and $B \subseteq \{0, 1, \ldots, m-1\}$ we let $M_{A,B}$ denote the $|B| \times |A|$–submatrix of M which represents

$r_B \circ M \circ i_A$. For $f \in \mathbb{C}^A$, we use the now customary notation $\|f\|_0 = |\operatorname{supp} f|$ where $\operatorname{supp} f = \{a \in A : f(a) \neq 0\}$. Clearly, $\|\cdot\|_0$ is not a norm.

Throughout this paper, G denotes a finite Abelian group. The identity element of G is denoted by e or by 0 in case that G is cyclic, in other words, if $G = \mathbb{Z}_n$ for some $n \in \mathbb{N}$. The dual group of characters $\widehat{G}$ of G is the set of continuous homomorphisms $\xi \in \mathbb{C}^G$ which map G into the multiplicative group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ [Ben97, Kat76, Ter99]. The set $\widehat{G}$ is an Abelian group under pointwise multiplication and, as is customary, we shall write this commutative group operation additively. Note that $G \cong \widehat{G}$ as groups and Pontryagin duality implies that $\widehat{\widehat{G}}$ can be canonically identified with G, a fact which is emphasized by writing $\langle \xi, x\rangle$ for $\xi(x)$.

The Fourier transform $\mathcal{F}f = \widehat{f} \in \mathbb{C}^{\widehat{G}}$ of $f \in \mathbb{C}^G$ is given by

$$\widehat{f}(\xi) = \sum_{x\in G} f(x)\,\overline{\xi(x)} = \sum_{x\in G} f(x)\,\overline{\langle \xi, x\rangle}, \quad \xi \in \widehat{G}\,.$$

The inversion formula for the Fourier transformation allows us to reconstruct the original function from its Fourier transform. Namely, for $f \in \mathbb{C}^G$ we have

$$f(x) = \tfrac{1}{|G|} \sum_{\xi\in\widehat{G}} \widehat{f}(\xi)\,\langle \xi, x\rangle, \quad x \in G\,.$$

The inversion formula implies that

$$\|f\|_2^2 = \tfrac{1}{|G|} \sum_{\xi\in\widehat{G}} |\widehat{f}(\xi)|^2 = \tfrac{1}{|G|}\|\widehat{f}\|_2^2, \tag{1}$$

where $\|f\|_2 := (\sum_{t\in G} |f(t)|^2)^{\frac{1}{2}}$. Further, (1) together with $\|\xi\|_2 = |G|^{\frac{1}{2}}$ for all $\xi \in \widehat{G}$ implies that the normalized characters in $\{|G|^{-\frac{1}{2}}\xi\}_{\xi\in\widehat{G}}$ form an orthonormal basis for $\mathbb{C}^G$, and $\sum_x \langle \xi, x\rangle = 0$ if $\xi \neq 0$ and $\sum_\xi \langle \xi, x\rangle = 0$ if $x \neq 0$.

For $n \in \mathbb{N}$ and $\omega = e^{2\pi i/n}$, the *discrete Fourier matrix* $W_{\mathbb{Z}_n}$ of the cyclic group $\mathbb{Z}_n$ is defined by $W_{\mathbb{Z}_n} = (\omega^{rs})_{r,s=0}^{n-1}$. Identifying $\mathbb{C}^{\mathbb{Z}_n}$ with $\mathbb{C}^n$, we have $\widehat{f} = W_{\mathbb{Z}_n} f$.

An arbitrary finite Abelian group G can be represented as a direct product of cyclic groups $G \cong \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \ldots \times \mathbb{Z}_{d_m}$ where $d_1, \ldots, d_m$ can be chosen to be powers of prime numbers. A character in the dual group $\widehat{G}$ is then given by

$$\langle (\xi_1, \xi_2, \ldots, \xi_m), (x_1, x_2, \ldots, x_m)\rangle = \langle \xi_1, x_1\rangle \langle \xi_2, x_2\rangle \ldots \langle \xi_m, x_m\rangle\,,$$

where $(\xi_1, \xi_2, \ldots, \xi_m) \in \widehat{\mathbb{Z}}_{d_1} \times \widehat{\mathbb{Z}}_{d_2} \times \ldots \times \widehat{\mathbb{Z}}_{d_m} \cong \widehat{G}$. The discrete Fourier matrix W_G for $G = \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \ldots \times \mathbb{Z}_{d_m}$ is the Kronecker product of the Fourier matrices for the groups $\mathbb{Z}_{d_1}, \mathbb{Z}_{d_2}, \ldots, \mathbb{Z}_{d_m}$, that is, $W_G = W_{d_1} \otimes W_{d_2} \otimes \ldots \otimes W_{d_m}$. For example, we have

$$W_{\mathbb{Z}_4} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \quad \text{and} \quad W_{\mathbb{Z}_2\times\mathbb{Z}_2} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Note that for appropriately chosen bijections $S_1 : \{0,1,\dots,|G|-1\} \longrightarrow G$ and $S_2 : \{0,1,\dots,|G|-1\} \longrightarrow \widehat{G}$ we have $\widehat{f}\circ S_2 = W_G(f\circ S_1)$ for $f\in\mathbb{C}^G$.

The translation operator $T_x, x\in G$ is the unitary operator on $\mathbb{C}^G$ given by $T_x f(y) = f(y-x)$, $y\in G$. Similarly, the modulation operator $M_\xi, \xi\in\widehat{G}$ is the unitary operator defined by $M_\xi f = f\cdot\xi$, where here and in the following $f\cdot g$ denotes the pointwise product of $f,g\in\mathbb{C}^G$. We have $\widehat{M_\xi f} = T_\xi\widehat{f}$. We refer to the unitary operators $\pi(\lambda) = M_\xi\circ T_x$ for $\lambda = (x,\xi)\in G\times\widehat{G}$ as time-frequency shift operators.

The *short–time Fourier transformation* $V_g : \mathbb{C}^G \longrightarrow \mathbb{C}^{G\times\widehat{G}}$ with respect to the window $g\in\mathbb{C}^G\backslash\{0\}$ is given by [FK98, FKL07, Grö01, Grö03]

$$V_g f(x,\xi) = \langle f, \pi(x,\xi)g\rangle = \sum_{y\in G} f(y)\overline{g(y-x)\langle\xi,y\rangle}, \quad f\in\mathbb{C}^G, \quad (x,\xi)\in G\times\widehat{G},$$

The inversion formula for the short–time Fourier transform is

$$f(y) = \frac{1}{|G|\,\|g\|_2^2}\sum_{(x,\xi)\in G\times\widehat{G}} V_g f(x,\xi)\, g(y-x)\langle\xi,y\rangle\,, \quad y\in G, \tag{2}$$

that is, f can be composed of time–frequency shifted copies of any $g\in\mathbb{C}^G\backslash\{0\}$. Further, $\|V_g f\|_2 = \sqrt{|G|}\,\|f\|_2\|g\|_2$. This equation resembles (1), but the so-called Gabor system $\{\pi(x,\xi)g\}_{(x,\xi)\in G\times\widehat{G}}$ is clearly not an orthonormal basis if $|G|>1$ since it consists of $|G|^2$ vectors in a $|G|$ dimensional space.

For a given group G, we shall use again the previously chosen enumerations $S_2 : \{0,1,\dots,|G|-1\} \longrightarrow \widehat{G}$ and $S_1 : \{0,1,\dots,|G|-1\} \longrightarrow G$ which relate the Fourier transform to the Fourier matrix W_G. For $g\in\mathbb{C}^G$ and $x\in G$, we define the $|G|\times|G|$–diagonal matrix

$$D_{x,g} = \begin{pmatrix} g(S_1(0)+x) & & & 0 \\ & g(S_1(1)+x) & & \\ & & \ddots & \\ 0 & & & g(S_1(|G|-1)+x) \end{pmatrix}.$$

Then, the $|G|\times|G|^2$–*full Gabor system matrix* with respect to g is given by

$$A_{G,g} = (D_{S_1(0),g}W_G \,|\, D_{S_1(1),g}W_G \,|\, \cdots \,|\, D_{S_1(|G|-1),g}W_G)^*, \tag{3}$$

where M^* denotes the adjoint of the matrix M. For example, for $G=\mathbb{Z}_4$,

$$A_{\mathbb{Z}_4,(1,2,3,4)} := \left(\begin{array}{rrrr|rrrr|rrrr|rrrr} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\ 2 & 2i & -2 & -2i & 3 & 3i & -3 & -3i & 4 & 4i & -4 & -4i & 1 & i & -1 & -i \\ 3 & -3 & 3 & -3 & 4 & -4 & 4 & -4 & 1 & -1 & 1 & -1 & 2 & -2 & 2 & -2 \\ 4 & -4i & -4 & 4i & 1 & -i & -1 & i & 2 & -2i & -2 & 2i & 3 & -3i & -3 & 3i \end{array}\right)^*.$$

Similarly, for the group $G=\mathbb{Z}_2\times\mathbb{Z}_2$ we have

$$A_{\mathbb{Z}_2\times\mathbb{Z}_2,(1,2,3,4)} := \left(\begin{array}{rrrr|rrrr|rrrr|rrrr} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\ 2 & -2 & 2 & -2 & 1 & -1 & 1 & -1 & 4 & -4 & 4 & -4 & 3 & -3 & 3 & -3 \\ 3 & 3 & -3 & -3 & 4 & 4 & -4 & -4 & 1 & 1 & -1 & -1 & 2 & 2 & -2 & -2 \\ 4 & -4 & -4 & 4 & 3 & -3 & -3 & 3 & 2 & -2 & -2 & 2 & 1 & -1 & -1 & 1 \end{array}\right)^*. \tag{4}$$

Using the enumeration $S : \{0, 1, \ldots, |G|^2-1\} \longrightarrow G\times\widehat{G}$ that is given by the lexicographic order on $G\times\widehat{G}$ induced by S_1 and S_2, we have $V_g f \circ S = A_{G,g} f$. Therefore, we shall refer to $A_{G,g}$ as short–time Fourier transform matrix with respect to the window g. Clearly, the rows of $A_{G,g}$ represent the vectors in the Gabor system $\{\pi(\lambda)g\}_{\lambda\in G\times\widehat{G}}$, and formula 2 implies that $A^*_{G,g}A_{G,g}$ is a multiple of the identity matrix.

3 Uncertainty principles for the Fourier transform on finite Abelian groups

The following uncertainty theorem for functions defined on finite Abelian groups is the natural starting point for our discussion [DS89].

Theorem 3.1 *Let $f \in \mathbb{C}^G\backslash\{0\}$, then $\|f\|_0 \cdot \|\widehat{f}\|_0 \geq |G|$.*

Proof. For $f \in \mathbb{C}^G$, $f \neq 0$, and without loss of generality $\|\widehat{f}\|_\infty = 1$, we compute

$$\begin{aligned} |G| &= |G|\|\widehat{f}\|_\infty^2 \leq |G|\left(\sum_{x\in G}|f(x)|\right)^2 \leq |G|\|f\|_0\sum_{x\in G}|f(x)|^2 \\ &= |G|\|f\|_0\frac{1}{|G|}\sum_{\xi\in\widehat{G}}|\widehat{f}(\xi)|^2 \leq \|f\|_0\|\widehat{f}\|_0\|\widehat{f}\|_\infty^2 = \|f\|_0\,\|\widehat{f}\|_0. \end{aligned}$$

□

A complementary result characterizes those f for which the bound in Theorem 3.1 is sharp [DS89, Smi90, MÖP04].

Proposition 3.2

1. *If k divides $|G|$, then there exists $f \in \mathbb{C}^G$ with $\|f\|_0 = k$ and $\|\widehat{f}\|_0 = \frac{|G|}{k}$.*
2. *If $\|f\|_0\|\widehat{f}\|_0 = |G|$ and $e \in \operatorname{supp} f$, then $\operatorname{supp} f$ is a subgroup of G.*

A generalization of Theorem 3.1 to non Abelian groups is given in [Mes92] and those f achieving the respective lower bounds are described in [Kan07].

3.1 Groups of prime order

The geometric mean of two positive numbers is dominated by their arithmetic mean; hence, Theorem 3.1 implies the weaker inequality

$$\|f\|_0 + \|\widehat{f}\|_0 \geq 2\sqrt{|G|}. \tag{5}$$

If $|G|$ is prime, that is, if G is a cyclic group of prime order, then (5) and also Theorem 3.1 can be improved significantly [Fre04, Tao05].

Theorem 3.3 *Let $G = \mathbb{Z}_p$ with p prime. Then $\|f\|_0 + \|\widehat{f}\|_0 \geq |G|+1$ holds for all $f \in \mathbb{C}^G \backslash \{0\}$.*

As illustrated in [Tao05], Theorem 3.3 follows from combining Chebotarev's theorem on roots of unity which states that every minor of the Fourier transform matrix $W_{\mathbb{Z}_p}$, p prime, is nonzero [EI76, SL96, Tao05, Fre04] with

Proposition 3.4 *Let $M \in \mathbb{C}^{m\times n}$. Then $\|f\|_0 + \|Mf\|_0 \geq m+1$ for all $f \in \mathbb{C}^n$ if and only if every minor of M is nonzero. Moreover, if every minor of $M \in \mathbb{C}^{m\times n}$ is nonzero and k, l are given with $k + l \geq m+1$, then there exists $f \in \mathbb{C}^n$ with $\|f\|_0 = k$ and $\|Mf\|_0 = l$.*

Proposition 3.4 in turn can be obtained from the following observation which will also be used in numerical experiments below.

Lemma 3.5 *For $M \in \mathbb{C}^{m\times n}$ and $1 \leq k \leq m$, $1 \leq l \leq n$, there exists $f \in \mathbb{C}^n$ with $\|f\|_0 = k$ and $\|Mf\|_0 = l$ if and only if there exist sets $A \subseteq \{0, \dots, n-1\}$ and $B \subseteq \{0, \dots, m-1\}$ with $|A| = k$, $|B| = m - l$, and for all $a \in A$ and $y \in B^c$, we have*

$$\operatorname{rank} M_{A\backslash\{a\},B} \ = \ \operatorname{rank} M_{A,B} \ = \ \operatorname{rank} M_{A,B\cup\{y\}} - 1 \ < \ |A| \,. \tag{6}$$

Proof. If $f \in \mathbb{C}^n$ with $\|f\|_0 = k$ and $\|Mf\|_0 = l$, then $A = \operatorname{supp} f$ and $B^c = \operatorname{supp} Mf$ satisfy $0 \neq r_A f \in \ker M_{A,B}$, so $\operatorname{rank} M_{A,B} < |A|$. Moreover, for $a \in A$, $\operatorname{supp} f = A$ implies $f \notin \{g : \|g\|_0 < |A|\} \supset i_{A\backslash\{a\}} \ker M_{A\backslash\{a\},B}$ and, hence,

$$f \in i_A \ker M_{A,B} \backslash i_{A\backslash\{a\}} \ker M_{A\backslash\{a\},B} \,.$$

So $\dim\ker M_{A,B} \geq \dim\ker M_{A\backslash\{a\},B} + 1$. We conclude that for all $a \in A$,

$$\begin{aligned} \operatorname{rank} M_{A\backslash\{a\},B} &\leq \operatorname{rank} M_{A,B} = |A| - \dim\ker M_{A,B} \\ &\leq |A| - \dim\ker M_{A\backslash\{a\},B} - 1 \\ &= \operatorname{rank} M_{A\backslash\{a\},B} \end{aligned}$$

which implies $\operatorname{rank} M_{A\backslash\{a\},B} = \operatorname{rank} M_{A,B}$. Also, $\operatorname{supp} Mf = B^c$, so for $y \in B^c$, $Mf(y) \neq 0$. Therefore, $f \notin \ker M_{A,B\cup\{y\}}$and so $f \in i_A \ker M_{A,B} \backslash i_A \ker M_{A,B\cup\{y\}}$. This implies

$$\operatorname{rank} M_{A,B} \ = \ |A| - \dim\ker M_{A,B} < |A| - \dim\ker M_{A,B\cup\{y\}} = \operatorname{rank} M_{A,B\cup\{y\}} \,.$$

The submatrices considered differ only by one column, so the rank can increase at most by one and we get $\operatorname{rank} M_{A,B} \ = \ \operatorname{rank} M_{A,B\cup\{y\}} - 1$.

