Measurement of Time–Variant Linear Channels

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ABSTRACT

The goal of channel measurement or operator identification is to obtain complete knowledge of a channel operator by observing the image of a finite number of input signals. We shall show that if the spreading support of the operator (that is, the support of the symplectic Fourier transform of the Kohn–Nirenberg symbol of the operator) has area less than one then the operator is identifiable. If the spreading support is larger than one, then the operator is not identifiable. The shape of the support region is essentially arbitrary thereby proving a conjecture of Bello. The input signal considered is a weighted delta train where the weights are the window function of a finite Gabor system whose elements satisfy a certain robust completeness property.

Keywords: Underspread operators, spreading function, Kohn–Nirenberg symbol, Channel identification, Gabor analysis

AMS subject: 47G02, 81S02 .

1. INTRODUCTION

The measurement of incompletely known linear channel operators based on the observation of a single input and the corresponding output signal is a traditional goal in *communications engineering.*

Starting in the late 1950s, Thomas Kailath analyzed the question whether an unknown time–varying channel operator H with a known restriction on time and frequency spread can be measured by applying the operator to a single known input signal f, i.e., whether the operator H can be identified by analyzing the single channel output Hf [Kai59, Kai62]. Kailath considered operators formally given by

$$(Hf)(x) = \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_0^a \eta_H(t,\nu)\, T_t M_\nu f(x)\, dt\, d\nu\,, \quad x \in \mathbb{R},$$

where T_t is a *time–shift* by t, i.e., $T_t f(x) = f(x-t)$, $t \in \mathbb{R}$, and M_ν is the *frequency shift* or *modulation* given by $\widehat{M_\nu f}(\gamma) = \widehat{f}(\gamma-\nu)$, $\nu \in \widehat{\mathbb{R}}$, i.e., $M_\nu f(x) = e^{2\pi i \nu x} f(x)$, where $\widehat{f}(\gamma) =$

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$\int f(x)\, e^{-2\pi i x\gamma}\, dx$, $\gamma \in \widehat{\mathbb{R}} = \mathbb{R}$. The function η_H is called *spreading function* of H, a denotes the *maximal time–delay* and $\frac{b}{2}$ is the *maximal doppler spread* of H.

Kailath published his results in the landmark paper *Time–Variant Communication Channels*, IEEE Transactions of Information Theory, in 1963 [Kai63]. There, he postulated that members in a collection of communication channels which are characterized by having common maximum delay a and common maximum Doppler spread $\frac{b}{2}$, i.e., all H considered do satisfy $\eta_H(t,\nu) = 0$ for $(t,\nu) \notin R = [0,a] \times [-\frac{b}{2}, \frac{b}{2}]$, would be identifiable by a single input signal if and only if the area of the rectangle R satisfies $S_R = ab \leq 1$. To show the necessity of this so–called *underspread condition*, Kailath provided ingenious arguments based on the comparison of the degrees of freedom of the operator, and degrees of freedom of the output signal. To count these degrees of freedom, Kailath used the theoretical construct of a bandlimited input signal with finite duration.

Being aware of the mathematical shortcomings of his approach, and understanding the contemporary and groundbreaking work of Slepian, Landau, and Pollak on "the dimensions of the space of essentially time- and bandlimited functions" [SP61, LP61, LP62], Kailath conjectured that the underspread condition $ab \leq 1$ is necessary in general: "Recent work by Landau and Shannon has shown that the concept of approximately $2TW$ degrees of freedom holds even in such cases. This leads us to believe that our proof of the necessity of the $BL \leq 1$ [$a = L, b = B$] condition is not merely a consequence of the special properties of strictly band-limited functions. It would be valuable to find an alternative method of proof" [Kai62]. Kailath's assertion has been proven in general only recently [KP05].

In 1969, Philip Bello published the paper *Measurement of Random Time–Variant Linear Channels*, IEEE Transactions of Information Theory, in which he postulates that the rectangular support condition $\mathrm{vol}\,(R) = S_R = ab \leq 1$ is too restrictive [Bel69]. Bello considers operators given by

$$(Hf)(x) = \iint_A \eta_H(t,\nu)\, T_t M_\nu f(x)\, dt\, d\nu\,, \quad x \in \mathbb{R},$$

where A is an essentially arbitrary bounded region in the time–frequency plane $\mathbb{R}\times\widehat{\mathbb{R}}$ and postulates: "Unfortunately the criterion $S_R < 1$ [of Kailath] has been uncritically accepted subsequently as the channel measurability criterion for random time–varying linear channels, without paying sufficiently careful attention to the conditions under which it was derived. In this paper we shall show that the criterion $S_R < 1$ is not the proper channel measurability criterion, and we shall propose a new criterion, $S_A < 1$ [S_A denotes the area of A], where the parameter S_A is called the area spread factor of the channel." In other words, Bello claimed that one could replace the rectangle spanned by maximum delay a and maximum Doppler spread $\frac{b}{2}$ by any bounded region A of time–frequency shifts. The corresponding operator class would be identifiable if the area of the region is smaller than one and not identifiable if it is larger than one. Clearly, Kailath's assertions can be seen as a special case of Bello's result, namely, when A is a rectangle.

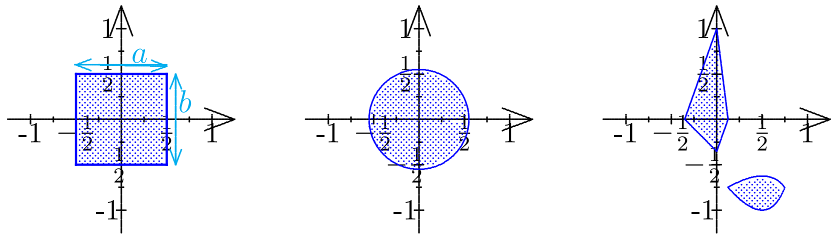

Figure 1. Spreading support regions of area less than or equal to one which characterize identifiable operator classes.

Similar to Kailath's approach, Bello discretizes the measurement setup in order to apply dimension counting arguments. In fact, he assumes that "the input to a channel is confined by a time gate to the time interval $0 < t < T$ and the output spectrum is confined by a bandpass filter to the frequency interval $-\frac{1}{2}W < t < \frac{1}{2}W$." Hence, Bello measures operators of the form $H_{W,T} = Q_W \circ H \circ P_T$ where $P_T f(x) = f(x)\,\mathbf{1}_{[0,T]}(x)$ is a *time–limiting operator*, $\widehat{Q_W f}(\nu) = \widehat{f}(\nu)\,\mathbf{1}_{[-\frac{W}{2},\frac{W}{2}]}(\nu)$ is a *frequency–limiting operator*, and the spreading function η_H of H is supported on A. The spreading function $\eta_{H_{W,T}}$ of $H_{W,T}$ is therefore not compactly supported and in particular not restricted to A. For his class of operators, Bello was able to prove necessity of $S_A \leq 1$ for identifiability, and to reduce the sufficiency condition of $S_A < 1$ to linear algebra, i.e., to the invertibility of a matrix of a finite number of time and frequency shifts of a prototype vector. Bello gave heuristic arguments for the existence of a prototype vector which guarantees the invertibility of this matrix. His assertion has been proven only recently [LPW05]. It is worth noting that the same prototype vectors play a crucial role in this paper.

Using an approach similar to Bello's together with more novel techniques from Gabor analysis we shall give a complete proof of both of Bello's assertions. Our approach does not require a time–gate for the input signal and a frequency–gate for the output signal. In fact, letting $\mathrm{vol}^-(M)$ and $\mathrm{vol}^+(M)$ denote the inner and outer Jordan content of M, respectively, (see (9) and (10)), we prove the following theorem:

THEOREM 1.1. *$\mathcal{H}_M$ is identifiable if $\mathrm{vol}^+(M) < 1$, and not identifiable if $\mathrm{vol}^-(M) > 1$.*

Here, $\mathcal{H}_M$ denotes a class of Hilbert–Schmidt operators with the property that their time–frequency spread is contained in the set M, i.e., those operators H with $\eta_H(t,\nu) = 0$ for $(t,\nu) \notin M$.

The operator classes discussed in this paper are relevant not only to communications engineering. In fact, the work of Kailath and Bello was greatly influenced by the work of Green and Price on radar measurements [Gre68][Sko80][VT71]. See [KP05] for remarks on radar and other applications.

A comparison of our result to Heisenberg's uncertainty principle is described in [KP05], in particular, we would like to point to connections with minimal rectangles in phase space as described in [FS97].

The mathematical framework used in this paper is described in Section 2. In Section 3, we shall prove that if the set A satisfies $\text{vol}^+(A) < 1$ then the corresponding operator class allows identification, and in Section 4, we shall prove that if $\text{vol}^-(A) > 1$, then the corresponding operator class does not allow identification. Note that if $\text{vol}^+(A) = \text{vol}^-(A) = \text{vol}(A)$ then A is Jordan measurable and $\text{vol}(A) = S_A$ equals the Lebesgue measure of A. (see Section 2.6).

2. PRELIMINARIES

In Section 2.1 we shall discuss the principles of channel measurement and operator identification. We shall describe our choice of domain space X and target space Y in Section 2.2 and the operator spaces $\mathcal{H}_M$ in Section 2.3. In Section 2.4 and Section 2.5, we shall present techniques from Gabor analysis that are used in this paper. In Section 2.6 we shall discuss Jordan domains and the inner and outer Jordan content of sets in euclidean space. These concepts will be used to describe spreading supports and their sizes.

Throughout this paper we are using the notation of [Grö01] and [KP05].

2.1. Channel measurements

The goal of operator (or channel) identification (or measurement) is to select, for given normed linear spaces X and Y and a normed linear space of bounded linear operators $\mathcal{H} \subset \mathcal{L}(X,Y)$, an element $f \in X$ which induces a bounded and injective, or better, a bounded and stable linear map $\Phi_f : \mathcal{H} \longrightarrow Y,\ H \mapsto Hf$ (see Figure 2). An operator is *stable* if it is invertible and the inverse operator is bounded. Consequently, we call $\mathcal{H}$ *identifiable by* $f \in X$, if there exist $A, B > 0$ with

$$A\,\|H\|_{\mathcal{H}} \leq \|Hf\|_Y \leq B\,\|H\|_{\mathcal{H}} \tag{1}$$

for all $H \in \mathcal{H}$. Note that the fact that we only consider bounded linear operators $\mathcal{H} \subset \mathcal{L}(X,Y)$ together with $\|H\|_{\mathcal{L}(X,Y)} \leq \|H\|_{\mathcal{H}}$ guarantees that for any $f \in X$, Φ_f is bounded. Hence B in

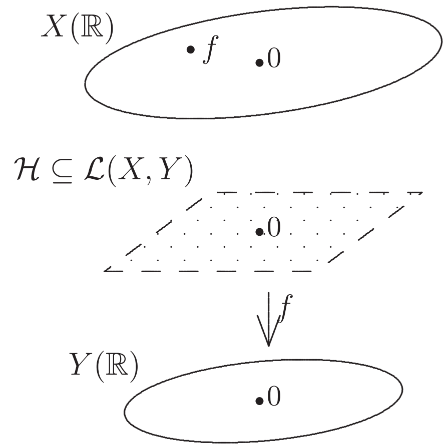

Figure 2. Identification of an operator class $\mathcal{H}$ by a vector $f \in X(\mathbb{R})$.

(1) always exists. Establishing identifiability is therefore equivalent to finding f so that for some positive A we have $A\,\|H\|_{\mathcal{H}} \leq \|Hf\|_Y$ for all $H \in \mathcal{H}$.

2.2. The Feichtinger Algebra

The identification problem considered in this paper requires the use of tempered distributions such as *Dirac's delta* $\delta : f \mapsto f(0)$ and *Shah distributions* (also called *comb–functions* or *delta trains*) $⧢_a = \sum_{n\in\mathbb{Z}^d} \delta_{an}$, where $\delta_{na} = T_{na}\delta$ and $a > 0$, as identifiers. Hence, we have to choose a domain space $X(\mathbb{R})$ which includes some tempered distributions and, therefore, we have to deviate from a standard $L^2(\mathbb{R})$[1] setup. [2]

Our choice for $X(\mathbb{R})$ is the dual $S_0'(\mathbb{R})$ of Feichtinger's Banach algebra $S_0(\mathbb{R})$ which has been introduced in [Fei81] and which has developed into a major tool in Gabor analysis.

The *Feichtinger algebra* $S_0(\mathbb{R}^d)$ is defined as follows. Let $A(\mathbb{R}^d)$ be the space of Fourier transforms of functions in $L^1(\mathbb{R}^d)$[3], with norm $\|\widehat{f}\|_A = \|f\|_{L^1}$, $l^1(\mathbb{Z}^d)$ the space of absolutely summable sequences

[1] $L^2(\mathbb{R})$ is the Hilbert space of complex valued, square–integrable functions, i.e., $f \in L^2(\mathbb{R})$ provided that $\|f\|_2^2 = \int_{\mathbb{R}} |f(x)|^2\, dx < \infty$.

[2] A possible choice for $X(\mathbb{R})$ would be the space of all tempered distributions $\mathcal{S}'(\mathbb{R}^d)$, which is the dual of the Fréchet space of Schwartz functions $\mathcal{S}(\mathbb{R}^d)$ and which is equipped with the weak–$*$ topology. Certainly, we would rather choose a Banach space as domain $X(\mathbb{R})$, since this would give us the convenience of expressing continuity (boundedness) and openess (stability) of linear operators by means of norm inequalities.

The results in this paper are consequences of the structure of the identification problem at hand, and not of topological subtleties. Our choice to work with the Banach spaces $S_0(\mathbb{R}^d)$ and $S_0'(\mathbb{R}^d)$ as opposed to the Fréchet space of of Schwartz functions $\mathcal{S}(\mathbb{R}^d) \subset S_0(\mathbb{R}^d)$ was made for convenience only.