Suppose now that $A \subseteq \{0, \dots, n-1\}$ and $B \subseteq \{0, \dots, m-1\}$ with $|A| = k$ and $|B| = m - l$ satisfy (6). This implies $\dim\ker M_{A,B} \geq 1$ and that for any $a \in A$,

$$\begin{aligned} \dim\ker M_{A\backslash\{a\},B} &= |A| - 1 - \operatorname{rank} M_{A\backslash\{a\},B} \\ &= |A| - 1 - \operatorname{rank} M_{A,B} \\ &= \dim\ker M_{A,B} - 1. \end{aligned}$$

So $i_{A\setminus\{a\}}\ker M_{A\setminus\{a\},B} \subsetneqq i_A \ker M_{A,B}$, and consequently, there exists some vector $f_a \in i_A \ker M_{A,B} \setminus i_{A\setminus\{a\}} \ker M_{A\setminus\{a\},B}$, so $f_a(a) \neq 0$, $f_a(x) = 0$ for $x \notin A$ and $\operatorname{supp} Mf_a \cap B = \varnothing$.

Similarly, (6) implies also that for any $y \in B^c$ we have $i_A \ker M_{A,B\cup\{y\}} \subsetneqq i_A \ker M_{A,B}$, so there exists g_y with $\operatorname{supp} g_y \subseteq A$ such that $Mg_y(y) \neq 0$ while $Mg_y(b) = 0$ for all $b \in B$.

To conclude this proof, we enumerate the vectors f_a, $a \in A$ and g_y, $y \in B^c$ and choose a linear combination

$$f = \sum_{a\in A} c_a f_a + \sum_{y\in B^c} c_y g_y = \sum_{r=0}^{k+l-1} d_r h_r \tag{7}$$

with the property that $\operatorname{supp} f = \bigcup_{a\in A} \operatorname{supp} f_a = A$ and $\operatorname{supp} Mf = \bigcup_{y\in B^c} \operatorname{supp} Mg_y = B^c$.

By construction we have $\operatorname{supp} f \subseteq A$ and $\operatorname{supp} Mf \subseteq B^c$. To get the reverse inequality, we assume without loss of generality that $\min_{x\in\operatorname{supp} h_r} |h_r(x)| = 1$ for all r, and choose $d_r = N^{2r}$, where $N-1 \geq \|h_r\|_\infty, \|Mh_r\|_\infty, \|Mh_r\|_\infty^{-1}$ for $r = 0, 1, \ldots, k+l-1$. Since $f_{a_0}(a_0) \neq 0$ we can find $s = \max\{r : \ h_r(a_0) \neq 0\}$. Then

$$\begin{aligned}
|f(a_0)| &= \Big|\sum_{r=0}^{s} d_r h_r(a_0)\Big| \\
&\geq |N^{2s} h_s(a_0)| - \Big|\sum_{r=0}^{s-1} N^{2r} h_r(a_0)\Big| \\
&\geq N^{2s} - (N-1)\sum_{r=0}^{s-1} (N^2)^r \\
&= N^{2s} - \frac{N^{2s}-1}{N+1} > 0,
\end{aligned}$$

so $a_0 \in \operatorname{supp} f$.

Similarly, $Mg_{y_0}(y_0) \neq 0$ for fixed $y_0 \in B^c$ implies that for $s = \max\{r : Mh_r(y_0) \neq 0\}$ we have

$$\begin{aligned}
|Mf(y_0)| &= \Big|\sum_{r=0}^{s} d_r Mh_r(y_0)\Big| \\
&\geq |N^{2s} Mh_s(y_0)| - \Big|\sum_{r=0}^{s-1} N^{2r} Mh_r(y_0)\Big| \\
&\geq \frac{N^{2s}}{N-1} - \frac{N^{2s}-1}{N+1} > 0.
\end{aligned}$$

We conclude that $\operatorname{supp} f = A$ and $\operatorname{supp} Mf = B^c$.

□

Proof of Proposition 3.4. If f has no zero minors, then (6) in Lemma 3.5 is equivalent to $|B| < |A|$, implying that there exists $f \in \mathbb{C}^n$ with $\|f\|_0 = k$ and $\|Mf\|_0 = l$ if and only if $k + l \geq m+1$.

It remains to show that $\|f\|_0 + \|Mf\|_0 \geq m+1$ for all f implies that M has no zero minors. To this end, assume that there is a $d \times d$ submatrix $M_{A,B}$ of M with $\det M_{A,B} = 0$. Then there exists a nonzero vector $f' \in \mathbb{C}^A$ such that $M_{A,B}f' = 0$. For $f = i_A f'$, $\|Mf\|_0 \leq m-d$ and therefore $\|f\|_0+\|Mf\|_0 \leq d+m-d = m < m+1$. □

Theorem 3.3 is a clear improvement to Theorem 3.1 but it applies only to cyclic groups of prime order since any other finite Abelian group G has proper subgroups leading to zero minors in W_G [MÖP04]. As example, we display in Table 1 counts on the ranks of square submatrices of $W_{\mathbb{Z}_5}$ and $W_{\mathbb{Z}_6}$. See [CR06] for estimates on the probability that for randomly chosen sets $T \subseteq G$ and $\Omega \subseteq \widehat{G}$ with $|T| + |\Omega| \leq G$ there exists $f \in \mathbb{C}^G$ with $\operatorname{supp} f = T$ and $\operatorname{supp} \widehat{f} = \Omega$. Due to their role in obtaining Theorem 3.3, we shall now collect facts regarding zero and nonzero minors of Fourier matrices in general.

	1	2	3	4	5
1	25	0	0	0	0
2	0	100	0	0	0
3	0	0	100	0	0
4	0	0	0	25	0
5	0	0	0	0	1

	1	2	3	4	5	6
1	36	36	0	0	0	0
2	0	189	48	0	0	0
3	0	0	352	36	0	0
4	0	0	0	189	0	0
5	0	0	0	0	36	0
6	0	0	0	0	0	1

Table 1: Counts of square submatrices of $W_{\mathbb{Z}_5}$ and $W_{\mathbb{Z}_6}$ with given size (column index) and rank (row index).

Let $M \in \mathbb{C}^{n \times n}$ and let $A, B \subset \{1, 2, \ldots, n\}$ such that $|A| = |B|$. Then $\det M_{A,B}$ defines a minor of M, and $\det M_{A^c,B^c}$ is called its *complementary minor.*

Proposition 3.6

1. *The complementary minor of any zero minor in a Fourier matrix W_G is also zero.*

2. *Let $d_0 > 1$ be the smallest divisor of $|G|$. Then for all $d_0 \leq r \leq n-d_0$, there exists an $r \times r$ zero minor of the Fourier matrix W_G. In particular, if $|G|$ is even, then there exist $r \times r$ zero minor for $r = 2, 3, \ldots, |G|-2$.*

3. *Any minor of the Fourier matrix $W_{\mathbb{Z}_n}$, $n \in \mathbb{N}$, that contains only adjacent rows or columns is nonzero.*

Proof. *1.* The adjugate of a matrix $M = (m_{kl})$ is adj $M = (M_{kl})$, where $M_{kl} = (-1)^{k+l} \det M_{\{k\}^c,\{l\}^c}$ is the cofactor of the element m_{kl}. Then for any sets $A, B \subset \{1, 2, \ldots n\}$ of cardinality r, a theorem by Jacobi (see [Pra94]) states that

$$\det M_{A,B} = (-1)^r \det(\text{adj } M)_{A^c,B^c} \cdot (\det M)^{r-1} \tag{8}$$

Furthermore, adj $M \cdot M = \det M \cdot I$ by Cramer's rule.

For any zero minor of $M = W_G$ on the left hand side of (8), Jacobi's theorem implies that the right hand side, representing a minor in adj W_G, is zero as well. Since $W_G \cdot \overline{W_G} = |G| \cdot I$, we have adj $W_G = \frac{\det(W_G)}{|G|} \cdot \overline{W_G}$. Thus the corresponding minor in $\overline{W_G}$ is zero, which implies that also the corresponding minor in W_G is zero.

2. Let d divide $|G|$. Part *1* in Proposition 3.2 allows us to choose f_d such that $\|f_d\|_0 = d$ and $\|\widehat{f_d}\|_0 = \frac{|G|}{d}$. Hence, for any r with $d \leq r \leq |G| - \frac{|G|}{d}$ we can pick sets $A \supseteq \operatorname{supp} f_d$ and $B \subseteq (\operatorname{supp} \widehat{f_d})^c$ such that $|A| = |B| = r$. Then $r_A f_d \in \ker M_{A,B}$ and the $r \times r$-minor $\det M_{A,B}$ is zero.

This way, we obtain $r \times r$ zero minors for $d_0 \leq r \leq \frac{|G|}{d_0}(d_0 - 1)$ and for $\frac{|G|}{d_0} \leq r \leq |G| - d_0$, where d_0 is the smallest nontrivial divisor of $|G|$. The result follows since $d_0 - 1 \geq 1$.

3. A minor with adjacent columns is a determinant of the type

$$\begin{aligned}
\det \begin{pmatrix} \omega^{k_1 l} & \omega^{k_1(l+1)} & \ldots & \omega^{k_1(l+m)} \\ \omega^{k_2 l} & \omega^{k_2(l+1)} & \ldots & \omega^{k_2(l+m)} \\ \vdots & \vdots & \ldots & \vdots \\ \omega^{k_m l} & \omega^{k_m(l+1)} & \ldots & \omega^{k_m(j+m)} \end{pmatrix} &= \omega^{k_1 l + k_2 l + \cdots + k_m l} \det \begin{pmatrix} 1 & \omega^{k_1} & \cdots & \omega^{mk_1} \\ 1 & \omega^{k_2} & \cdots & \omega^{mk_2} \\ \vdots & \vdots & \ldots & \vdots \\ 1 & \omega^{k_m} & \cdots & \omega^{mk_m} \end{pmatrix} \\
&= \omega^{k_1 l + k_2 l + \cdots + k_m l} \prod_{i<j\leq m} (\omega^{k_j} - \omega^{k_i}) \neq 0
\end{aligned}$$

The second determinant was evaluated using the formula for Vandermonde determinants and the result does not equal 0, as always $0 < k_j - k_i < n$ and ω is a primitive n-th root of unity. □

3.2 Groups of non-prime order

Meshulam improved the bound in the classical uncertainty relation presented in Theorem 3.1 for nontrivial finite Abelian groups of nonprime order [Mes06]. He defined for $0 < k \leq |G|$ the function

$$\theta(G, k) = \min \left\{ \|\widehat{f}\|_0 : \ f \in \mathbb{C}^G \text{ and } 0 < \|f\|_0 \leq k \right\}.$$

Using this notation, Theorem 3.3 implies that $\theta(\mathbb{Z}_p, k) = p - k + 1$. The main result in [Mes06] is

Theorem 3.7 *For $k \leq |G|$, let d_1 be the largest divisor of $|G|$ which is less than or equal to k and let d_2 be the smallest divisor of $|G|$ which is larger than or equal to k. Then*

$$\theta(G,k) \geq \frac{|G|}{d_1 d_2}(d_1 + d_2 - k). \tag{9}$$

Tao realized that this theorem simply states that all possible lattice points $(\|f\|_0, \|\widehat{f}\|_0)$ lie in the convex hull of the points $(|H|, |G/H|)$, where H ranges over all subgroups of G [Mes06]. To see this, recall that for any divisor d of $|G|$ exists a subgroup H of G with $d = |H|$. Furthermore, the right hand side of expression (9) is linear between two successive divisors and the slope is increasing when k increases. Hence (9) characterizes the convex hull of the points $(|H|, |G|/|H|)$. Proposition 3.2, part *1*, implies that the vertex points $(|H|, |G|/|H|)$ are attained, but little more is known about the set $\{(\|f\|_0, \|\widehat{f}\|_0),\ f \in \mathbb{C}^G\}$.

The proof of Theorem 3.7 in [Mes06] is inductive and uses three facts: first, it uses Theorem 3.3 as induction seed, and second, it uses the submultiplicativity of the right hand side of (9). That is, if we denote this right hand side by $u(n,k)$ for $n = |G|$, then it uses that $u(n,k) \leq u(\frac{n}{d}, t)u(d,s)$ for d dividing n and $st \leq k$. The third ingredient is

Proposition 3.8 *Let H be a subgroup of G. For $k \leq |G|$ there exist $s \leq |H|$, $t \leq |G/H|$ with $st \leq k$ and*

$$\theta(G,k) \geq \theta(H,s)\,\theta(G/H,t)\,.$$

Meshulam's proof of Proposition 3.8 relies on algebraic notation and does not give good insight from the point of view of Fourier analysis. For this reason, and for completeness sake, we give a streamlined version of Meshulam's proof of Proposition 3.8. See also [LM05] for an elegant and non-inductive proof of Theorem 3.7.

But first, note that if $G \cong H \times G/H$, then Proposition 3.8 can be proven using the fact that then $\widehat{G} \cong \widehat{H} \times \widehat{G/H}$, and, therefore, $\widehat{f}$ can be calculated by performing two partial Fourier transforms. For example, such argument can be applied to $G = \mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$, $\gcd(m,n) = 1$, and $H = \mathbb{Z}_m \times \{e\}$. Even simpler is the special case discussed in Proposition 3.9. We state and prove this result to illustrate the main idea used to prove Proposition 3.8.

We can use the following tensor argument to motivate the upcoming proposition. Let $G \simeq H_1 \times H_2$. Let $f \in \mathbb{C}^G$ be the tensor product $f_1 \otimes f_2, f_1 \in \mathbb{C}^{H_1}, f_2 \in \mathbb{C}^{H_2}$. Then

$$\|f\|_0 = \|f_1\|_0 \cdot \|f_2\|_0.$$

The Fourier transform of f is computed in the following way,

$$\begin{aligned}
\widehat{f}(\xi_1,\xi_2) &= \sum_{x_1\in H_1, x_2\in H_2} f_1(x_1)f_2(x_2)\overline{\langle(\xi_1,\xi_2),(x_1,x_2)\rangle} \\
&= \sum_{x_1\in H_1, x_2\in H_2} f_1(x_1)f_2(x_2)\overline{\langle\xi_1,x_1\rangle\langle\xi_2,x_2\rangle} \\
&= \sum_{x_1\in H_1} f_1(x_1)\overline{\langle\xi_1,x_1\rangle} \sum_{x_2\in H_2} f_2(x_2)\overline{\langle\xi_2,x_2\rangle} \\
&= \widehat{f_1}(\xi_1)\cdot\widehat{f_2}(\xi_2)
\end{aligned}$$

Thus $\widehat{f}=\widehat{f_1}\otimes\widehat{f_2}$. Therefore,

$$\|\widehat{f}\|_0 = \|\widehat{f_1}\|_0\cdot\|\widehat{f_2}\|_0.$$

Hence, for product groups we can show existence of admissible pairs $(\|f\|_0, \|\widehat{f}\|_0)$ by looking at the building blocks of the group.

Proposition 3.9 *Let $A_1\subseteq G_1$ and $A_2\subseteq G_2$ and $f\in\mathbb{C}^{G_1\times G_2}$ be given with* $\operatorname{supp} f\subseteq A_1\times A_2$. *Then* $\|\widehat{f}\|_0 \geq \theta(G_1,|A_1|)\,\theta(G_2,|A_2|)$.

Proof. We picture f as a $|G_1|\times|G_2|$ matrix and note that $\operatorname{supp} f\subseteq A_1\times A_2$ implies that f has exactly $|G_2\backslash A_2|$ zero columns and $|A_2|$ columns with at least $|G_1\backslash A_1|$ zeros.

The function $\mathcal{F}_1 f$ is obtained by applying the G_1–Fourier transformation to each column. Hence, $\mathcal{F}_1 f$ has $|G_2\backslash A_2|$ zero columns and, at most, $|G_1|-\theta(G_1,|A_1|)$ zeros in the remaining A_2 columns. It is easy to see that in the scenario which leads to the weakest bound for $\|\widehat{f}\|_0$, we have $|G_1|-\theta(G_1,|A_1|)$ zeros in each of these $|A_2|$ columns and that they are lined up to form $|G_1|-\theta(G_1,|A_1|)$ zero rows in $\mathcal{F}_1 f$. In this case, the remaining $\theta(G_1,|A_1|)$ rows contain exactly $|G_2\backslash A_2|$ zeros, that is, $|A_2|$ nonzero elements.

Now, we calculate $\mathcal{F}f$ by taking a G_2–Fourier transform along each row of $\mathcal{F}_1 f$. As a result, $|G_1|-\theta(G_1,|A_1|)$ zero rows remain, and in the other $\theta(G_1,|A_1|)$ rows, at least $\theta(G_2,|A_2|)$ zeros are present. We conclude that

$$\|\widehat{f}\|_0 \;\geq\; \theta(G_1,|A_1|)\,\theta(G_2,|A_2|).$$

□

The property that the $G=G_1\times G_2$–Fourier transformation "splits" into a G_1–Fourier transformation and a G_2–Fourier transformation is the basis of the simple proof of Proposition 3.9. In the proof of Proposition 3.8 we shall see that the general case follows from small adjustments to the arguments used to prove Proposition 3.9.

Proof of Proposition 3.8. Let $H=\{x_i\}$ be a subgroup of G and, abusing notation, we let $\{x_j\}$ be a set of coset representatives of the quotient group G/H.

Then each element in G has a unique representation as x_i+x_j. We let $H^\perp$ denote the characters $\{\xi_j \in \widehat{G} : \xi_j(H) = 1\}$. $H^\perp$ is a subgroup of $\widehat{G}$, and we denote by $\{\xi_i\}$ a set of coset representatives of the quotient group $\widehat{G}/H^\perp$. Every element $\xi \in \widehat{G}$ has a unique decomposition as $\xi_i+\xi_j$.

The Pontryagin duality theorem implies $\widehat{G}/H^\perp \cong \widehat{H}$. This allows us to assign a character $\xi_i' \in \widehat{H}$ to each $\xi_i \in \widehat{G}/H^\perp$ with $\xi_{i_1}'+\xi_{i_2}' = (\xi_{i_1}+\xi_{i_2})'$ [Kat76].[1] Further, $\langle \xi_i, x_i\rangle_G = \langle \xi_i', x_i\rangle_H$ for all $x_i \in H$ and all $\xi_i \in \widehat{G}/H^\perp$. Similarly, we use $\widehat{G/H} \cong H^\perp$ to assign to each ξ_j an element $\xi_j' \in \widehat{G/H}$ with $\langle \xi_j, x_j\rangle_G = \langle \xi_j', x_j+H\rangle_{G/H}$ for all x_j.

For $f \in \mathbb{C}^G$ and any $\xi = \xi_i+\xi_j \in \widehat{G}$, we calculate

$$\begin{aligned}
\widehat{f}(\xi) = \widehat{f}(\xi_i+\xi_j) &= \sum_{x_j}\sum_{x_i} f(x_i+x_j)\overline{\langle \xi_i+\xi_j, x_i+x_j\rangle}_G \\
&= \sum_{x_j}\sum_{x_i} f(x_i+x_j)\overline{\langle \xi_i, x_i\rangle}_G\overline{\langle \xi_i, x_j\rangle}_G\overline{\langle \xi_j, x_i\rangle}_G\overline{\langle \xi_j, x_j\rangle}_G \\
&= \sum_{x_j}\left(\sum_{x_i} f(x_i+x_j)\overline{\langle \xi_i', x_i\rangle}_H\right)\overline{\langle \xi_i, x_j\rangle}_G \;\; \overline{\langle \xi_j', x_j+H\rangle}_{G/H}
\end{aligned}$$

where the last equality follows since $\xi_j \in H^\perp$ implies $\langle \xi_j, x_i\rangle_G = 1$.

We set $f_1(\xi_i', x_j) := \sum_{x_i\in H} f(x_i+x_j)\overline{\langle \xi_i', x_i\rangle}_H$, which, for fixed x_j, is the H–Fourier transform $\mathcal{F}_H$ on the coset x_j+H in G, and $f_2(\xi_i', x_j) = f_1(\xi_i', x_j)\overline{\langle \xi_i, x_j\rangle}_G$. Further f_1 and f_2 have the same support sets. We summarize that $\widehat{f}$ can be obtained from f via two partial Fourier transformations and an enclosed unitary multiplication operator, as illustrated in Figure 1.

Let us now fix $f \in \mathbb{C}^G$ with $\|f\|_0 \le k$ and $\|\widehat{f}\|_0 = \theta(G,k)$.

Let $t := |\{x_j : \operatorname{supp} f \cap (x_j+H) \ne \varnothing\}|$. Note that the support of f contains at most k elements which are distributed among t cosets of H. Hence, there must be a coset $x_{j_0}+H$ which contains $s' \le s = \lfloor \frac{k}{t} \rfloor$ elements of $\operatorname{supp} f$. Therefore,

$$\|f_2(\cdot, x_{j_0})\|_0 = \|\mathcal{F}_H f(\cdot + x_{j_0})\|_0 \ge \theta(H, s') \ge \theta(H,s)$$

This implies that $\Xi = \{\xi_i \in \widehat{G}/H^\perp : f_2(\xi_i', \cdot) \not\equiv 0\}$ satisfies $|\Xi| \ge \theta(H,s)$. In fact, the definition of t implies that for $\xi_i \in \Xi$, we have $0 < \operatorname{supp} f_2(\xi_i', \cdot) \le t$. We conclude

$$\theta(G,k) = \|\widehat{f}\|_0 = \sum_{\xi_i} \|\mathcal{F}_{G/H} f_2(\xi_i', \cdot)\|_0 \ge \sum_{\xi_i\in\Xi} \|\mathcal{F}_{G/H} f_2(\xi_i', \cdot)\|_0 \ge \theta(H,s)\theta(G/H,t).$$

□

In the following, we discuss the question whether the inequality (9) in Theorem 3.7 is sharp, or, more precisely, we shall check whether for some given Abelian

[1] In particular, in the case $G = \mathbb{Z}_{mn}$, $\gcd(m,n) = 1$, $\mathbb{Z}_m^\perp \cong \mathbb{Z}_n$ and $\mathbb{Z}_m^\perp \cong \mathbb{Z}_n$.

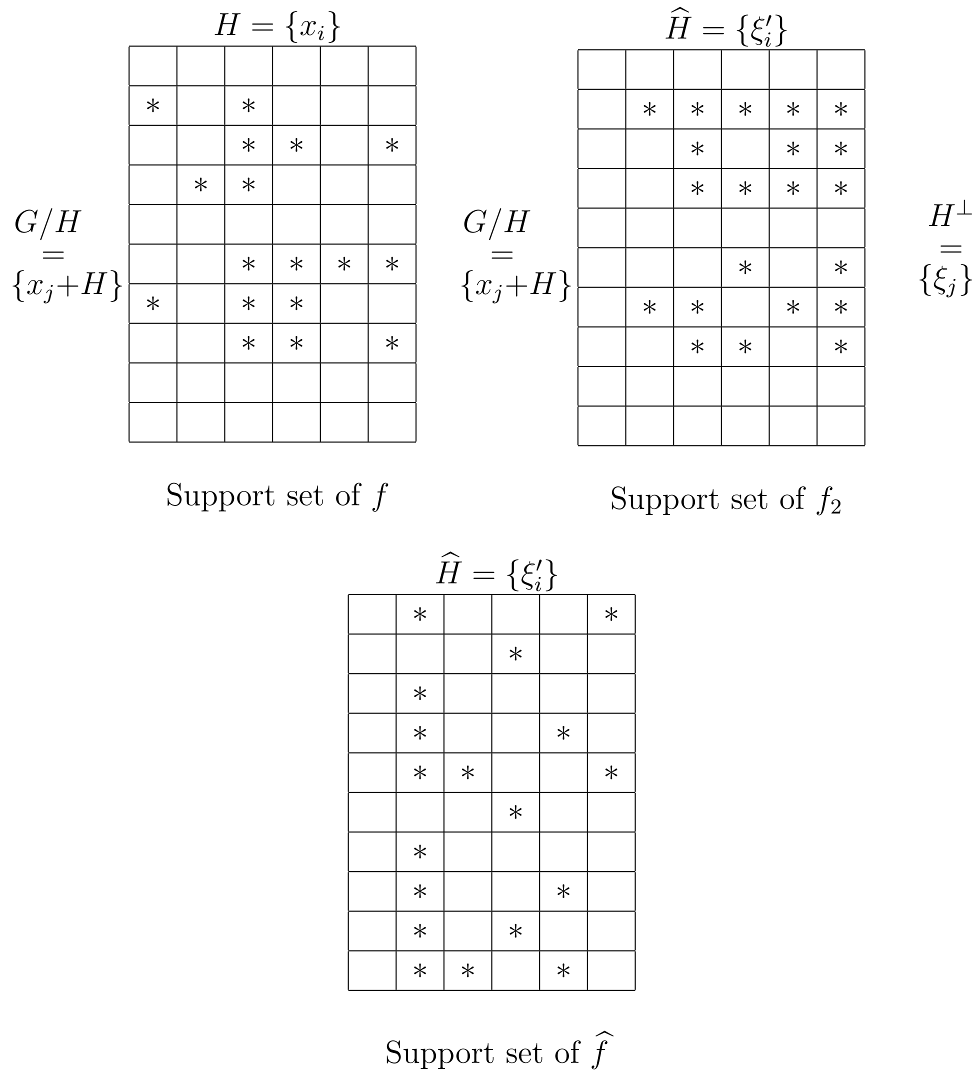


Figure 1: Illustration of the proof of Proposition 3.8 for $G = \mathbb{Z}_{10} \times \mathbb{Z}_6$ and $k = 17$. The function f_2 is obtained by the application of H–Fourier transformations to the rows of f which is succeeded by an unitary multiplication operator . To calculate $\widehat{f}$ we apply G/H–Fourier transformations to the columns of f_2. For clarity, we choose synthetic support sets of f, f_2, and $\widehat{f}$. Here $t = 6$ and $s = \lfloor \frac{17}{6} \rfloor = 2$.

group G and (k, l) chosen with $l \geq \theta(G, k) \geq \frac{|G|}{d_1 d_2}(d_1 + d_2 - k)$ there exists a function $f \in \mathbb{C}^G$ with $\|f\|_0 = k$ and $\|\widehat{f}\|_0 = l$. This question has been considered earlier in [FKLM05] where the set $\big\{(\|f\|_0, \|\widehat{f}\|_0),\ f \in G\big\}$ has been described for $G = \mathbb{Z}_6$ and $G = \mathbb{Z}_8$.

First, we state an affirmative positive result for cyclic groups. It follows from Example 5.6 in [Smi90] and the proof of Proposition 4.5 in [Kut03].

Proposition 3.10 *Let $G = \mathbb{Z}_n$, $n \in \mathbb{N}$. If $0 < k, l \leq |G|$ satisfy $l + k \geq |G| + 1$, then there exists a function $f \in \mathbb{C}^G$ with $\|f\|_0 = k$ and $\|\widehat{f}\|_0 = l$.*

However, it does not hold for other types of groups, as the following example for $\mathbb{Z}_2 \times \mathbb{Z}_2$ shows, We show that for $f \in \mathbb{C}^{\mathbb{Z}_2 \times \mathbb{Z}_2}$ with $\|f\|_0 = 3$, $\|\widehat{f}\|_0 \neq 2$. Without loss of generality we assume $f = (a, b, c, 0)^T$.

$$\widehat{f} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} f = \begin{pmatrix} a+b+c \\ a-b+c \\ a+b-c \\ a-b-c \end{pmatrix},$$

which we can rewrite as a matrix

$$\widehat{f} = \begin{pmatrix} a+b+c & a-b+c \\ a+b-c & a-b-c \end{pmatrix}$$

If the two zero entries are in the same row or column, then either $b = 0$ or $c = 0$. If they are in one of the two diagonals, then $c \pm b = 0$, implying $a = 0$.

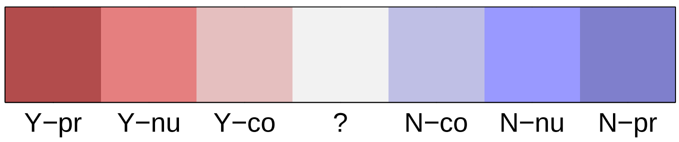
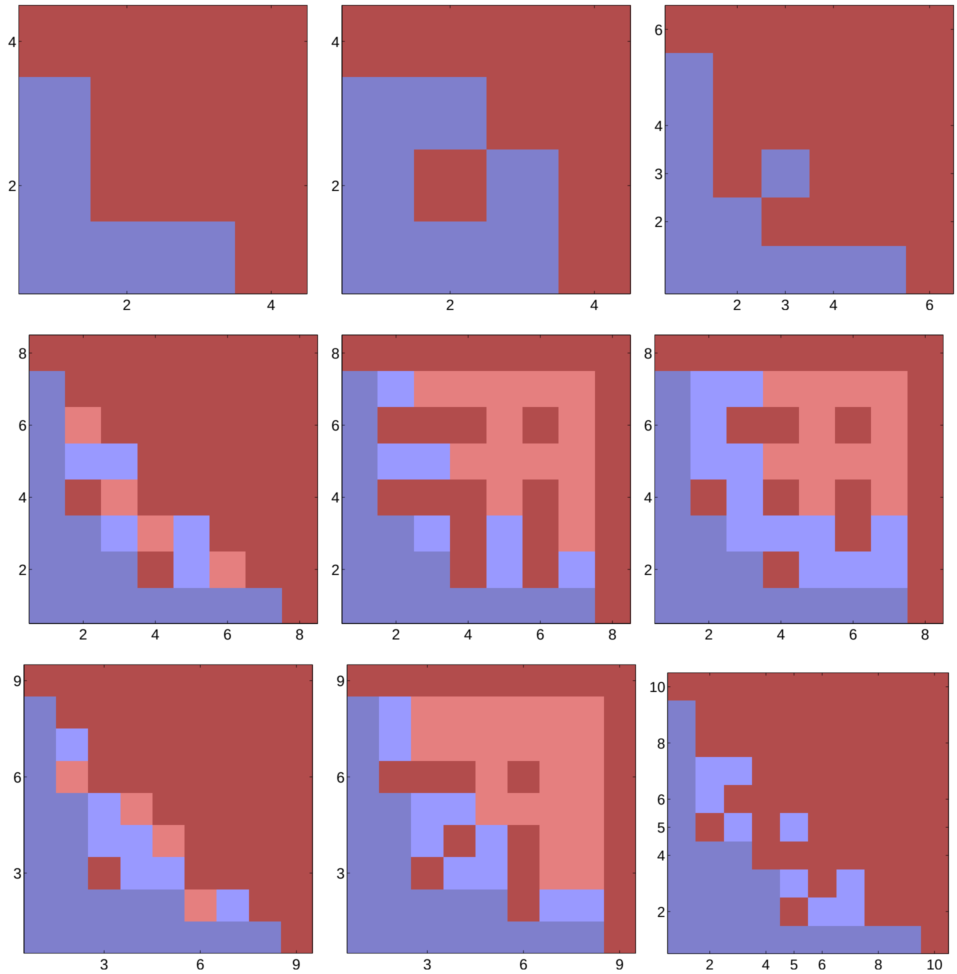


Figure 2: Color coding which is used in Figures 3–7 to describe subsets of $\mathbb{N}^2$ or $\mathbb{N}^3$. The color determines whether a given value is in the set under discussion. Y-pr indicates that there is proof that the corresponding value is in the set considered. Y-nu implies that there is numerical evidence that the value is in the set and Y-co indicates that we conjecture that the value is in the set. N-pr indicates that there is proof that the corresponding value is not in the set, and N-nu and N-co are defined accordingly. The color adjacent to ? implies that no judgement is made here.

The numerical results collected in Figure 3 are based on Lemma 3.5 and they show that the set of all possible pairs $(\|f\|_0, \|\widehat{f}\|_0)$ is not easily described in general. The computations needed to obtain Figure 3 are quite involved. For example, the computations showing that there is no vector on $\mathbb{Z}_{16}$ with five nonzero entries and whose Fourier transform has nine nonzero entries include the calculation of the singular values of $\binom{16}{5}\binom{16}{7} = 49969920$ five by seven matrices.

In addition, we give all possible pairs $(\|f\|_0, \|\widehat{f}\|_0)$ for the group $G = \mathbb{Z}_6$ and give a partial result for the groups $G = \mathbb{Z}_{2p}$ for $p \geq 5$ prime.

Proposition 3.11 *For $1 \leq k, l \leq 6$ exists $f \in \mathbb{C}^{\mathbb{Z}_6}$ with $\|f\|_0 = k$ and $\|\widehat{f}\|_0 = l$ if and only if $kl \geq 6$ and $(k, l) \neq (3, 3)$.*



Proof. Theorem 3.1 and Proposition 3.10 cover all cases except those for which $(k,l) \in \{(2,4),(3,3),(4,2)\}$. For $\omega = e^{2\pi i/6}$, we have $\mathcal{F}(1,-1,0,1,-1,0)^T = (0,0,1-\omega^2,0,1-\omega^4,0)^T$, and only the case $(k,l)=(3,3)$ remains to be excluded.