[3] $L^1(\mathbb{R})$ is the Banach space of complex valued, integrable functions, i.e., $f \in L^1(\mathbb{R})$ provided that $\|f\|_1 = \int_{\mathbb{R}} |f(x)|\, dx < \infty$.

[Kat76] and suppose that $\psi \in A(\mathbb{R}^d)$ has compact support and satsifies $\sum_{n\in\mathbb{Z}^d} T_n\psi = 1$. Then $f \in S_0(\mathbb{R}^d)$ provided that $\sum_{n\in\mathbb{Z}^d} \|f \cdot T_n\psi\|_A < \infty$. Moreover, for each such $\psi \in A(\mathbb{R}^d)$,

$$\|f\|_{S_0} = \sum_{n\in\mathbb{Z}^d} \|f \cdot T_n\psi\|_A$$

defines an equivalent norm on $S_0(\mathbb{R}^d)$. Intuitively, $f \in S_0(\mathbb{R}^d)$ if and only if f is locally in $A(\mathbb{R}^d)$ with global decay of l^1–type. $S_0(\mathbb{R}^d)$ is therefore the same as the *Wiener amalgam space* $W(A(\mathbb{R}^d), l^1(\mathbb{Z}^d))$ (see e.g., [Grö01]).

An equivalent characterization of $S_0(\mathbb{R}^d)$ is the following.

$$S_0(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d) : V_{g_0}f\,(t,\nu) = \int f(x)\; e^{-2\pi i\nu x} g_0(x-t)\,dx \;\in L^1(\mathbb{R}^d \times \widehat{\mathbb{R}}^d) \right\}$$

where $V_{g_0}f$ is the *short time Fourier transform* of f with respect to the gaussian window $g_0(x) = e^{-\pi\|x\|_2^2}$, $x \in \mathbb{R}^d$. In fact, $\|f\| = \|V_{g_0}f(\cdot,\cdot)\|_{L^1(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)}$ is an equivalent norm on $S_0(\mathbb{R}^d)$.

We shall now mention some important properties of $S_0(\mathbb{R}^d)$. First of all, we have the inclusions $\mathcal{S}(\mathbb{R}^d) \subset S_0(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d) \supset S_0'(\mathbb{R}^d) \supset L^2(\mathbb{R}^d)$, where $\mathcal{S}(\mathbb{R}^d)$ denotes the space of Schwartz functions and $\mathcal{S}'(\mathbb{R}^d)$ its dual, i.e., the space of tempered distributions. The Fourier transform, and the modulation (M_ν, $\nu \in \widehat{\mathbb{R}}^d$) and translation ($T_t$, $t \in \mathbb{R}^d$) operators, are isometric isomorphisms on $S_0(\mathbb{R}^d)$ and hence on its dual $S_0'(\mathbb{R}^d)$. Moreover, the Feichtinger algebra $S_0(\mathbb{R}^d)$ can be continuously embedded in any Banach space which has these properties and which contains at least one, and therefore all, non–trivial Schwartz function [FZ98]. The usefulness of $S_0(\mathbb{R}^d)$ stems from the fact that it is in this sense the smallest Banach space allowing for meaningful time–frequency analysis and as a consequence, this property extends to its correspondingly large dual Banach space $S_0'(\mathbb{R}^d)$. Finally $S_0(\mathbb{R}^d)$ is a Banach algebra under convolution and pointwise multiplication.

The dual space $S_0'(\mathbb{R}^d)$ of the Feichtinger algebra satisfies $S_0'(\mathbb{R}^d) = W(A'(\mathbb{R}^d), l^\infty(\mathbb{Z}^d))$[FG85]. Hence, $S_0'(\mathbb{R}^d)$ contains Dirac's delta δ and Shah distributions.

2.3. Hilbert–Schmidt operators with bandlimited symbols

A *Hilbert–Schmidt operator* H is a bounded linear operator on $L^2(\mathbb{R}^d)$ which can be represented as an integral operator

$$Hf(x) = \int \kappa_H(x,t) f(t)\,dt = \int \kappa_H(x, x-t) f(x-t)\,dt \quad (a.e.), \tag{2}$$

with kernel $\kappa_H \in L^2(\mathbb{R}^{2d})$ [Die70, Gaa73]. The linear space of Hilbert–Schmidt operators $HS(L^2(\mathbb{R}^d))$ is endowed with the Hilbert space structure of $L^2(\mathbb{R}^{2d})$ by setting

$$\langle H_1, H_2\rangle_{\mathrm{HS}} = \langle \kappa_{H_1}, \kappa_{H_2}\rangle_{L^2}.$$

The spreading function η_H of a Hilbert–Schmidt operator H is given by

$$\eta_H(t,\nu) = \int \kappa_H(x, x-t) e^{-2\pi i \nu x}\, dx \quad (a.e.) \tag{3}$$

and leads to a representation of H as an operator valued integral by means of

$$H = \iint \eta_H(t,\nu)\, T_t M_\nu \, dt\, d\nu\,. \tag{4}$$

Note that throughout this paper, operator valued integrals are interpreted weakly, i.e., if $H(z)$ is an operator valued function on $\mathbb{R}^d$ then the action of $\int H(z)dz$ on $f \in L^2(\mathbb{R})$, is defined by

$$\Big\langle \int H(z)dz f,\, g\Big\rangle_{L^2(\mathbb{R}^d)} = \int \langle H(z)f, g\rangle_{L^2(\mathbb{R}^d)}\, dz \quad \text{for all } g \in L^2(\mathbb{R}^d).$$

The Kohn–Nirenberg symbol σ_H of a Hilbert–Schmidt operator H is given by

$$\sigma_H(x,\xi) = \int \kappa_H(x, x-y)\, e^{-2\pi i y \xi}\, dy \quad (a.e.)$$

[KN65, Fol89]. Note that

$$\|\sigma_H\|_{L^2} = \|\eta_H\|_{L^2} = \|\kappa_H\|_{L^2} = \|H\|_{HS}.$$

As previously mentioned, our results require the use of Shah distributions as identifiers (see Section 2.2). Not all Hilbert–Schmidt operators in $\mathcal{L}(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))$ can be extended to act on a space of distributions containing the Shah distribution, hence, we shall narrow the class of operators considered to those which satisfy a regularity condition on their kernels. Since $⧢_a \in S_0'(\mathbb{R})$, it is natural to choose

$$\mathcal{H} = \left\{ H \in HS(L^2(\mathbb{R}^d)) :\ \kappa_H \in S_0(\mathbb{R}^{2d}) \right\},$$

since then $\mathcal{H} \subseteq \mathcal{L}(S_0'(\mathbb{R}), S_0(\mathbb{R})) \subset \mathcal{L}(S_0'(\mathbb{R}), L^2(\mathbb{R}))$ [FZ98].