The assumption $\|f\|_0 = 3$ leads to three different cases.

Case 1. If $f=(c_0,0,c_2,0,c_4,0)^T$ then $\widehat{f}(\xi)=\widehat{f}(\xi+3)$ and if $f=(0,c_1,0,c_3,0,c_5)^T$ then $\widehat{f}(\xi)=-\widehat{f}(\xi+3)$. In either case, $\|\widehat{f}\|_0$ is even and cannot be 3.

Case 2. If two entries whose indices differ by 3 are both nonzero, then the support of the Fourier transform cannot be 3 either. To see this, consider without loss of generality, $f=(c_0,*,*,c_3,*,*)^T$. Then, for c_k, located at position k, being the

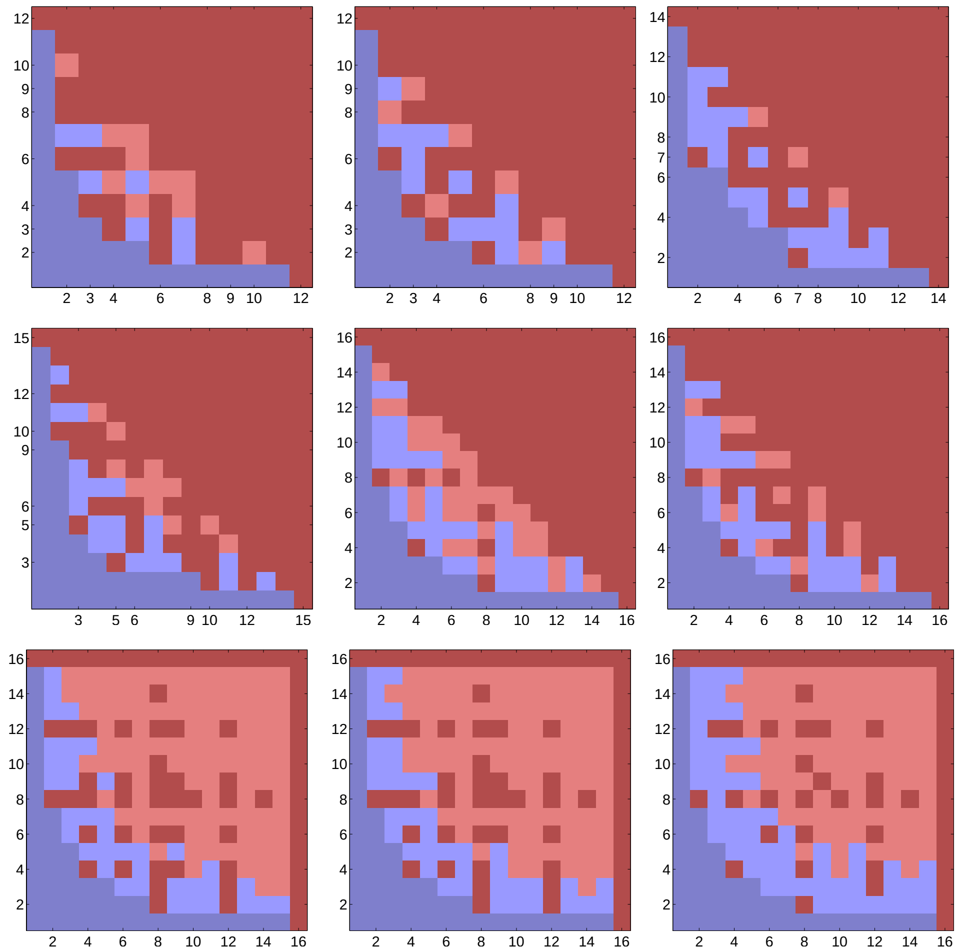


Figure 3: The set $\{(\|f\|_0, \|\widehat{f}\|_0),\ f \in \mathbb{C}^G \backslash \{0\}\}$ for the groups (from left to right) $\mathbb{Z}_4$, $\mathbb{Z}_2^2$, $\mathbb{Z}_6$, $\mathbb{Z}_8$; $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2^3$, $\mathbb{Z}_9$, $\mathbb{Z}_3^2$; $\mathbb{Z}_{10}$, $\mathbb{Z}_{12}$, $\mathbb{Z}_2 \times \mathbb{Z}_6$, $\mathbb{Z}_{14}$; $\mathbb{Z}_{15}$, $\mathbb{Z}_{16}$, $\mathbb{Z}_2 \times \mathbb{Z}_8$; $\mathbb{Z}_4^2$, $\mathbb{Z}_2^2 \times \mathbb{Z}_4$, $\mathbb{Z}_2^4$. The color code used is described in Figure 2. The graphs are based on the results in Section 3.

third nonzero entry, we have

$$\widehat{f} = \begin{pmatrix} c_0 + c_3 + c_k \\ c_0 - c_3 + \omega^k c_k, \\ c_0 + c_3 + \omega^{2k} c_k \\ c_0 - c_3 + \omega^{3k} c_k \\ c_0 + c_3 + \omega^{4k} c_k \\ c_0 - c_3 + \omega^{5k} c_k \end{pmatrix} \tag{10}$$

If three coordinates of $\widehat{f}$ are 0, then two of the respective sums in (10) contain either both $c_0 + c_3$ or both $c_0 - c_3$. Without loss of generality, we assume that $\widehat{f}(l_1) = c_0 + c_3 + \omega^{l_1 k} c_k \neq 0 \neq c_0 + c_3 + \omega^{l_2 k} c_k = \widehat{f}(l_2)$, $l_1 < l_2$. Since $c_k \neq 0$ we have

$\omega^{l_1 k} = \omega^{l_2 k}$ and $\omega^{(l_2-l_1)k} = 1$. Since $k = 1, 2, 4$ or 5, we must have 3 divides $l_1 - l_2$, but that is a contradiction, as of two entries with distance 3, one must contain the summand $c_3 - c_0$ and one $c_0 + c_3$.

Case 3. If all three nonzero entries are adjacent, then $\widehat{f}$ must have three adjacent entries as well, as otherwise, we could just exchange the roles of f and $\widehat{f}$ and return to Case 1 or Case 2. Without loss of generality we assume $f = (c_0, c_1, c_2, 0, 0, 0)$. A modulation in f results in a translation in $\widehat{f}$, so without loss of generality, we can also assume the first three entries of $\widehat{f}$ to be 0. Hence,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & -\omega \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = 0 \quad \text{but} \quad \det \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & -\omega \end{pmatrix} = -1 \neq 0$$

and, therefore, $f = 0$. □

The following result for $\mathbb{Z}_{2p}$, $p \geq 5$ prime, shows that the bound in Theorem 3.7 is not sharp, a fact that was observed for the case $G = \mathbb{Z}_8$ in [FKLM05].

Proposition 3.12 *For $p \geq 5$ prime there exists no $f \in \mathbb{C}^{\mathbb{Z}_{2p}}$ with $\|f\|_0 = 3$ and $\|\widehat{f}\|_0 = p-1$.*

Proof. The group $\mathbb{Z}_{pq}$ has $(p-1)(q-1)$ automorphisms, each of them mapping one of the $(p-1)(q-1)$ elements of order pq to 1. The $p-1$ automorphisms on the group $\mathbb{Z}_{2p} = \{0, 1, 2, \ldots, 2p-1\}$ will allow us to consider only f with well-"concentrated" nonzero entries.

Every automorphism σ on $\mathbb{Z}_{pq}$ induces an automorphism $\tilde{\sigma}$ on the character group $\widehat{\mathbb{Z}}_{pq}$, which satisfies $\langle \tilde{\sigma}(\xi), x \rangle = \langle \xi, \sigma^{-1}(x) \rangle$. Further,

$$\begin{aligned} \widehat{f \circ \sigma}(\xi) &= \frac{1}{pq} \sum_{x \in \mathbb{Z}_{pq}} f(\sigma(x)) \overline{\langle \xi, x \rangle} \\ &= \frac{1}{pq} \sum_{y \in \mathbb{Z}_{pq}} f(y) \overline{\langle \xi, \sigma^{-1}(y) \rangle} \\ &= \frac{1}{pq} \sum_{y \in \mathbb{Z}_{pq}} f(y) \overline{\langle \tilde{\sigma}(\xi), y \rangle} \\ &= \widehat{f}(\tilde{\sigma}(\xi)) \end{aligned}$$

Let $f \in \mathbb{C}^{\mathbb{Z}_{2p}}$, $p \geq 5$ prime, be given with $\|f\|_0 = 3$. Then at least two of the addresses of the non-zero elements have the same parity. By a translation of f we can move those elements to positions $0, 2k$, where $k \in \mathbb{Z}_{2p}$. The support of $\widehat{f}$ is not affected by this. If k is odd, then k is a generator of $\mathbb{Z}_{2p}$ and we choose σ_1 with $\sigma_1(k) = 1$. If k is even, then $p + k$ is odd and we pick σ_1 with $\sigma_1(p + k) = 1$. In either case $\sigma_1(2k) = 2$. The corresponding automorphism $\tilde{\sigma}_1$ in $\widehat{\mathbb{Z}}_{2p}$ will affect $\operatorname{supp} \widehat{f}$, but $\|\widehat{f}\|_0$ does not change.

Let the third non-zero element have address r. If $\sigma_1(r) \neq p{+}1$, then there are either $p-1$ adjacent zeroes among the addresses $3, \ldots, p+1$ or among $p+1, \ldots, 2p-1$.

In case that $\sigma_1(r) = p{+}1$, then we apply another automorphism σ_2 in a similar way as above. If $\frac{p+1}{2}$ is a generator for $\mathbb{Z}_{2p}$, then $\sigma_2(\frac{p+1}{2}) = 1$, $\sigma_2(2) = \sigma_2(4\frac{p+1}{2}) = 4\sigma_2(\frac{p+1}{2}) = 4$, and $\sigma_2(p+1) = \sigma_2(2\frac{p+1}{2}) = 2\sigma_2(\frac{p+1}{2}) = 2$. Otherwise, we choose σ_2 such that $\sigma_2(p{+}\frac{p+1}{2}) = 1$, so $\sigma_2(p{+}1) = 2\sigma_2(p{+}\frac{p+1}{2}) = 2$ and $\sigma_2(2) = 2\sigma_2(p{+}1) = 4$. In both cases, $\operatorname{supp}(f{\circ}\sigma_2{\circ}\sigma_1) = \{0, 2, 4\}$, so the vector contains a string of at least $p-1$ consecutive zeros on addresses $5, \ldots, 2p-1$.

The following lemma from [DS89] implies that $\|\widehat{f{\circ}\sigma'{\circ}\sigma}\|_0 > p{-}1$ and, therefore, $\|\widehat{f}\|_0 \geq p$.

Lemma 3.13 *If $\widehat{f}$ has N nonzero elements, then f cannot have N consecutive zeros.*

□

4 Uncertainty principles for short–time Fourier transforms on finite Abelian groups

We now turn to discuss minimum support conditions on time-frequency representations of elements in $\mathbb{C}^G$, in particular, for the short–time Fourier transform of a function $f \in \mathbb{C}^G$ with respect to a window $g \in \mathbb{C}^G$. For background on uncertainty principles in joint time-frequency representations see [Grö03, HL05].

But first, we consider the simplest joint time–frequency representation of f which is given by the tensor product $f{\otimes}\widehat{f}$. Similarly, in electrical engineering the so-called Rihaczek distribution $R : G{\times}\widehat{G} \longrightarrow \mathbb{C}$ given by $Rf(x,\omega) = f(x)\overline{\widehat{f}(\omega)}\,\overline{\langle \omega, x\rangle}$ is considered. Theorem 3.1 implies that $\|Rf\|_0 = \|f{\otimes}\widehat{f}\|_0 = \|f\|_0\|\widehat{f}\|_0 \geq |G|$. Figure 4 lists all possible pairs $(\|f\|_0, \|Rf\|_0)$ for $f \in \mathbb{C}^{\mathbb{Z}_4}$ and $f \in \mathbb{C}^{\mathbb{Z}_2^2}$.

The following result resembles Theorem 3.1. It is given for functions on the real line as so-called weak uncertainty principle in [Grö03].

Proposition 4.1 $\|V_g f\|_0 \geq |G|$ *for* $f, g \in \mathbb{C}^G\backslash\{0\}$ *with equality for* $f = g = \delta$.

Proof. Clearly $\|V_\delta \delta\|_0 = |G|$. For $f, g \in \mathbb{C}^G\backslash\{0\}$,

$$|G|\ \|f\|_2^2\ \|g\|_2^2 = \|V_g f\|_2^2 \leq \|V_g f\|_0\ \|V_g f\|_\infty^2 \leq \|V_g f\|_0\ \|f\|_2^2\ \|g\|_2^2$$

and the result follows. □

We shall now seek lower bounds on $\|V_g f\|_0$ depending on $\|f\|_0$, $\|\widehat{f}\|_0$, $\|g\|_0$, and $\|\widehat{g}\|_0$.

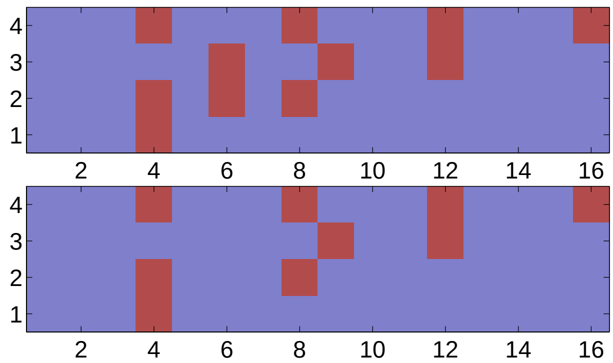


Figure 4: For groups $G = \mathbb{Z}_4$ and $\mathbb{Z}_2^2$ all possible pairs $(\|f\|_0, \|Rf\|_0)$ are colored red, those pairs that are not achieved by some $f \in \mathbb{C}^G$ are colored blue in accordance with the color code given in Figure 2.

Proposition 4.2 *For $f, g \in \mathbb{C}^G \backslash \{0\}$, we have*

$$\|V_g f\|_0 \geq \max\{\ \theta(G, \|g\|_0)\, \theta(G, \|\widehat{f}\|_0)\, ,\, \theta(G, \|f\|_0)\, \theta(G, \|\widehat{g}\|_0)\ \}\, , \qquad (11)$$

and, therefore,

$$\|V_g f\|_0 \geq \tfrac{1}{2} \Big(\theta(G, \|g\|_0)\, \theta(G, \|\widehat{f}\|_0) + \theta(G, \|f\|_0)\, \theta(G, \|\widehat{g}\|_0) \Big)\, , \qquad (12)$$

and

$$\|V_g f\|_0 \geq \sqrt{\theta(G, \|f\|_0)\, \theta(G, \|\widehat{f}\|_0)\, \theta(G, \|g\|_0)\, \theta(G, \|\widehat{g}\|_0)}\, . \qquad (13)$$

Proof. We shall prove $\|V_g f\|_0 \geq \theta(G, \|f\|_0)\theta(\widehat{G}, \|\widehat{g}\|_0)$. Then (11) follows from $\|V_g f\|_0 = \|V_{\widehat{g}} \widehat{f}\|_0$ and $\theta(G, k) = \theta(\widehat{G}, k)$ for any k, or, alternatively from $\|V_g f\|_0 = \|V_f g\|_0$. Further, (11) implies (12) and (13) since the maximum of two positive numbers dominates their arithmetic and geometric means.

To see (11), observe first that the so-called symplectic Fourier transformation $\mathcal{F}_s = R \circ \mathcal{F}_{\widehat{G}}^{-1} \circ \mathcal{F}_G$, that is, the composition of a Fourier transformation $\mathcal{F}_G$ on G, an inverse Fourier transformation $\mathcal{F}_{\widehat{G}}^{-1}$ on $\widehat{G}$, and the axis transformation $R : F \mapsto F \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ obeys the same uncertainty principle as the Fourier transformation

		1	2	2	2	2	3	3	3	3	4	4	4	4	4	5	5	5	5	5	6	6	6	6	6	6
		6	3	4	5	6	2	4	5	6	2	3	4	5	6	2	3	4	5	6	1	2	3	4	5	6
1	6	6	24	18	12	6	30	18	12	6	30	24	18	12	6	30	24	18	12	6	36	30	24	18	12	6
2	3	24	20	20	20	20	25	16	16	16	25	20	15	12	12	25	20	15	10	8	30	25	20	15	10	5
2	4	18	20	15	15	15	25	15	12	12	25	20	15	10	9	25	20	15	10	6	30	25	20	15	10	5
2	5	12	20	15	10	10	25	15	10	8	25	20	15	10	6	25	20	15	10	5	30	25	20	15	10	5
2	6	6	20	15	10	5	25	15	10	5	25	20	15	10	5	25	20	15	10	5	30	25	20	15	10	5
3	4	18	16	15	15	15	20	12	12	12	20	16	12	9	9	20	16	12	8	6	24	20	16	12	8	4
3	5	12	16	12	10	10	20	12	8	8	20	16	12	8	6	20	16	12	8	4	24	20	16	12	8	4
3	6	6	16	12	8	5	20	12	8	4	20	16	12	8	4	20	16	12	8	4	24	20	16	12	8	4
4	4	18	15	15	15	15	15	12	12	12	15	12	9	9	9	15	12	9	6	6	18	15	12	9	6	3
4	5	12	12	10	10	10	15	9	8	8	15	12	9	6	6	15	12	9	6	4	18	15	12	9	6	3
4	6	6	12	9	6	5	15	9	6	4	15	12	9	6	3	15	12	9	6	3	18	15	12	9	6	3
5	5	12	10	10	10	10	10	8	8	8	10	8	6	6	6	10	8	6	4	4	12	10	8	6	4	2
5	6	6	8	6	5	5	10	6	4	4	10	8	6	4	3	10	8	6	4	2	12	10	8	6	4	2
6	6	6	5	5	5	5	5	4	4	4	5	4	3	3	3	5	4	3	2	2	6	5	4	3	2	1

Table 2: Numerical representation of (11) for $G = \mathbb{Z}_6$. Rows represent possible pairs $(\|f\|_0, \|\widehat{f}\|_0)$, columns possible pairs $(\|g\|_0, \|\widehat{g}\|_0)$, and the table entries give the lower bound on $\|V_g\|_0$.

on the group $G\times\widehat{G}$. For $f, g \in \mathbb{C}^G$, we calculate

$$\begin{aligned}
\mathcal{F}_s V_g f(r,\rho) &= \sum_{x\in G}\sum_{\xi\in\widehat{G}} V_g f(x,\xi)\overline{\langle\rho,x\rangle}\langle\xi,r\rangle \\
&= \sum_{x\in G}\sum_{\xi\in\widehat{G}}\sum_{t\in G} f(t)\overline{g(t-x)}\,\overline{\langle\xi,t\rangle}\;\overline{\langle\rho,x\rangle}\langle\xi,r\rangle \\
&= \sum_{x\in G}\sum_{t\in G} f(t)\overline{g(t-x)}\,\overline{\langle\rho,x\rangle}\sum_{\xi\in\widehat{G}}\langle\xi,r-t\rangle \\
&= |G|\sum_{x\in G} f(r)\overline{g(r-x)}\,\overline{\langle\rho,x\rangle} \\
&= |G|\overline{\langle\rho,r\rangle}f(r)\overline{\widehat{g}(\rho)}
\end{aligned}$$

and note that $\operatorname{supp}\mathcal{F}_s V_g f = \operatorname{supp} f\times\operatorname{supp}\widehat{g}$. Proposition 3.9 implies that $\|V_g f\|_0 = \|\mathcal{F}_s^{-1}\big(\mathcal{F}_s V_g f\big)\|_0 \geq \theta(G,\|f\|_0)\theta(\widehat{G},\|\widehat{g}\|_0)$. □

For $G = \mathbb{Z}_6$, we list in Table 2 the lower bounds on $\|V_g f\|_0$ given by (11) for different values of $\|f\|_0$, $\|\widehat{f}\|_0$, $\|g\|_0$ and $\|\widehat{g}\|_0$.

Corollary 4.3 *For $f, g \in \mathbb{C}^{\mathbb{Z}_p}\backslash\{0\}$, p prime,*

$$\|V_g f\|_0 \geq \max\{\ (p+1-\|g\|_0)(p+1-\|\widehat{f}\|_0)\,,\ (p+1-\|f\|_0)(p+1-\|\widehat{g}\|_0)\ \}$$

and $$\|V_g f\|_0 \geq (p+1)^2 - \tfrac{1}{2}(p+1)(\|f\|_0+\|\widehat{f}\|_0+\|g\|_0+\|\widehat{g}\|_0) + \tfrac{1}{2}\Big(\|\widehat{f}\|_0\|g\|_0 + \|f\|_0\|\widehat{g}\|_0\Big).$$

Now, we give an improvement to the lower bound on $\|V_g f\|_0$ that is given in Corollary 4.3.

Proposition 4.4 *Let $G = \mathbb{Z}_p$, p prime. For $f, g \in \mathbb{C}^G\backslash\{0\}$,*

$$\|V_g f\|_0 \geq \begin{cases} |G|(|G|+1) - \|f\|_0\|g\|_0 & \text{if } \|f\|_0 + \|g\|_0 > |G|; \\ |G|(|G|+1) - (|G|+1-\|f\|_0)(|G|+1-\|g\|_0) & \text{if } \|f\|_0 + \|g\|_0 \leq |G|. \end{cases}$$

Proof. Note that for all $x \in G$, $V_g f(x, \cdot) = \langle f, \pi(x, \cdot) g \rangle$ represents the Fourier transform of a vector of the form $f \cdot T_x \bar{g}$, that is,

$$V_g f(x, \xi) = \langle f, \pi(x, \xi) g \rangle = \sum_{y \in G} f(y) \overline{g(y-x) \langle \xi, y \rangle} = \widehat{f \cdot T_x \bar{g}}(\xi) \quad x \in G, \xi \in \widehat{G}\,.$$

As long as $f \cdot T_x \bar{g} \neq 0$, Theorem 3.3 applies and so $\|f \cdot T_x \bar{g}\|_0 + \|\widehat{f \cdot T_x \bar{g}}\|_0 \geq |G| + 1$. For $K := \{x : f \cdot T_x \bar{g} \neq 0\}$ we get

$$\|V_g f\|_0 = \sum_{x \in K} \|\widehat{f \cdot T_x \bar{g}}\|_0 \;\geq\; |K|(|G|+1) - \sum_{x \in G} \|f \cdot T_x \bar{g}\|_0 = |K|(|G|+1) - \|f\|_0 \|g\|_0,$$

where $\sum_{x \in G} \|f \cdot T_x \bar{g}\|_0 = \|f\|_0 \|g\|_0$ follows from a simple counting argument.

We shall now estimate $|K|$ using the Cauchy-Davenport inequality, which states that for non-empty subsets A and B of $G = \mathbb{Z}_p$, p prime, $|A+B| \geq \min(|A|+|B|-1, |G|)$, where $A + B = \{a + b : a \in A, b \in B\}$ [Kár05]. Now $K = \{x : f \cdot T_x \bar{g} \neq 0\} = \{x : \{(\operatorname{supp} \bar{g}) + x\} \cap \operatorname{supp} f \neq \varnothing\} = \operatorname{supp} f - \operatorname{supp} \bar{g}$. We set $A = \operatorname{supp} f$, $B = \operatorname{supp} \bar{g}$, and obtain $|K| = |\operatorname{supp} f - \operatorname{supp} \bar{g}| \geq \min(\|f\|_0 + \|g\|_0 - 1, |G|)$.

If $\|f\|_0 + \|g\|_0 \geq |G| + 1$, then $|K| = |G|$ and, hence, $\|V_g f\|_0 \geq |G|(|G|+1) - \|f\|_0 \|g\|_0$. If $\|f\|_0 + \|g\|_0 \leq |G|$, then $|K| \geq \|f\|_0 + \|g\|_0 - 1$ and so

$$\begin{aligned} &\|V_g f\|_0 \geq (\|f\|_0 + \|g\|_0 - 1)(|G|+1) - \|f\|_0 \|g\|_0 \\ = \;& |G|(|G|+1) - (|G| + 1 - \|f\|_0)(|G| + 1 - \|g\|_0)\,. \end{aligned}$$

□

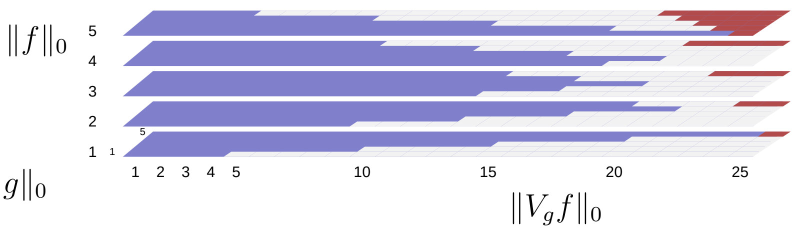


Figure 5: The set $\{(\|f\|_0, \|g\|_0, \|V_g f\|_0),\ f, g \in \mathbb{C}^G \backslash \{0\}\}$ for $G = \mathbb{Z}_5$. The color code used is described in Figure 2. The graphs are based on Proposition 4.4 and Theorem 4.5.

The lower bound on $\|V_g f\|_0$ given in Proposition 4.4 is illustrated for $G = \mathbb{Z}_5$ in Table 5. To establish results similar to Proposition 3.11 for the short–time Fourier transformations for a given group G analytically is quite tedious since it requires to check all combinations of $\|f\|_0$ and $\|g\|_0$. For the case $G = \mathbb{Z}_3$, however, we have assembled all possible and impossible combinations in Figure 6. A derivation of the entries can be found in the appendix.

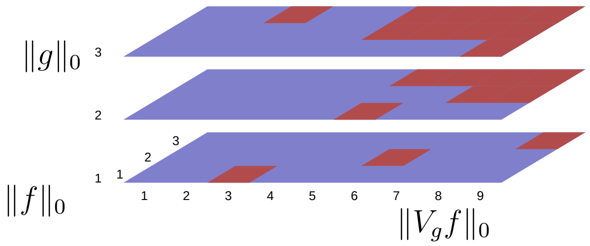


Figure 6: Same as Figure 5 for $G = \mathbb{Z}_3$.

4.1 Groups of prime order

In the following, we shall fix the window g and vary only the analyzed function f. First we provide a short–time Fourier transform version of Theorem 3.3.

Theorem 4.5 *Let $G = \mathbb{Z}_p$, p prime. For almost every $g \in \mathbb{C}^G$, we have*

$$\|f\|_0 + \|V_g f\|_0 \geq |G|^2 + 1 \tag{14}$$

for all $f \in \mathbb{C}^G \setminus \{0\}$. Moreover, for $1 \leq k \leq |G|$ and $1 \leq l \leq |G|^2$ with $k + l \geq |G|^2 + 1$ there exists f with $\|f\|_0 = k$ and $\|V_g f\|_0 = l$.

We picture this result for $G = \mathbb{Z}_5$ and $G = \mathbb{Z}_7$ in Figure 7. Note that Theorem 4.5 follows from Proposition 3.4 together with Theorem 4 from [LPW05] which we state as

Theorem 4.6 *For almost every $g \in \mathbb{C}^{\mathbb{Z}_p}$, p prime, we have that every minor of $A_{\mathbb{Z}_p,g}$ is nonzero.*

Outline of a proof of Theorem 4.6. It suffices to show that each square submatrix $(A_{\mathbb{Z}_p,g})_{A,B}$ has determinant nonzero for almost every g.

To this end, choose $A \subseteq \mathbb{Z}_p$ and $B \subseteq \mathbb{Z}_p \times \widehat{\mathbb{Z}_p}$ with $|A| = |B|$ and set $P_{A,B}(z) = \det(A_{\mathbb{Z}_p,z})_{A,B}$, $z = (z_0, z_1, \ldots, z_{p-1})$. To show that $P_{A,B} \neq 0$, we shall locate a term in the polynomial in standard form which has a nonzero coefficient. To construct this term, we determine first the maximal possible exponent of z_0 in one of the terms of P that are not trivially zero. Next, we determine the maximal exponent that z_1 can have in a monomial where the maximal exponent of z_0 is attained and so on.

Using generalized Vandermonde determinants, it can then be shown that the coefficient of this "maximal" term within $P_{A,B}$ can be expressed as a product of different minors of the discrete Fourier matrix $W_{\mathbb{Z}_p}$. For p prime, all these minors are nonzero, so the polynomial P has a nonzero coefficient for this "maximal term", hence is not identically 0, and nonzero almost everywhere. We have $P = \prod_{A,B:\,|B|=|A|} P_{A,B} \not\equiv 0$, which implies that for $g \notin Z_P = \{z : \; P(z) = 0\}$, every minor of $A_{\mathbb{Z}_p,g}$ is nonzero. Since $P \not\equiv 0$, Z_P has Lebesgue measure 0. $\square$

Clearly, this proof of Theorem 4.6 is based on Chebotarev's Theorem on roots of unity. Also, Chebotarev's Theorem on roots of unity and therefore Theorem 3.3 can be obtained as a corollary to Theorem 4.6 as shown in the Appendix.

It is easy to see that if $g \in \mathbb{C}^{\mathbb{Z}_p}$ satisfies (14) then $\|g\|_0 = \|\widehat{g}\|_0 = p$, that is, $g(x) \neq 0$ for all $x \in \mathbb{Z}_p$ and $\widehat{g}(\xi) \neq 0$ for all $\xi \in \widehat{\mathbb{Z}}_p$ [LPW05]. In addition, we have

Proposition 4.7 *There exists a unimodular $g \in \mathbb{C}^{\mathbb{Z}_p}$, p prime, that is, a g with $|g(x)| = 1$ for all $x \in G$ satisfying the conclusions of Theorem 4.5.*

Proof. Theorem 4.6 implies that all minors of $A_{\mathbb{Z}_p,g}$ are nonzero polynomials in the polynomial ring $\mathbb{C}[z_0, ..., z_{n-1}]$. Let P be the product of all these minor polynomials, which, by assumption, is nonzero. We have to show that $P(g) \neq 0$ for some $g \in \mathbb{C}^{\mathbb{Z}_p}$ with $|g(x)| = 1$ for all $x \in \mathbb{Z}_p$.

This follows since the only polynomial P with $P(g) = 0$ whenever $|g(x)| = 1$ for all $x \in \mathbb{Z}_p$ is trivial, that is, $P \equiv 0$, which we show below using induction over the number of variables n.

The case $n = 1$ follows since any nonzero polynomial in one variable has only finitely many zeros; only $P \equiv 0$ vanishes for all $z \in S^1 = \{z : \ |z| = 1\}$. Next, we consider a polynomial P of n variables which we regard as a polynomial in z_{n-1} with coefficients in the polynomial ring $\mathbb{C}[z_0, ..., z_{n-2}]$, that is,

$$P(z_{n-1}) = Q_m(z_0, ..., z_{n-2}) z_{n-1}^m + Q_{m-1}(z_0, ..., z_{n-2}) z_{n-1}^{m-1} + \cdots + Q_0(z_0, ..., z_{n-2})$$

For any fixed $(c_0, \ldots, c_{n-2}) \in (S^1)^{n-1}$ we have

$$Q_m(c_0, ..., c_{n-2}) z_{n-1}^m + Q_{m-1}(c_0, ..., c_{n-2}) z_{n-1}^{m-1} + \cdots + Q_0(c_0, ..., c_{n-2}) = 0$$

for all $z_{n-1} \in S^1$, hence, all its coefficients $Q_k(c_0, ..., c_{n-2})$, $k = 0, \ldots, m$ vanish. In other words, we have that $Q_k \in \mathbb{C}[z_0, ..., z_{n-2}]$, $k = 0, \ldots, m$ vanish on $(S^1)^{n-1}$, which, by induction hypothesis, implies that all $Q_k \equiv 0$ and therefore $P \equiv 0$. □

Table 3 together with Lemma 3.5 show that the condition "$G = \mathbb{Z}_p$ with p prime" is necessary for the existence of $g \in \mathbb{C}^G$ satisfying (14).

Proposition 4.8 *If $|G|$ is not prime, then $A_{G,g}$ has zero minors for all $g \in \mathbb{C}^G$.*

Proof. Let $|G| = k \cdot m$, $k, m \neq 1$. We consider only $G = \mathbb{Z}_{km}$, the general case follows since the Fourier matrix W_G for any non-cyclic G is a Kronecker product of Fourier matrices of cyclic groups.