Identifiability will be shown for operator classes with compactly supported spreading function, i.e., we consider operator classes of the form

$$\mathcal{H}_M = \{ H \in \mathcal{H} :\ \operatorname{supp} \eta_H \subseteq M\}\ , \quad M \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

Since σ_H is the symplectic Fourier transform of η_H, i.e.,

$$\sigma_H(x,\xi) = \iint \eta_H(t,\nu) e^{2\pi i(x\nu - t\xi)}\, dt\, d\nu \ (a.e.),$$

each Hilbert–Schmidt operator considered here has therefore a bandlimited Kohn–Nirenberg symbol.

Note that $\mathcal{H}_M \subseteq \mathcal{H}_{M'}$ if $M \subseteq M'$, and that the linear spaces $\mathcal{H}$ and $\mathcal{H}_M$, $M \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, are not closed as linear subspaces of the space of Hilbert–Schmidt operators.

Equation (4) illustrates that support restrictions on η_H reflect limitations on the maximal time and frequency shifts which the input signals undergo: Hf is a continuous superposition of time–frequency shifted versions of f with weighting function η_H.

Further, note that if an operator H satisfies $\eta_H(\cdot,\nu) \subseteq [0,a]$ for all $\nu \in \widehat{\mathbb{R}}$, then $\kappa_H(x, x-t)$ vanishes for $x \in \mathbb{R}$ and $t \notin [0,a]$, and for f with $\operatorname{supp} f \subseteq [0,T]$ we have $\operatorname{supp} Hf \subseteq [0, T{+}a]$. Similarly, if $\eta_H(t,\cdot) \subseteq [-\frac{b}{2}, \frac{b}{2}]$ for all $t \in \mathbb{R}$, then for f with $\operatorname{supp} \widehat{f} \subseteq [-\Omega, \Omega]$ we have $\operatorname{supp} \widehat{Hf} \subseteq [-(\Omega{+}\frac{b}{2}), \Omega{+}\frac{b}{2}]$. Hence, the condition

$$\operatorname{supp} \eta_H \subseteq Q_{a,b} = [0,a]^d \times [-\tfrac{b}{2}, \tfrac{b}{2}]^d \tag{5}$$

for some $a, b > 0$, reflects a limitation on the maximal time delay a and the maximal frequency spread $\frac{b}{2}$ produced by H. An operator which satisfies (5) for $a, b > 0$ is called *underspread* if $ab \leq 1$ and *overspread* if $ab > 1$.

A comparison of (2) to a time–invariant convolution operator K given by $Kf(x) = \int \kappa_K(t) f(x - t)\, dt$ — whose kernel κ_K is independent of the time variable x — together with (3) shows that the condition $\eta_H(t,\cdot) \subseteq [-\frac{b}{2}, \frac{b}{2}]$ for all $t \in \mathbb{R}$, excludes high frequencies and therefore rapid change of $\kappa(x, x-t)$ as a function of x. This further illuminates the role of underspread and overspread operators in the analysis of *slowly time–varying* communications channels.

The previous paragraphs emphasize the usefulness of η_H in the time–frequency analysis of operators. Additional remarks on the use of Hilbert–Schmidt operators as model of physical time–varying linear systems, as they appear in radar and in mobile communications can be found in [FL96, Yoo02, KP05] .

2.4. Gabor analysis on $L^2(\mathbb{R}^d)$

One of the fundamental questions in Gabor theory is to show when a function $f \in L^2(\mathbb{R}^d)$ can be stably recovered from its *Gabor coefficients* $\{\langle f, M_{kb} T_{la} g \rangle\}_{k,l \in \mathbb{Z}^d}$ (here a, $b > 0$ and $g \in L^2(\mathbb{R}^d)$ are fixed) or whether any f can be approximated by finite linear combinations of elements of the *Gabor* *system* $(g, a, b) = \{M_{kb}T_{la}g\}_{k,l\in\mathbb{Z}^d}$ [Gab46, Grö01]. Specifically, we ask whether the system (g, a, b) is a *frame* for $L^2(\mathbb{R}^d)$, that is, whether there exist $A, B > 0$ such that

$$A\|f\|_{L^2}^2 \leq \sum |\langle f, M_{kb}T_{la}g\rangle|^2 \leq B\|f\|_{L^2}^2 \quad \text{for all } f \in L^2(\mathbb{R}^d)\,. \tag{6}$$

If (6) holds then every $f \in L^2(\mathbb{R}^d)$ has a stable representation

$$f = \sum_k \sum_l \langle f, M_{kb}T_{la}\gamma\rangle\, M_{kb}T_{la}g = \sum_k \sum_l \langle f, M_{kb}T_{la}\gamma\rangle\, M_{kb}T_{la}g \ \text{ in } L^2(\mathbb{R}^d) \tag{7}$$

where $\gamma \in L^2(\mathbb{R}^d)$ and (γ, a, b) is called a dual frame of (g, a, b). A frame is *tight* if $A = B$ and is *exact* if it ceases to be a frame upon the removal of a single element. A frame which is not exact is also called *overcomplete*.

For each Gabor system (g, a, b) define the *analysis map* C_g on $L^2(\mathbb{R}^d)$ by

$$C_g(f) = \{\langle f, M_{kb}T_{la}g\rangle\}_{m,n\in\mathbb{Z}^d}$$

and the *synthesis map* D_g on $l_0(\mathbb{Z}^{2d})$ (the space of finite sequences on $\mathbb{Z}^{2d}$) by

$$D_g(\{c_{k,l}\}) = \sum_{k,l} c_{k,l}\, M_{kb}T_{la}g.$$

Then the following holds.

Theorem 2.1. *If (g, a, b) is a frame for $L^2(\mathbb{R}^d)$ then*

(a) *$C_g \colon L^2(\mathbb{R}^d) \longrightarrow l^2(\mathbb{Z}^{2d})$ is well–defined, bounded, stable and has closed range.*

(b) *$D_g \colon l^2(\mathbb{Z}^{2d}) \to L^2(\mathbb{R}^d)$ is bounded and is an isomorphism when restricted to* Range(C_g).

(c) *The operator $D_g \circ C_g \colon L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$, called the* frame operator *is a linear isomorphism of $L^2(\mathbb{R}^d)$ and $\gamma = (D_g \circ C_g)^{-1}g$ satisfies (7).*

Some of the fundamental results in Gabor frame theory related to quantity ab and in particular to the critical value $ab = 1$. Below the critical value the following holds.

Theorem 2.2. *If (g, a, b) is a frame for $L^2(\mathbb{R}^d)$, then the following are equivalent*

(a) *$ab < 1$.*

(b) *(g, a, b) is* overcomplete *in $L^2(\mathbb{R})$.*

(c) $C_g \colon L^2(\mathbb{R}^d) \to l^2(\mathbb{Z}^{2d})$ *is not surjective.*

(d) $D_g \colon l^2(\mathbb{Z}^{2d}) \to L^2(\mathbb{R}^d)$ *is not injective, that is there is a nontrivial sequence* $\{c_{k,l}\} \in l^2(\mathbb{Z}^{2d})$ *such that* $\sum_{n,m} c_{k,l} M_{kb} T_{la} g = 0$.