For a primitive $|G|$-th root of unity ω, we have $(\omega^k)^m = \omega^{|G|} = 1$, so the discrete Fourier matrix W_G has a 1 in its (k, m)-entry. Now the matrix given by the first $|G|$ columns of $A_{G,g}$ results from W_G by multiplying the i-th row by c_i. So the minor given by the columns 0 and k and the rows 0 and m of A is $\det \begin{pmatrix} c_0 & c_0 \\ c_m & c_m \end{pmatrix} = 0$. Hence $A_{G,g}$ has a zero minor. □

4.2 Groups of non-prime order

Recall Proposition 4.1, namely, the fact that for any G the estimates $|G| \leq \|V_g f\|_0 \leq |G|^2$ are sharp. In other words, for all G and $0 < k \leq |G|$ we have

$$\min_{g \in \mathbb{C}^G \setminus \{0\}} \min \left\{ \|V_g f\|_0 : \; f \in \mathbb{C}^G \text{ and } 0 < \|f\|_0 \leq k \right\} = |G| \,,$$

and

$$\max_{g \in \mathbb{C}^G \setminus \{0\}} \max \left\{ \|V_g f\|_0 : \; f \in \mathbb{C}^G \text{ and } 0 < \|f\|_0 \leq k \right\} = |G|^2 \,.$$

Certainly, $\|V_g f\|_0 = |G|$ is a rare event. In fact, it is reasonable to assume that $\|V_g f\|_0 = |G|^2$ for almost every pair (f, g). We shall now address the question whether for an appropriately chosen window g, we can achieve $\|V_g f\|_0 \geq l$ for some $|G| < l \leq |G|^2$ and all $f \in \mathbb{C}^G$.

	1	2	3	4	5
1	125	0	0	0	0
2	0	3000	0	0	0
3	0	0	23000	0	0
4	0	0	0	63250	0
5	0	0	0	0	53130

	1	2	3	4	5	6
1	216	216	0	0	0	0
2	0	9234	1368	0	0	0
3	0	0	141432	2106	0	0
4	0	0	0	881469	0	0
5	0	0	0	0	2261952	0
6	0	0	0	0	0	1947792

Table 3: Count of numerically computed ranks of minors of $A_{\mathbb{Z}_5,g}$ and $A_{\mathbb{Z}_6,g}$ for randomly generated g. Columns correspond to the dimension of square submatrices and rows to the rank of submatrices considered.

To this end, we define for $1 \leq k \leq |G|$,

$$\phi(G,k) := \max_{g \in \mathbb{C}^G \setminus \{0\}} \min \left\{ \|V_g f\|_0 : \; f \in \mathbb{C}^G \text{ and } 0 < \|f\|_0 \leq k \right\}. \tag{15}$$

Using this notation, Theorem 4.5 indicates that $\phi(\mathbb{Z}_p, k) = p^2 - k + 1$ for p prime. Taking max and min is justified due to the compactness of the unit ball in $\mathbb{C}^G$. In fact, we have

Proposition 4.9 *For almost every $g \in \mathbb{C}^G$, $\min\limits_{0<\|f\|_0 \leq k} \|V_g f\|_0 = \phi(G,k)$ for all $k \leq |G|$.*

In the following, we set $Q_{A,B}(z) = \det (A_{G,z})^*_{A,B} (A_{G,z})_{A,B}$, $z = (z_0, z_1, \ldots, z_{|G|-1})$, for $A \subseteq G$ and $B \subseteq G \times \widehat{G}$. $Q_{A,B}$ is a homogeneous polynomial in $z_0, z_1, \ldots, z_{|G|-1}$ of degree $2|A|$.

Lemma 4.10 *The vector $g \in \mathbb{C}^G$ satisfies $\min\limits_{0<\|f\|_0 \leq k} \|V_g f\|_0 \geq l$ if and only if $Q_{A,B}(g) \neq 0$ for all $A \subseteq G$ with $|A| = k$ and all $B \subseteq G \times \widehat{G}$ with $|B| = |G|^2 - l + 1$.*

Proof. Fix $A \subseteq G$ with $|A| = k$ and $g \in \mathbb{C}^G$. Then g satisfies $\|V_g f\|_0 \geq l$ for all f with $\operatorname{supp} f \subseteq A$ if and only if $\langle f|_A, \pi(\lambda)g|_A\rangle = \langle f, \pi(\lambda)g\rangle \neq 0$ for at least l elements $\lambda \in G\times\widehat{G}$ for all f with $\operatorname{supp} f \subseteq A$, that is, for at most $|G|^2 - l$ vectors in $\{\pi(\lambda)g\}$ we have $\langle f, \pi(\lambda)g\rangle = 0$ for $\operatorname{supp} f \subseteq A$. This is equivalent to $\{\pi(\lambda)g|_A\}_{\lambda\in B}$ spans $\mathbb{C}^A$ whenever $|B| = |G|^2 - l + 1$. That is, if and only if $\operatorname{rank}(A_{G,g})_{A,B} = |A|$ for all B with $|B| = |G|^2 - l + 1$. But this is equivalent to $Q_{A,B}(g) \neq 0$ for all $|B| = |G|^2 - l + 1$. The result follows since for each f with $\|f\|_0 \leq k$ exists $A \subseteq G$ with $|A| = k$ and $\operatorname{supp} f \subseteq A$. □

Proof of Proposition 4.9. Lemma 4.10 and $\min_{0<\|f\|_0\leq k} \|V_{g_k} f\|_0 \geq \phi(G,k)$, $k \leq |G|$, for some $g_k \in \mathbb{C}^G\setminus\{0\}$ imply that $Q_{A,B} \not\equiv 0$ for all pairs $A \subseteq G$ and $B \subseteq G\times\widehat{G}$ with $|B| = |G|^2 - \phi(G,|A|) + 1$. Hence, $Q = \prod_{A,B:\,|B|=\phi(G,|A|)+1} Q_{A,B} \not\equiv 0$. This implies that $Q(g) \neq 0$ for almost every $g \in \mathbb{C}^G$ and therefore, for almost every $g \in \mathbb{C}^G$ we have $\min_{0<\|f\|_0\leq k} \|V_g f\|_0 \geq \phi(G,k)$ for all $k \leq |G|$, from which the desired equality follows. □

To obtain bounds on $\phi(G,k)$ for groups of non-prime order, we shall follow the roadmap used in to show Theorem 3.7 [Mes06]. The proof is inductive and the induction step is based on

Proposition 4.11 *Let H be a subgroup of the finite Abelian group G. For $k \in \mathbb{N}$ exist $s,t \in \mathbb{N}$ with $st \leq k$ such that*

$$\phi(G,k) \geq \phi(H,s)\phi(G/H,t) \tag{16}$$

Proof. In the following, we express the short–time Fourier transformation for functions defined on G as two consecutive short–time Fourier transformations. We apply again the notation from the proof of Theorem 3.7, that is, $H = \{x_i\} = \{y_i\}$ and $\{x_j\} = \{y_j\}$ is a set of coset representatives of the quotient group G/H. As before $H^\perp = \{\xi_j \in \widehat{G} : \xi_j(H) = 1\}$ and $\{\xi_i\}$ is a set of coset representatives of $\widehat{G}/H^\perp$.

Set

$$\phi_H(G,k) = \max_{g_1\in\mathbb{C}^H,\ g_2\in\mathbb{C}^{G/H}} \min\left\{\|V_{g_1\otimes g_2} f\|_0 :\ f \in \mathbb{C}^G \text{ and } 0 < \|f\|_0 \leq k\right\},$$

where $g_1\otimes g_2(x_i + x_j) = g_1(x_i)g_2(x_j + H)$. Clearly $\phi(G,k) \geq \phi_H(G,k)$, so (16) follows from $\phi_H(G,k) \geq \phi(H,s)\phi(G/H,t)$, which we shall show below. First, note that a similar argument as is used in Proposition 4.9 gives that for almost every pair (g_1,g_2),

$$\phi_H(G,k) = \min_{0<\|f\|_0\leq k} \|V_{g_1\otimes g_2} f\|_0, \quad 1 \leq k \leq |G|.$$

Therefore, we can pick g_1 and g_2 so that for all possible k, s, t,

$$\begin{aligned} \phi_H(G,k) &= \min_{0<\|f\|_0\le k} \|V_{g_1\otimes g_2} f\|_0, \\ \phi(H,s) &= \min_{0<\|f_1\|_0\le s} \|V_{g_1} f_1\|_0, \\ \phi(G/H,t) &= \min_{0<\|f_2\|_0\le t} \|V_{g_2} f_2\|_0\,. \end{aligned} \tag{17}$$

We fix $x = x_i{+}x_j$ and $\xi = \xi_i{+}\xi_j$, and compute as in the proof of Proposition 3.8

$$\begin{aligned} & V_{g_1\otimes g_2} f(x,\xi) \\ = & \sum_{y_j}\sum_{y_i} f(y_i{+}y_j)\, \overline{g_1(y_i{-}x_i)}\, \overline{g_2(y_j{-}x_j+H)}\, \overline{\langle \xi_i, y_i\rangle}_H \overline{\langle \xi_i, y_j\rangle}_G \overline{\langle \xi_j, y_j+H\rangle}_{G/H} \\ = & \sum_{y_j} \overline{g_2(y_j{-}x_j+H)}\, \overline{\langle \xi_i, y_j\rangle}_G \overline{\langle \xi_j, y_j+H\rangle}_{G/H} \sum_{y_i} f(y_i{+}y_j)\, \overline{g_1(y_i{-}x_i)\langle \xi_i, y_i\rangle}_H \end{aligned}$$

where we used $\xi_j \in H^\perp$, that is, $\langle \xi_j, y_i\rangle_G = 1$. For

$$F_H(x_i,\xi_i,y_j) := \overline{\langle \xi_i, y_j\rangle}_G \sum_{y_i} f(y_i{+}y_j)\, \overline{g_1(y_i{-}x_i)}\, \overline{\langle \xi_i, y_i\rangle}_H$$

we have

$$F_H(x_i,\xi_i,y_j) = \overline{\langle \xi_i, y_j\rangle}_G V_{g_1} T_{-y_j} f(x_i,\xi_i')$$

and $V_g f(x,\xi) = \big(V_{g_2} F_H(x_i,\xi_i,\,\cdot)\big)(x_j{+}H,\xi_j)$.

We fix now f such that $\|f\|_0 \le k$. Let $t = |\{y_j : \operatorname{supp} f \cap y_j{+}H \ne \varnothing\}|$. If for some y_j, $\operatorname{supp} f \cap y_j + H = \varnothing$, then $F_H(\cdot\,,\cdot\,,y_j) \equiv 0$ too. Therefore, $\|F_H(x_i,\xi_i,\cdot\,)\|_0 \le t$ and using (17) we obtain $\|V_{g_2} F_H(x_i,\xi_i,\cdot\,,\cdot\,)\|_0 \ge \phi(G/H,t)$. Also, by distributing $\operatorname{supp} f$ over t cosets of H in G, there is a coset $y_{j_0}{+}H$ with $|\operatorname{supp} f \cap y_{j_0}{+}H| = s \le k/t$. Because $F_H(\cdot\,,\cdot\,,y_{j_0})$ is, up to a nonzero factor, the partial short–time Fourier transform of $T_{-y_{j_0}} f$ with window g_1 on that coset,

$$\|F_H(\cdot\,,\cdot\,,y_{j_0})\|_0 = \|V_{g_1} T_{-y_{j_0}} f\|_0 \ge \phi(H,s).$$

We have obtained that the set $\Lambda = \{(x_i,\xi_i') \in H{\times}\widehat{H} : F_H(x_i,\xi_i,y_{j_0}) \ne 0\}$ has at least $\phi(H,s)$ elements so

$$\begin{aligned} \|V_g f(x_i{+}x_j,\xi_i{+}\xi_j)\|_0 &= \sum_{(x_i,\xi_i')\in H\times\widehat{H}} \|V_g f(x_i,\xi_i,\cdot\,,\cdot)\|_0 \\ &\ge \sum_{(x_i,\xi_i)\in\Lambda} \|V_{g_2} F_H(x_i,\xi_i,\cdot\,,\cdot)\|_0 \\ &\ge \phi(H,s)\phi(G/H,t)\,. \end{aligned}$$

This inequality holds for all $V_g f$ with $0 < \|f\|_0 \le k$ and therefore, $\phi_H(G,k) \ge \phi(H,s)\phi(G/H,t)$. □

Theorem 4.12 *For any finite Abelian group G and $k \leq |G|$, let d_1 be the largest divisor of $|G|$ which is less than or equal to k and let d_2 be the smallest divisor of $|G|$ which is larger than or equal to k. Then*

$$\phi(G,k) \geq \frac{|G|^2}{d_1 d_2}(d_1 + d_2 - k). \tag{18}$$

Proof. The function $v(n,k) = n\,u(n,k) = \frac{n^2}{d_1 d_2}(d_1+d_2-k)$, is submultiplicative since $u(n.k) = \frac{n}{d_1 d_2}(d_1+d_2-k)$ in [Mes06] is submultiplicative, in other words, we have $v(a,b)v(c,d) \geq v(ac,bd)$. We proceed by induction on $|G| = n$. Suppose (18) holds for $|G| = 1,\ldots,n-1$. If n is prime, then Proposition 4.5 implies $v(n,k) = n(1+n-k) < n^2-k+1 = \phi(\mathbb{Z}_p,k)$ for all k. Else, we choose a nontrivial divisor d of n, and let H be a subgroup of G of order d. By Proposition 4.11, there exist s,t with $1 \leq s \leq d$, $1 \leq t \leq \min\{\frac{k}{s}, \frac{n}{d}\}$ such that $\phi(G,k) \geq \phi(H,s)\phi(G/H,t)$. Therefore, $\phi(G,k) \geq v(d,s)v(\frac{n}{d},t) \geq v(n,st) \geq v(n,k)$. □

For the case $G = \mathbb{Z}_{pq}$, we can improve this estimate by finding the convex hull of all pairs $(|H|,|G/H|)$ for all subgroups H of G as in [Mes06].

Proposition 4.13 *Let $G = \mathbb{Z}_{pq}$ with $q < p$ and p, q prime. Then*

$$\phi(G,k) \geq \begin{cases} p^2(q^2-k+1) & \text{if } k < q; \\ (p^2 - \frac{k}{q}+1)(q^2-q+1) & \text{else.} \end{cases} \tag{19}$$

Proof. Proposition 3.8 implies that there exists s,t such that $st \leq k$ and $\phi(G,k) \geq \phi(H,s)\phi(G/H,t)$. For $G = \mathbb{Z}_{pq}$ and $|H| = p$, we have $\phi(H,s) = p^2-s+1$ and $\phi(G/H,t) = q^2-t+1$. As $st \leq k$, we can find $\bar{t} \in \mathbb{R}$ such that $q \geq \bar{t} \geq t$ and $p \geq \frac{k}{\bar{t}} \geq s$. Hence,

$$\phi(G,k) \;\geq\; (p^2-s+1)(q^2-t+1) \geq (p^2 - \tfrac{k}{\bar{t}}+1)(q^2-\bar{t}+1).$$

So $\phi(G,k)$ must exceed the minimum of $M(u) = (p^2 - \frac{k}{u}+1)(q^2-u+1)$, where u ranges from $\frac{k}{p}$ to q since we assume $\frac{k}{u} \leq p$ and $u \leq q$. We have $M'(u) = -(p^2+1) + \frac{k(q^2+1)}{u^2} = 0$ if and only if $u = \pm\sqrt{k\frac{q^2+1}{p^2+1}}$. As $M(u) \to -\infty$ for $u \to 0^+$ and $u \to \infty$, the only positive extremum is a maximum and the minimum is attained in a boundary point. A simple calculation gives that $M(q) \leq M\left(\frac{k}{p}\right)$.

For $k < q$, the condition $1 \leq s$, $1 \leq t$, implies that t ranges only from 1 to k. The same arguments as used above show again that the minimum is attained at a boundary point and that $M(1) \geq M(k)$. □

At $k = q$, the two lower bounds in (19) coincide and lead to what a geometric argument shows to be the optimal value that can be obtained using $g = g_1 \otimes g_2$. So the two straight lines give a convex hull similar to [Mes06]. However, as expected, the computational results are better than those given in (19), since since tensor windows cannot be used to find optimal bounds for $\phi(G,k)$. See Table 4 for an illustration of (19) for $G = \mathbb{Z}_6$.

$\|f\|_0$	1	2	3	4	5	6
Theorem 4.12	36	18	12	10	8	6
Proposition 4.13	36	26	25	23	22	20
Numerical results	36	33	32	32	32	31

Table 4: Lower bounds for $\|V_g f\|_0$ given by Theorem 4.12 and Proposition 4.13 for $G = \mathbb{Z}_6$ and by numerical experiments and randomly chosen $g \in \mathbb{C}^{\mathbb{Z}_6}$.

4.3 Outlook

For $|G|$ prime, Theorem 4.5 characterizes all pairs $(\|f\|_0, \|V_g f\|_0)$, $f \in \mathbb{C}^G$ which are achieved for almost every window function $g \in \mathbb{C}^G$. However, for general Abelian groups it is quite difficult to establish lower bounds for $\|V_g f\|$. Further, our limited numerical results for cyclic groups indicate a close correspondence between the achieved pairs $(\|f\|_0, \|\widehat{f}\|_0)$ and the achieved pairs $(\|f\|_0, \|V_g f\|_0)$ for a given window g. Consequently, we pose

Question 4.14 *For every cyclic group G and almost every $g \in \mathbb{C}^G$, is it true that*

$$\big\{(\|f\|_0, \|V_g f\|_0),\ f \in \mathbb{C}^G \backslash \{0\}\big\} = \big\{(\ \|f\|_0\ ,\ \|\widehat{f}\|_0 + |G|^2 - |G|\ ),\ f \in \mathbb{C}^G \backslash \{0\}\big\}\ ?$$

The basis for this question is illustrated in Figure 7 by considering the cyclic groups $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_6$, $\mathbb{Z}_7$, and $\mathbb{Z}_8$. The statement does not hold for noncyclic groups; for example, in the diagram for $\mathbb{Z}_2^2$ in Figure 7 the existence of $4{\times}4$ zero minors in $A_{G,g}$ in (4) that is, the minor given by columns 1,3,13,14, leads to the possible pair (4, 12).

We state some preliminary observations regarding Question 4.14.

For example, the technique used to prove Theorem 4.5 possesses certain degrees of freedom, that is, we only need to show that a particular product of minors is nonzero. Nevertheless, these degrees of freedom do not allow us to give a positive answer to Question 4.14. For example, for $G = \mathbb{Z}_4$, we can choose the 4×4 submatrix

$$M(z) = \big(A_{\mathbb{Z}_4}, (z_0, z_1, z_2, z_3)\big)_{\{0,1,4,12\},\{0,1,2,3\}} = \begin{pmatrix} z_0 & z_0 & z_3 & z_1 \\ z_1 & -z_1 & z_0 & z_2 \\ z_2 & z_2 & z_1 & z_3 \\ z_3 & -z_3 & z_2 & z_0 \end{pmatrix}$$

In this submatrix, none of the monomials that is "maximal" in the sense described above, namely the monomials $z_0^3 z_2$, $z_1^3 z_3$, $z_2^3 z_0$, and $z_3^3 z_1$, has a nonzero coefficient in the polynomial $P(z_0, z_1, z_2, z_3) = \det M(z) = -2z_0^2 z_1^2 - 2z_1^2 z_2^2 - 2z_0^2 z_3^2 - 2z_2^2 z_3^2 - 4z_0 z_1 z_2 z_3 \not\equiv 0$.

Using Proposition 3.6, we derive a partial result on nonzero minors of $A_{\mathbb{Z}_n,g}$.

Proposition 4.15 *For every n, any minor of the full Gabor system matrix $A_{\mathbb{Z}_n,g}$, where the columns corresponding to each fixed translation are adjacent with respect*

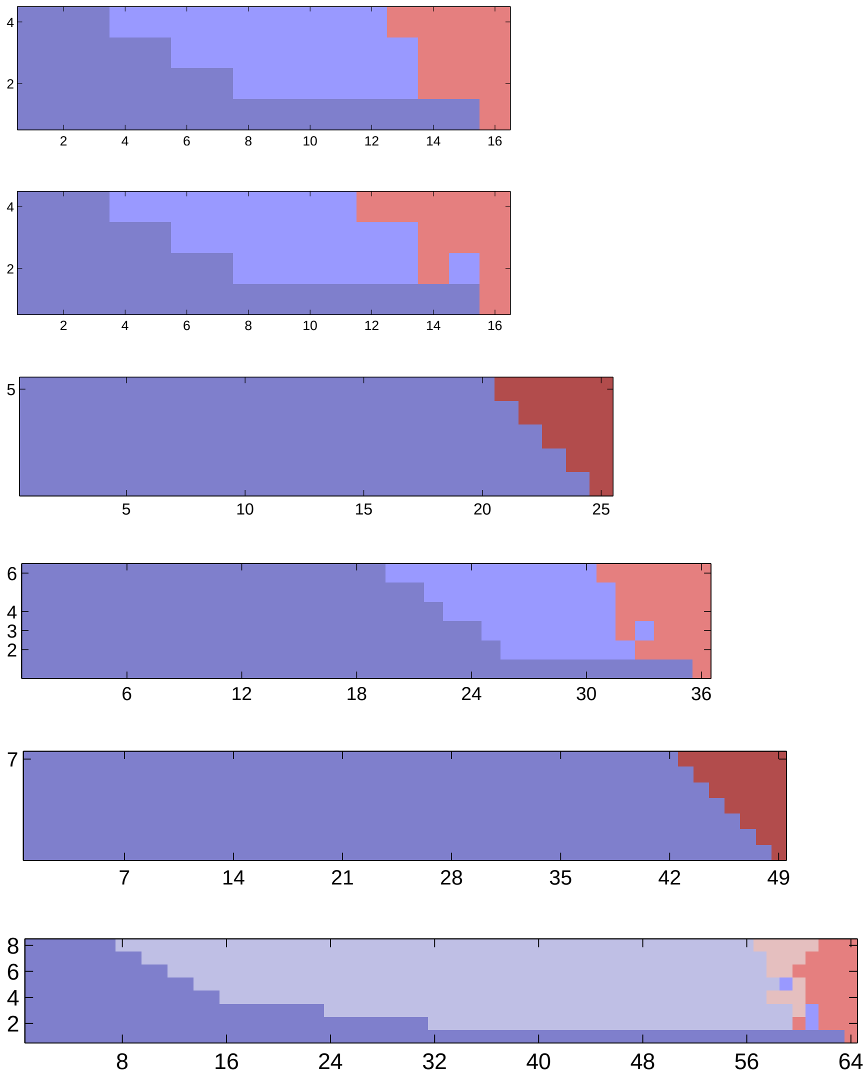


Figure 7: The set $\{(\|f\|_0, \|V_g f\|_0),\ f \in \mathbb{C}^G \setminus \{0\}\}$ for appropriately chosen g for the groups $\mathbb{Z}_4, \mathbb{Z}_2^2, \mathbb{Z}_5, \mathbb{Z}_6, \mathbb{Z}_7, \mathbb{Z}_8$. The color coding from Figure 2 is applied in accordance with numerical experiments based on Lemma 3.5.

to modulation is nonzero for almost every g. The same holds for a minor corresponding to a submatrix of size $n \times n$, where the columns corresponding to each fixed modulation are adjacent with respect to translation.

Proof. As in the proof of Theorem 4.6, choose $A \subseteq G$ and $B \subseteq G \times \widehat{G}$ with $|A| = |B|$ and set $P_{A,B}(z) = \det(A_{\mathbb{Z}_n,z})_{A,B}$, $z = (z_0, z_1, \ldots, z_{n-1})$. In that proof, we identified a "maximal" term within $P_{A,B}$, the coefficient of which can be expressed as a product of different minors of the discrete Fourier matrix $W_{\mathbb{Z}_n}$. Each of these minors arise from the columns of $P_{A,B}(z)$ that correspond to a specific translation. By assumption, these columns are adjacent with respect to modulation in $A_{\mathbb{Z}_n,z}$.

So each of these minors is a minor of the DFT matrix corresponding to adjacent columns, where each row is multiplied by some factor z_i. Using the multilinearity of the determinant, we can pull the factors outside. By Proposition 3.6, we conclude that these minors of the DFT-matrix are nonzero, hence also their product. So the "maximal" term has a nonzero coefficient.

To obtain the dual statement, take the Fourier transform of each column of $A_{\mathbb{Z}_n,g}$. By linearity, the resulting matrix can have no size-n zero minors either, as that would mean that one column of the corresponding submatrix is a linear combination of other columns. As $\widehat{M_\xi T_x g} = T_\xi M_{-x}\widehat{g}$, the resulting matrix will correspond to $A_{\mathbb{Z}_n,\widehat{g}}$, except that modulations and translations have exchanged their roles. So modulation adjacency becomes translation adjacency, which implies the dual statement. □

5 Applications

We shall now turn to applications of the results stated in Section 4 to communications engineering and, in the subsequent section, to the problem of recovering sparse signals from incomplete data.

5.1 Gabor frames, erasures, and the identification of operators

In generic communication systems, information is transmitted in the form of the entries of a vector $f \in \mathbb{C}^G$ over a channel in such a way that recovery of the information at the receiver is robust to errors introduced by the channel. Here, we will focus on two inherent problems. First, we shall discuss transmission over a channel with erasure, that is, some of the vector entries may be lost during transmission. Second, we discuss the so-called identification problem for another class of operators, namely, of linear time–varying operators which play a central role in wireless and mobile communications. Clearly, knowledge of the operator at hand would help to counteract disturbances that were caused during transmission.

But first, we give some preliminaries on frames in finite dimensional vector spaces, which will be used in this section. For details on frames and, in particular, Gabor frames we refer to the excellent expositions [Chr03, Grö01, KC06]. The geometry of finite frames is discussed in [BF03].

Definition 5.1 *Let G be a finite Abelian group and let K be a finite or countably infinite index set. A family of functions $\{\varphi_k\} \subset \mathbb{C}^G$ with*

$$A\|f\|_2^2 \leq \sum_k |\langle f, \varphi_k\rangle|^2 \leq B\|f\|_2^2, \quad f \in \mathbb{C}^G,$$

for positive A and B is called a frame for $\mathbb{C}^G$. A is called a lower frame bound and B is called an upper frame bound of the frame $\{\varphi_k\}$.

A frame is called tight if we can choose $A = B$. If we can choose $A = B = 1$, then the frame is called Parseval tight frame. If $\|\varphi_k\| = C > 0$ for all k, then the frame $\{\varphi_k\}$ is called equal norm frame and if in addition $C = 1$, then we have a unit norm frame.

In the following, we shall refer to a Gabor system which forms a frame as Gabor frame. A direct consequence of (2) is

Proposition 5.2 *For any $g \in \mathbb{C}^G \backslash \{0\}$, the collection $\{\pi(\lambda)g\}_{\lambda \in G \times \widehat{G}}$ is an equal norm tight frame for $\mathbb{C}^G$ with frame bound $A = B = |G| \, \|g\|_2^2$.*

The usefulness of frames stems largely from the existence of a reconstruction formula similar to (1) and (2).

Proposition 5.3 *Let $\{\varphi_k\}$ be a frame for $\mathbb{C}^G$. Then exists a so-called dual frame $\{\widetilde{\varphi}_k\}$, with*

$$f = \sum_k \langle f, \varphi_k \rangle \widetilde{\varphi}_k = \sum_k \langle f, \widetilde{\varphi}_k \rangle \varphi_k \,, \quad f \in \mathbb{C}^G \,. \tag{20}$$

Note that Parseval frames are self dual, that is, we can choose $\widetilde{\varphi}_k = \varphi_k$ for all k.

Now we are in a position to briefly discuss the recovery of information from a vector that suffered erasures [CK03, PK05, GK01, SH03]. In data transmission, rather then sending the information in raw form, that is, sending vector entries one-by-one, information is being coded prior to transmission. For example, we can choose a frame $\{\varphi_k\}_{k \in K}$ for $\mathbb{C}^G$ and send the coefficients $\langle f, \varphi_k \rangle$, $k \in K$. If none of the transmitted coefficients are lost, the receiver can use a dual frame $\{\widetilde{\varphi}_k\}$ of $\{\varphi_k\}$ and recover f using (20). But even if some coefficients are lost and only $\langle f, \varphi_k \rangle$ is received for $k \in K' \subset K$, then the information can still be recovered if and only if $\{\varphi_k\}_{k \in K'}$ remains a frame. This necessitates that $|K'| \geq |G| = \dim \mathbb{C}^G$.

Definition 5.4 *A frame $\mathcal{F} = \{\varphi_k\}_{k \in K}$ in $\mathbb{C}^G$ is maximally robust to erasures if the removal of any $l \leq |K| - |G|$ vectors from $\mathcal{F}$ leaves a frame.*

Similarly, we give

Definition 5.5 *A set of m vectors in $\mathbb{C}^G$ is in general position, if any collection of at most $|G|$ of these vectors are linearly independent.*

Before giving slight generalizations of results from [LPW05] on Gabor frames that are maximally robust to erasure in Theorem 5.7, we introduce some vocabulary and notation regarding the previously mentioned operator identification problem.

Definition 5.6 *A linear space of operators $\mathcal{H}$ mapping $\mathbb{C}^A$ to $\mathbb{C}^B$ is called identifiable with identifier $g \in \mathbb{C}^A$ if the linear map $\varphi_g : \mathcal{H} \longrightarrow \mathbb{C}^B$, $H \mapsto Hg$ is injective, that is, if $Hg \neq 0$ for all $H \in \mathcal{H} \backslash \{0\}$.*

Time–variant communication channels, for example, multipath channels in wireless telephony, are often modeled through a combination of translation operators (time–shift, delay) and modulation operators (frequency shifts that are caused by the Doppler effect). Therefore, identification of $\mathcal{H}_\Lambda = \{\sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda),\ c_\lambda \in \mathbb{C}\}$ for $\Lambda \subseteq G\times\widehat{G}$ is quite a relevant goal (see [PRT07] and references therein).

Theorem 5.7 *For $g \in \mathbb{C}^G\backslash\{0\}$, the following are equivalent:*

1. *Every minor of $A_{G,g}$ of order $|G|$ is nonzero.*
2. *The vectors from the Gabor system $\{\pi(\lambda)g\}_{\lambda\in G\times\widehat{G}}$ are in general position.*
3. *The Gabor system $\{\pi(\lambda)g\}_{\lambda\in G\times\widehat{G}}$ is an equal norm tight frame which is maximally robust to erasures.*
4. *For all $f \in \mathbb{C}^G\backslash\{0\}$ we have $\|V_g f\|_0 \geq |G|^2-|G|+1$.*
5. *For all $f \in \mathbb{C}^G$, $V_g f(\lambda)$, and, therefore, f, is completely determined by its values on any set Λ with $|\Lambda| = |G|$.*
6. *$\mathcal{H}_\Lambda$ is identifiable by g if and only if $|\Lambda| \leq |G|$*

For $|G|$ prime, Theorem 4.5 ensures the existence of $g \in \mathbb{C}^G$ which satisfy parts *1-6* in Theorem 5.7 and Proposition 4.7 allows us to choose g to be unimodular. A positive answer to Question 4.14 would also confirm the existence of $g \in \mathbb{C}^{\mathbb{Z}_n}$ satisfying Theorem 5.7, part *4*, and therefore Theorem 5.7, parts *1-6*, for cyclic groups.

Remark 5.8 To our knowledge, the only known *equal norm tight frames that are maximally robust to erasures* are so-called harmonic frames (see Conclusions in [CK03]). Harmonic frames for $\mathbb{C}^n$ with $m \geq n$ elements are obtained by deleting identical $m-n$ components of the characters of $\mathbb{Z}_m$ [CK03]. Similarly, Theorem 4.6 together with Propostition 4.7 provides us with equal norm tight frames with p^2 elements in $\mathbb{C}^n$ for $n \leq p$. Namely, we can choose a $g \in (S^1)^p$ and remove $p-n$ components of the equal norm tight frame $\{\pi(\lambda)g\}_{\lambda\in G\times\widehat{G}}$ in order to obtain an equal norm tight frame which is maximally robust to erasure. Note that this frame is not a Gabor frame proper. Reducing the number of vectors in the frame to $m \leq p^2$ vectors leaves an equal norm frame which is maximally robust to erasure but which might not be tight. This holds for harmonic frames too. With the restriction to frames with p^2 elements, p prime, we have shown the existence of Gabor frames which share the usefulness of harmonic frames when it comes to transmission of information through erasure channels.