For example, we have that for any a, b with $ab < 1$ (g_0, a, b) where $g_0 : \mathbb{R} \to \mathbb{R}^+, \quad x \mapsto e^{-\pi x^2}$ is the Gaussian, is a frame [Lyu92, SW92, Sei92].

At the critical value $ab = 1$, if (g, a, b) is a frame then it must be an exact frame. A *Riesz basis* is the image of an orthonormal basis under a linear isomorphism and the class of exact frames is identical to the class of Riesz bases for $L^2(\mathbb{R}^d)$. A direct characterization of Riesz bases is the following. A collection $\{f_n\}$ is a Riesz basis for $L^2(\mathbb{R}^d)$ if there exist constants A, $B > 0$ such that for all finite sequences (c_n),

$$A \sum_n |c_n|^2 \le \left\| \sum_n c_n f_n \right\|_{L^2}^2 \le B \sum_n |c_n|^2. \tag{8}$$

Clearly an orthonormal basis is also a Riesz basis and in fact the Gabor system $\left(a^{-\frac{1}{2}} \mathbf{1}_{[0,a)}, a, \frac{1}{a}\right)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$ (here $\mathbf{1}_A(x)$ denotes the *characteristic function* of the set A).

In terms of the analysis and synthesis operators, the following holds.

Theorem 2.3. *If (g, a, b) is a frame for $L^2(\mathbb{R}^d)$ and $ab = 1$ then the following are equivalent.*

(a) $ab = 1$.

(b) (g, a, b) *is a* Riesz basis *for* $L^2(\mathbb{R})$.

(c) C_g *is an isomorphism of* $L^2(\mathbb{R})$ *onto* $l^2(\mathbb{Z}^2)$.

(d) D_g *is an isomorphism of* $l^2(\mathbb{Z}^2)$ *onto* $L^2(\mathbb{R})$.

Above the critical value, that is, when $ab > 1$, (g, a, b) is not complete in $L^2(\mathbb{R}^d)$ for any $g \in L^2(\mathbb{R}^d)$. In other words, the analysis map C_g is not injective. However, the following holds.

Theorem 2.4. *If $(g, 1/b, 1/a)$ is a frame for $L^2(\mathbb{R}^d)$, then the following are equivalent.*

(a) $ab > 1$.

(b) (g, a, b) *is a* Riesz basis *for its closed linear span in* $L^2(\mathbb{R})$.

(c) *The synthesis map D_g is is well–defined, bounded, stable and has closed range in $L^2(\mathbb{R}^d)$.*

(d) *There exist $A, B > 0$ such that for every finite sequence $\{c_{k,l}\}_{k,l\in\mathbb{Z}^d} \in l_0(\mathbb{Z}^{2d})$*

$$A\|\{c_{k,l}\}\|_{l^2} \leq \left\| \sum_{k,l\in\mathbb{Z}^d} c_{k,l}\, M_{kb}T_{la}g \right\|_{L^2} \leq B\|\{c_{k,l}\}\|_{l^2}.$$

In particular, if g_0 is the Gaussian, then for every a, b with $ab > 1$ the system (g, a, b) is a Riesz basis for its closed linear span in $L^2(\mathbb{R}^{2d})$.

More details on time–frequency analysis with some relevance to this paper can be found in [Grö01].

Operator–theoretic applications of Gabor theory as presented in this paper have drawn increasing interest in applied harmonic analysis, see, for example, [Dau88, HRT97, FK98, Koz98, RT98, Lab01, FN03, CG03, Hei03, GLM04].

2.5. Gabor analysis on $\mathbb{C}^L$

Discrete Gabor systems on finite dimensional spaces can be defined in a natural way and properties of such systems will be used in Section 3. Let $L \in \mathbb{N}$ be fixed and let $\omega = e^{-2\pi i/L}$. The translation operator T is the unitary operator on $\mathbb{C}^L$ given by $Tx = T(x_0, \ldots, x_{L-1}) = (x_{L-1}, x_0, x_1, \ldots, x_{L-2})$, and the modulation operator M is the unitary operator given by $Mx = M(x_0, \ldots, x_{L-1}) = (\omega^0 x_0, \omega^1 x_1, \ldots, \omega^{L-1} x_{L-1})$. Given a vector $c \in \mathbb{C}^L$ the *full Gabor system with window* c is the collection $\{M^l T^k c\}_{l,k=0}^{L-1}$.

Full Gabor systems as defined above have many nice properties. In particular any such system is a *uniform tight frame for* $\mathbb{C}^L$ with frame bound

$$L^2 \sum_{n=0}^{L-1} |c_n|^2.$$

That is, all vectors in the system have the same norm and for any $x \in \mathbb{C}^L$,

$$\sum_{l=0}^{L-1} \sum_{k=0}^{L-1} |\langle x, M^l T^k c\rangle|^2 = \left(L^2 \sum_{n=0}^{L-1} |c_n|^2 \right) \|x\|^2.$$

The class of uniform, tight finite frames is important in communication and coding theory (see e.g., [GKK01, LPW05]).

The property of full Gabor systems of interest in this paper is the following. We say that a family $\mathcal{F}$ of vectors in $\mathbb{C}^L$ with $|\mathcal{F}| \geq L$ has the *Haar property* (cf. [Che98]) if any subset $\mathcal{F}' \subset \mathcal{F}$ with $|\mathcal{F}'| = L$ is linearly independent. The following theorem was recently proved in [LPW05].

Theorem 2.5. *If L is prime then there is a dense open set E in $\mathbb{C}^L$ of full measure (i.e., $|\mathbb{C}^L \setminus E| = 0$) such that for every $c \in E$, the Gabor system $\{M^l T^k c\}_{l,k=0}^{L-1}$ has the Haar property.*

A restatement of Theorem 2.5 in slightly different language is useful for the calculation in Section 3 below. The *discrete $L \times L$ Fourier matrix W_L* is defined by $W_L = (\omega^{pq})_{p,q=0}^{L-1}$. Let $c = (c_0, c_1, \ldots, c_{L-1}) \in \mathbb{C}^L$ be given and for $k = 0, 1, \ldots, L-1$, let D_k be the diagonal matrix

$$D_k = \begin{pmatrix} c_k & & & & & & \\ & c_{k+1} & & & & & \\ & & \ddots & & & & \\ & & & c_{L-1} & & & \\ & & & & c_0 & & \\ & & & & & \ddots & \\ & & & & & & c_{k-1} \end{pmatrix}.$$

Define the $L \times L^2$ *full Gabor system matrix $\mathcal{A}$* by

$$\mathcal{A} = (D_0 \cdot W_L \,|\, D_1 \cdot W_L \,|\, \cdots \,|\, D_{L-1} \cdot W_L) \equiv (A_0 \,|\, A_1 \,|\, \cdots \,|\, A_{L-1}).$$