Background and more details on frames and erasures can be found in [CK03, GK01, SH03] and the references cited therein.

5.2 Signals with sparse representations

In Section 5.1 we discussed the recovery of signals or operators from $|G|$ known complex numbers. Here, we will use the functions ϕ and θ which were defined in Section 3.2 and Section 4.2 to refine some of these findings. That is, we show that a function/signal which can be represented as a linear combination of a small number of pure frequencies or of a small number of time–frequency shifts of a fixed function g, can be recovered from fewer than $|G|$ of its values. Our brief discussion is based on the most basic ideas and results from the theory of sparse signal recovery [Don06, Rau07, CRT06].

There exist a number of entry points to the theory of sparse signal recovery. Here, we shall consider dictionaries $\mathcal{D} = \{g_0, g_1, \ldots, g_{N-1}\}$ of N vectors in $\mathbb{C}^n$, or equivalently, in $\mathbb{C}^G$. For $k \leq n = |G|$ we shall examine the sets

$$\Sigma_k^{\mathcal{D}} = \{f \in \mathbb{C}^n : \ f = M_{\mathcal{D}}\, c = \sum_r c_r g_r, \ \text{with } \|c\|_0 \leq k\}\,.$$

The central question is: *how many values of $f \in \Sigma_k^{\mathcal{D}}$ need to be known (or stored), in order that $c \in \mathbb{C}^N$ with $f = \sum_r c_r g_r$ and $\|c\|_0 \leq k$, and therefore f, is uniquely determined by the known data?*

To this end, we set

$$\psi(\mathcal{D}, k) = \min \left\{ \|f\|_0 : \ f \in \Sigma_k^{\mathcal{D}} \right\},$$

and observe the following well known result.

Proposition 5.9 *Any $f \in \Sigma_k^{\mathcal{D}}$ is fully determined by any choice of $n - \psi(\mathcal{D}, 2k) + 1$ values of f.*

Note that unlike in Theorem 5.7, we do not assume knowledge of the set $\operatorname{supp} c$ for c with $M_{\mathcal{D}}c = f$, $\|f\|_0$ in Proposition 5.9 and in the following.

Proof. Assume that for some $B \subset \mathbb{C}^n$ with $|B| = n - \psi(\mathcal{D}, 2k) + 1$, two coefficient vectors $c_1, c_2 \in \mathbb{C}^N$ exist that satisfy $r_B M_{\mathcal{D}} c_1 = r_B f = r_B M_{\mathcal{D}} c_2$ and $\|c_1\|_0, \|c_2\|_0 \leq k$. Then $\|c_2 - c_1\|_0 \leq 2k$ with $\|M_{\mathcal{D}}(c_2 - c_1)\|_0 \leq n - |B| = n - (n - \psi(\mathcal{D}, 2k) + 1) = \psi(\mathcal{D}, 2k) - 1$, a contradiction. □

A classical dictionary for $\mathbb{C}^G$ is $\mathcal{D}_G = \{\xi\}_{\xi \in \widehat{G}}$, where G is a finite Abelian group. Then

$$\psi(\mathcal{D}, k) = \min \left\{ \|f\|_0 : \ f \in \Sigma_k^{\mathcal{D}} \right\} = \min \left\{ \|\widehat{f}\|_0 : \ \|f\|_0 \leq k \right\} = \theta(G, k)\,.$$

This equality together with Proposition 5.9 demonstrates the relevance of the results cited in Section 3 for the recovery of signals with limited spectral content. For example, Theorem 3.7 shows that for any finite Abelian group of order 16 we have $\theta(G, 6) \geq 3$. In fact, our computations that are illustrated in Figure 3 show that $\theta(G, 6) = 4$ for $|G| = 16$, and, hence, any $f \in \Sigma_3^{\mathcal{D}_G} = \{f : \ \|\widehat{f}\|_0 \leq 3\}$ can be recovered from any choice of $|G| - \theta(G, 2 \cdot 3) + 1 = 16 - 4 + 1 = 13$ values of

f. For $f \in \Sigma_3^{\mathcal{D}_{\mathbb{Z}_{17}}}$ on the other side, Theorem 3.3 implies that f is already fully determined by $|\mathbb{Z}_{17}| - \theta(\mathbb{Z}_{17}, 2\cdot 3) + 1 = 17 - (17 - 6 + 1) + 1 = 6$ of its values.

The results in Section 4 involving the function ϕ are relevant to determine vectors which have sparse representations in the dictionary $\mathcal{D}_{A_{G,g}}$ which consists of the columns of $A_{G,g}$. In fact, we have $F \in \Sigma_k^{\mathcal{D}_{A_{G,g}}}$ if and only if $F = V_g f$ for some $f \in \mathbb{C}^G$ with $\|f\|_0 \leq k$ and, therefore,

$$\psi(\mathcal{D}_{A_{G,g}}, k) = \min\left\{\|V_g f\|_0 : \ \|f\|_0 \leq k\right\} = \phi(G, k)\,.$$

For $|G|$ prime for example, this leads to the following short–time Fourier transform version of Theorem 1.1 in [CRT06].

Theorem 5.10 *Let $g \in \mathbb{C}^{\mathbb{Z}_p}$, p prime, satisfy the conclusion of Theorem 4.5. Then any $f \in \mathbb{C}^{\mathbb{Z}_p}$ with $\|f\|_0 \leq \frac{1}{2}|\Lambda|$, $\Lambda \subset \mathbb{Z}_p \times \widehat{\mathbb{Z}_p}$, is uniquely determined by Λ and $r_\Lambda V_g f$.*

In terms of sparse representations, the Gabor frame dictionary $\{\pi(\lambda)g\}_{\lambda \in G\times\widehat{G}}$ of time–frequency shifts of a prototype vector g, that is, the dictionary consisting of the rows of $A_{G,g}$, appears to be more interesting. Rudimentary numerical experiments based on Lemma 3.5 give some indication that for any Abelian group G, and almost every $g \in \mathbb{C}^G$, we have for $k \leq |G|$,

$$\psi(\{\pi(\lambda)g\}_{\lambda\in G\times\widehat{G}}, k) = \theta(G, k).$$

Note that this does not hold for all Abelian groups of finite order. For example, for any $g \in \mathbb{C}^{\mathbb{Z}_2\times\mathbb{Z}_2}$ we have $\psi(\{\pi(\lambda)g\}_{\lambda\in(\mathbb{Z}_2\times\mathbb{Z}_2)\times(\mathbb{Z}_2\times\mathbb{Z}_2)}, 4) = 0$ while $\theta(\mathbb{Z}_2\times\mathbb{Z}_2, 4) = 1$.

For $|G|$ prime, Theorem 4.6 implies that $\psi(\{\pi(\lambda)g\}_{\lambda\in G\times\widehat{G}}, k) = p - k + 1 = \theta(G, k)$, and analogous to Theorem 5.10, we obtain

Theorem 5.11 *Let $g \in \mathbb{C}^{\mathbb{Z}_p}$, p prime, satisfy the conclusion of Theorem 4.5. Then any $f \in \mathbb{C}^{\mathbb{Z}_p}$ with $f = \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) g$, $\Lambda \subset \mathbb{Z}_p\times\widehat{\mathbb{Z}_p}$ is uniquely determined by B and $r_B f$ whenever $|B| \geq 2|\Lambda|$.*

Note that similar to before, the recovery of f from $2|\Lambda|$ samples of f in Theorem 5.11 does not require knowledge of Λ.

6 Appendix

6.1 Justification of Figure 6

Let $\omega = e^{2\pi i/3}$. For $\|f\|_0 = 1$, we calculate

$$V_{(a,b,c)}(d,0,0) = (d\overline{a}, \omega^2 d\overline{a}, \omega d\overline{a}, d\overline{c}, \omega^2 d\overline{c}, \omega d\overline{c}, d\overline{b}, \omega^2 d\overline{b}, \omega d\overline{b})$$

So in any case, $\|V_g f\|_0 = 3\|g\|_0$, which justifies all cases involving $\|f\|_0 = 1$ or $\|g\|_0 = 1$.

For the case $\|f\|_0 = 2$ and $\|g\|_0 = 2$, we note $\|V_{(1,1,0)}(1,-1,0)\|_0 = 8$ and $\|V_{(1,1,0)}(1,10,0)\|_0 = 9$, which justifies the two red fields. Now assume that there are f and g with $\|f\|_0 = \|g\|_0 = 2$ and $\|V_g f\|_0 \le 7$. Then $V_g f$ has at least two zero entries. Note that the scalar product of f and another vector with support size 2 can only vanish, if $\operatorname{supp} f = \operatorname{supp} g$. So the zero entries in $V_g f$ must correspond to the same translation. If we set without loss of generality $f = (a,b,0), g = (c,d,0)$, then zeros at two different modulations M_{j_1} and M_{j_2} imply $a\overline{c} + \overline{\omega}^{j_1} b\overline{d} = 0 = a\overline{c} + \overline{\omega}^{j_2} b\overline{d}$, which clearly admits no nontrivial solution.

For the case $\|f\|_0 = 2$ and $\|g\|_0 = 3$ which is equivalent to the case $\|f\|_0 = 3$ and $\|g\|_0 = 2$, we note that $\|V_{(1,1,1)}(1,-1,0)\|_0 = 6$, $\|V_{(2,-4,8)}(2,1,0)\|_0 = 7$, $\|V_{(1,2,3)}(2,-1,0)\|_0 = 8$ and $\|V_{(1,2,3)}(1,2,0)\|_0 = 9$, which justifies the four red fields. Now assume, there are f and g with $\|f\|_0 = 2$, $\|g\|_0 = 3$ and $\|V_g f\|_0 \le 5$. Then $V_g f$ has at least four zero entries, in particular two that correspond to the same translation. Without loss of generality, we assume that this is the zero-translation and that f is supported in the first two coordinates, that is, $f = (a,b,0)$, $g = (c,d,e)$. Then we get as before $a\overline{c} + \overline{\omega}^{j_1} b\overline{d} = 0 = a\overline{c} + \overline{\omega}^{j_2} b\overline{d}$ which has no nontrivial solutions.

For the case $\|f\|_0 = 3$ and $\|g\|_0 = 3$, we note that $\|V_{(1,1,1)}(1,1,1)\|_0 = 3$, $\|V_{(1,1,1)}(1,1,-2)\|_0 = 6$, $\|V_{(1,2,5)}(10,5,2)\|_0 = 7$, $\|V_{(1,2,3)}(-5,1,1)\|_0 = 8$ and $\|V_{(1,2,3)}(1,2,3)\|_0 = 9$, which justifies the five red fields. Multiplying f or g by a constant does not change $\|V_g f\|_0$, so we can normalize $f(0) = g(0) = 1$. Hence we can set $f = (1,a,b)$, $g = (1,c,d)$. Then again, $\|V_g f\|_0 \le 5$ implies that $V_g f$ has two zero entries that correspond to the same translation and we shall assume without loss of generality and for the remainder of this section that those appear at $x = 0$ and $\xi = 1, 2$, that is, we have

$$1 + \omega a\overline{c} + \omega^2 b\overline{d} = 0 = 1 + \omega^2 a\overline{c} + \omega b\overline{d}$$

and hence $b\overline{d} = a\overline{c} = 1$ and $g = \left(1, \frac{1}{\overline{a}}, \frac{1}{\overline{b}}\right)$.

Before continuing, we state

Lemma 6.1 *Let S be a shearing on $\mathbb{C}^{\mathbb{Z}_3 \times \mathbb{Z}_3}$, that is, S translates the $(x = 1)$-row of an element in $\mathbb{C}^{3\times 3}$ by 1 and the $(x = 2)$-row by 2. Then given $f, g \in \mathbb{C}^{\mathbb{Z}_3}$, there exist $\tilde{f}, \tilde{g} \in \mathbb{C}^{\mathbb{Z}_3}$, such that $\operatorname{supp} V_{\tilde{g}}\tilde{f}$ is the image of $\operatorname{supp}(V_g f)$ under S.*

Proof. Suppose, two vectors $f = (u,v,w)$ and $g = (x,y,z)$ are given, and consider the vectors $\tilde{f} = (u,v,\omega w)$ and $\tilde{g} = (x,y,\omega z)$. Then

$$V_{\tilde{g}}\tilde{f}(0,\xi) = u\overline{x} + \overline{\omega}^{\xi} v\overline{y} + \overline{\omega}^{2\xi}(\omega w)(\overline{\omega z}) = u\overline{x} + \overline{\omega}^{\xi} v\overline{y} + \overline{\omega}^{2\xi}\overline{z}w = V_g f(0,\xi)\,,$$

$$V_{\tilde{g}}\tilde{f}(1,\xi) = u\overline{y} + \overline{\omega}^{\xi} v\overline{\omega z} + \overline{\omega}^{2\xi}(\omega w)(\overline{x}) = u\overline{y} + \overline{\omega}^{\xi+1} v\overline{z} + \overline{\omega}^{2\xi+2}\overline{x}w = V_g f(1,\xi+1)\,,$$

and

$$V_{\tilde{g}}\tilde{f}(2,\xi) = u\overline{\omega z} + \overline{\omega}^{\xi} v\overline{x} + \overline{\omega}^{2\xi}\omega w\overline{y} = \overline{\omega}(u\overline{z} + \overline{\omega}^{\xi+2} v\overline{x} + \overline{\omega}^{2\xi+1} w\overline{y}) = \overline{\omega} V_g f(2,\xi+2)\,.$$

As a multiplication by $\overline{\omega}$ does not change the support, we get the sheared image of the original support set as desired. $\square$

We now use Lemma 6.1 to show that in the case $\|f\|_0 = \|g\|_0 = 3$, no support size of 4 is possible. In fact this would imply that the short–time Fourier transform has five zeroes, so there is a second row with two zeroes (without loss of generality the row $x = 1$). By shearing we can move them to $\xi = 1, 2$ without changing the first row, that is,

$$\tfrac{1}{a} + \overline{\omega}\tfrac{a}{b} + \overline{\omega}^2 b = 0 = \tfrac{1}{a} + \overline{\omega}^2\tfrac{a}{b} + \overline{\omega}b\,.$$

This implies $\frac{1}{a} = \frac{a}{b} = b$ and hence $a = 1$, $a = \omega$ or $a = \omega^2$, and $b = \overline{a}$ accordingly. This reduces to the the example for $\|V_g f\|_0 = 3$ given above. Thus, $\|V_g f\|_0 = 4$ is impossible.

For a support size of 5, we can use the same argument to exclude that the remaining two zeroes occur at the same x. So in addition to the two zeros for $x = 0$, we can have zeroes at $x = 1, 2$ and either $\xi = 0$ for both or $\xi = 1$ for both. All other combinations can be reduced to these two by shearing and conjugation (using $\omega^2 = \bar{\omega}$).

These two cases correspond to solving

$$a + \overline{\omega}^k \tfrac{b}{a} + \overline{\omega}^{2k}\tfrac{1}{b} = 0 = \tfrac{1}{a} + \overline{\omega}^k\tfrac{a}{b} + \overline{\omega}^{2k} b$$

for $k = 0, 1$. These equations can be solved exactly using Mathematica. The only solutions are modulations of shearings of the solution with $\|V_g f\|_0 = 3$ considered above. So again, it follows that a short–time Fourier transform with support size 5 is not possible.

6.2 Proof of Chebotarev's Theorem 3.3 based on Thoerem 4.6.

Fix $A, \widetilde{A} \subseteq \mathbb{Z}_p$ with $|A| = |\widetilde{A}|$. We have to show that the restricted Fourier transformation $\mathcal{F}_{A\to\widetilde{A}} : \mathbb{C}^A \to \mathbb{C}^{\widetilde{A}}$ is an isomorphism. For g such that $A_{\mathbb{Z}_p,g}$ has no zero minors, define $M_g : \mathbb{C}^p \longrightarrow \mathbb{C}^p$ to be the pointwise multiplication operator with the vector g. Since g has no zero components, M is an isomorphism, and, moreover, M_g restricts to an isomorphism on $\mathbb{C}^A$. Set $B = \{0\} \times \widetilde{A}$. Therefore, $V_g : \mathbb{C}^A \longrightarrow \mathbb{C}^B$ is an isomorphism since $|B| = |\widetilde{A}|$. The result follows since the restricted Fourier transformation $\mathcal{F}_{A\to\widetilde{A}}$ is nothing but $P \circ V_g \circ M_g$ where P is the projection of $B = \{0\} \times \widetilde{A}$ onto $\widetilde{A}$.

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