In fact,

$$A_k = (c_{p+k}\, \omega^{qp})_{p,q=0}^{L-1}$$

where the subscript of c is taken modulo L. It is clear that the columns of $\mathcal{A}$ are the vectors $\{M^l T^k c\}_{l,k=0}^{L-1}$. In this notation, Theorem 2.5 can be stated as

Proposition 2.6. *If L is prime then there is a dense open set E in $\mathbb{C}^L$ of full measure such that for every $c \in E$, every minor of order L of $\mathcal{A}$ (that is, the determinant of every $L \times L$ submatrix of $\mathcal{A}$) is nonzero.*

2.6. Jordan domains and Jordan content

In [KP05] it was shown that if $M \subset \mathbb{R}\times\widehat{\mathbb{R}}$ is a parallelogram, then $\mathcal{H}_M = \{H \in \mathcal{H} : \operatorname{supp} \eta_H \subseteq M\}$ is identifiable if and only if the area of M is less than or equal to one. In order to extend this result to essentially arbitrary regions, we need to establish some technical restrictions on the regions considered. We choose a definition of Jordan content and Jordan domains which differs from but is equivalent to those found in most textbooks, e.g., see [Fol99].

Definition 2.7. *For $K, L \in \mathbb{N}$ set $R_{K,L} = [0, \frac{1}{K}] \times [0, \frac{K}{L}]$ and*

$$\mathcal{U}_{K,L} = \left\{ \bigcup_{j=1}^{J} \Big(R_{K,L} + (\tfrac{m_j}{K}, \tfrac{n_j K}{L}) \Big) : \ m_j, n_j \in \mathbb{Z}, J \in \mathbb{N} \right\}.$$

Let $M \subset \mathbb{R}\times\widehat{\mathbb{R}}$ be bounded and let μ be the Lebesgue measure on $\mathbb{R}\times\widehat{\mathbb{R}}$. The inner content *of M is defined as*

$$\operatorname{vol}^-(M) = \sup\{\mu(U) : U \subset M \text{ and } U \in \mathcal{U}_{K,L} \text{ for some } K, L \in \mathbb{N}\} \tag{9}$$

and the outer content *of M is given by*

$$\text{vol}^+(M) = \inf\{\mu(U) : U \supset M \text{ and } U \in \mathcal{U}_{K,L} \text{ for some } K, L \in \mathbb{N}\}. \tag{10}$$

Clearly, we have $\text{vol}^-(M) \leq \text{vol}^+(M)$ *and if* $\text{vol}^-(M) = \text{vol}^+(M)$*, then we say that M is a* Jordan domain *with* Jordan content $\text{vol}(M) = \text{vol}^-(M) = \text{vol}^+(M)$.

In the following proposition we collect some relevant facts on Jordan content.

PROPOSITION 2.8. *Let $M \subset \mathbb{R}\times\widehat{\mathbb{R}}$*

1. *If M is a Jordan domain, then M is Lebesgue measurable with* $\mu(M) = \text{vol}(M)$.
2. *If M is Lebesgue measurable and bounded and its boundary ∂M is a Lebesgue zero set, i.e., $\mu(\partial M) = 0$, then M is a Jordan domain.*
3. *If M is open, then* $\text{vol}^-(M) = \mu(M)$ *and if M is compact, then* $\text{vol}^+(M) = \mu(M)$.
4. *If $\mathcal{P} \subset \mathbb{N}$ is unbounded, then replacing the quantifier* "for some $L \in \mathbb{N}$" *with* "for some $L \in \mathcal{P}$" *in (9) and in (10) leads to equivalent definitions of inner and outer Jordan content.*

It should be noted that not all Lebesgue measurable sets are Jordan domains. For example let M be a Cantor set of positive Lebesgue measure in $\mathbb{R}\times\widehat{\mathbb{R}}$. In this case $\text{vol}^-(M) = 0$ whereas $\text{vol}^+(M) = \mu(M) > 0$. Such pathologies lead to cases in which classes of operators whose spreading functions have arbitrarily large support (in the sense of Lebesgue measure) are still identifiable. For example, for $n \in \mathbb{N}$ choose M_n to be a Cantor set with $\mu(M_n) = n$. Since M_n is nowhere dense and all operators in $\mathcal{H}$ have continuous spreading symbol, we have $\mathcal{H}_{M_n} = \{0\}$, which is clearly identifiable regardless of n. However, such an example is clearly not physically realistic.

3. SUFFICIENCY OF $VOL(M) < 1$ FOR THE IDENTIFIABILITY OF $\mathcal{H}_M$

In this section, we shall prove the following theorem.

THEOREM 3.1. *The class $\mathcal{H}_M$ is identifiable if* $\text{vol}^+(M) < 1$.

The case that M is a rectangle, i.e., $M = [a_1, a_2]\times[b_1, b_2]$ for some $a_2 > a_1 > 0$ and $b_2 > b_1 > 0$, has been considered by Kailath [Kai59, Kai62]. If $\text{vol}^+(M) = (a_2 - a_1)(b_2 - b_1) \leq 1$, then $⧢_a$ identifies $\mathcal{H}_M$ whenever $a_2 - a_1 \leq a \leq (b_2 - b_1)^{-1}$, since $H⧢_a$ records samples of $\kappa_H(\cdot, \cdot - t)$ at least at the critical sampling rate $(b_2 - b_1)^{-1}$, while $a \geq a_2 - a_1$ guarantees that no aliasing of samples

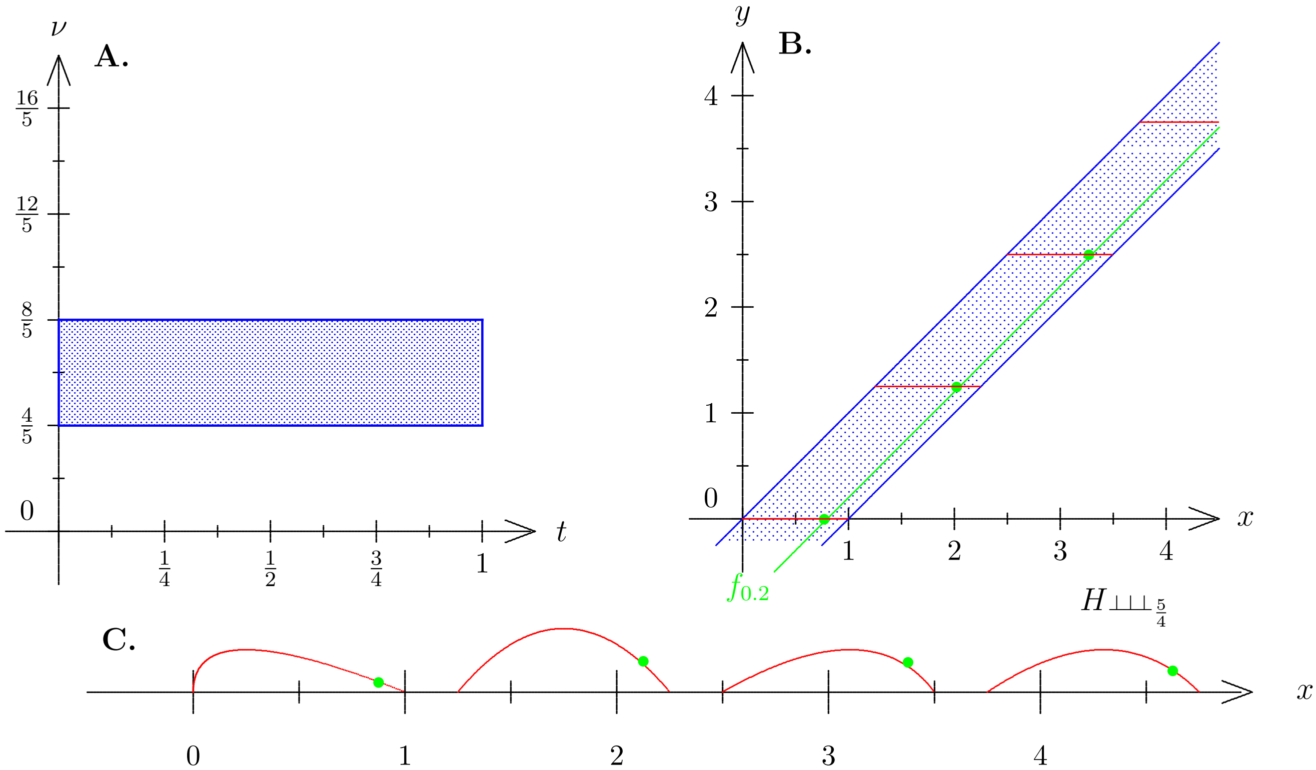

Figure 3. Identification of $\mathcal{H}_M$ for $M = [\frac{4}{5}, \frac{8}{5}] \times [0,1] \in \mathcal{U}_{4,5}$. **A.** Spreading support set M, $\operatorname{vol} M = \frac{4}{5} < 1$. **B.** Support of kernel κ_H of $H \in \mathcal{H}_M$. The function κ_H is bandlimited along the diagonals, i.e., $\operatorname{supp} \widehat{\kappa_H(\cdot, \cdot - t)} = \operatorname{supp} \eta_H(\cdot, t) \subseteq [\frac{4}{5}, \frac{8}{5}]$ for all $t \in [0,1]$. Here, $f_{0.2}(x) = \kappa_H(x, x-(1-0.2))$, $x \in \mathbb{R}$. **C.** Channel output $H Ш_{\frac{5}{4}}$, which contains all sampling values of κ_H needed to reconstruct κ_H and therefore H. Sampling values of $f_{0.2}$ are singled out.

takes place. See Figure 3 for details. The situation for M not being contained in a rectangle of volume at most one is more complicated (see Figure 4).

We divide the proof of Theorem 3.1 into four parts. In Section 3.1 we slightly simplify our setup. For fixed M, we construct in Section 3.2 a distribution $f \in S_0'(\mathbb{R})$ which will be used for identification later in the proof. This f induces a linear map

$$\Phi_f : \mathcal{H}_M \longrightarrow L^2(\mathbb{R}),\ H \mapsto Hf$$

whose injectiveness is shown in Section 3.3. In Section 3.4 we shall see that that Φ_f is indeed bounded and stable, implying, finally, that $\mathcal{H}_M$ is identifiable by f.

3.1. Assume $M \subseteq [0,1] \times [0,\infty)$

As the first step in the proof, we assume without loss of generality that if $(t,\nu) \in M$ then $t \in [0,1]$ and $\nu \geq 0$. To see why this can be done, suppose that for some a, $b > 0$, $M \subseteq [0,a] \times [-\frac{b}{2}, \frac{b}{2}]$ and $H \in \mathcal{H}_M$. Define $\widetilde{H} \in \mathcal{H}$ via

$$\kappa_{\widetilde{H}}(x,y) = a\,\kappa_H(ax,ay)\,e^{\pi i abx}.$$

Then it is easy to see that

$$\begin{aligned}
\eta_{\widetilde{H}}(t,\nu) &= \int \kappa_{\widetilde{H}}(x,x-t)\,e^{-2\pi i\nu x}\,dx \\
&= \int a\kappa_H(ax,a(x-t))\,e^{\pi i abx}\,e^{-2\pi i\nu x}\,dx \\
&= \int \kappa_H(x,x-at)\,e^{-2\pi i(\frac{\nu}{a}-\frac{b}{2})x}\,dx \\
&= \eta_H(at, \tfrac{\nu}{a} - \tfrac{b}{2})
\end{aligned}$$

and that if $\operatorname{supp}\eta_H \subseteq [0,a] \times [-\frac{b}{2}, \frac{b}{2}]$ then $\operatorname{supp}\eta_{\widetilde{H}} \subseteq [0,1] \times [0,ab]$.

3.2. Construction of the identifier f

The identifying distribution will have the form

$$f = \sum_k c_k\,\delta_{\frac{k}{K}} \tag{11}$$

for some $K \in \mathbb{N}$ and some $c = (c_0, c_1, \ldots, c_{L-1}) \in \mathbb{C}^L$, where indices of c_k in the sum are taken modulo L. The goal is to choose K, L, and c so that the aliasing in Figure 4.**C** can be controlled in such a way that κ_H and therefore H can be recovered from Hf. Below we will determine appropriate parameters K, L, and c needed to define f.

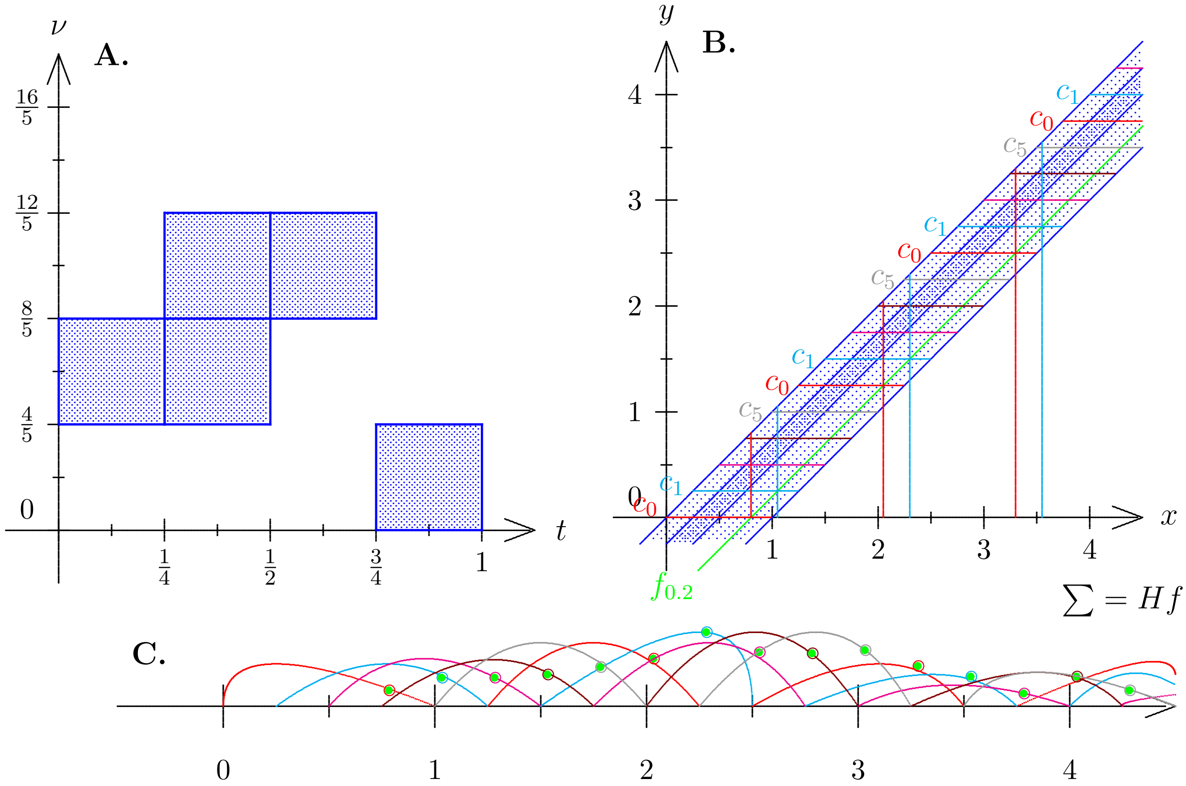

Figure 4. Identification of $\mathcal{H}_M$ for $M \in \mathcal{U}_{4,5}$ not being a rectangle. **A.** Spreading support set M, $\operatorname{vol} M = 1$. **B.** Support of kernel κ_H of $H \in \mathcal{H}_M$. The bandlimitation of $\kappa_H(\cdot, \cdot - t)$ along the diagonals depends on t. **C.** The channel output $H \sum_k c_k \delta_{\frac{k}{4}}$ is the sum over all functions displayed here, leading to aliasing of samples in the channel output $H \sum_k c_k \delta_{\frac{k}{4}}$. Samples of $f_{0.2}$ contributing to the weighted sum are marked.

Assume that $\mathrm{vol}^+(M) < 1$. With $\mathcal{P}$ denoting the set of prime numbers, Proposition 2.8(4) says that we can choose $K, L \in \mathbb{N}$ with L prime so that (i) $M \subseteq [0,1] \times [0,K]$, (ii) $L \geq K$, and (iii) $M \subseteq U_M$ with $U_M \in \mathcal{U}_{K,L}$ and $\mathrm{vol}\,(U_M) \leq 1$, i.e.,

$$U_M = \bigcup_{j=0}^{J-1} \Big(R_{K,L} + (\tfrac{m_j}{K}, \tfrac{n_j K}{L}) \Big), \qquad m_j, n_j \in \mathbb{Z}, J \in \mathbb{N},$$

where $R_{K,L} = [0, \frac{1}{K}] \times [0, \frac{K}{L}]$ and where $(m_j, n_j) \neq (m_{j'}, n_{j'})$ if $j \neq j'$. Note further that $1 \geq \mathrm{vol}\,(U_M) = J\,\mathrm{vol}\,(R_{K,L}) = \frac{J}{L}$ implies $J \leq L$. Since $\mathcal{H}_M \supset \mathcal{H}_{M'}$ if $M \supset M'$, the identifiability of $\mathcal{H}_M$ implies the identifiability of $\mathcal{H}_M \supset \mathcal{H}_{M'}$, and, by adding some additional cells to M if necessary, we can assume in what follows that $J = L$.

For any $c = (c_0, c_1, \ldots, c_{L-1}) \in \mathbb{C}^L$, let $A(c)$ denote the $L \times KL$ matrix

$$A(c) = [A_0 \;\; A_1 \;\; \cdots \;\; A_{K-1}]$$

where the $L \times L$ matrices A_k are defined by (2.5) and have the form

$$A_k = (c_{p+k}\, \omega^{qp})_{p,q=0}^{L-1}$$

where $\omega = e^{-2\pi i/L}$ and where the subscripts on c are taken modulo L. Note that since $K \leq L$, the matrix A is a submatrix of the full Gabor system matrix $\mathcal{A}$. In light of Proposition 2.6 there exists $c \in \mathbb{C}^L$ so that every $L \times L$ submatrix of $\mathcal{A}$ is invertible. Since A is a submatrix of $\mathcal{A}$, the same is true of every $L \times L$ submatrix of A.

Choose such a c and define f as in (11).

3.3. Determining $H \in \mathcal{H}_M$ from $H(f)$.

The operator $H \in \mathcal{H}_M$ is completely determined by its kernel κ_H. Therefore it is sufficient to show that κ_H can be recovered from $H(f)$.

For $t \in [0, \frac{1}{K}]$ and $0 \leq k < K$ define the function

$$\kappa_k(t,x) = \kappa_H(1 - t - \tfrac{k}{K} + x, x).$$

By our assumptions on κ_H, $\kappa_k(t, \cdot)$ is bandlimited to $[0, K]$ for each t and k. The Fourier transform of $\kappa_k(t,x)$ in the second variable is

$$e^{2\pi i \nu (1 - t - \frac{k}{K})}\, \eta_H(1 - t - \tfrac{k}{K}, \nu) \equiv \eta_k(t, \nu)$$

so that η_H and subsequently κ_H is completely determined by $\eta_k(t, \nu + \frac{pK}{L})$ for $(t, \nu) \in R_{K,L}$, $0 \leq k < K$ and $0 \leq p < L$.

With f as in (11),

$$(Hf)(x) = \left\langle \kappa(x, \cdot), \sum_k c_k \delta_{\frac{k}{K}}(\cdot) \right\rangle = \sum_k c_k\, \kappa_H(x, \tfrac{k}{K}).$$

As previously mentioned, in this sum and subsequently, all subscripts of c are taken modulo L. For $t \in [0, \frac{1}{K}]$ let $x_n(t) = (1-t) + \frac{n}{K}$. By our assumptions on κ_H, $(Hf)(x_n(t))$ reduces to a finite sum and we can write

$$\begin{aligned} s_n(t) &\equiv (Hf)(x_n(t)) \qquad (12)\\ &= \sum_k c_k\, \kappa_H(x_n(t), \tfrac{k}{K}) \\ &= \sum_{k=0}^{K-1} c_{n+k}\, \kappa_H(1 - t + \tfrac{n}{K}, \tfrac{n+k}{K}) \\ &= \sum_{k=0}^{K-1} c_{n+k}\, \kappa_H(1 - t - \tfrac{k}{K} + \tfrac{n+k}{K}, \tfrac{n+k}{K}) \\ &= \sum_{k=0}^{K-1} c_{n+k}\, \kappa_k(t, \tfrac{n+k}{K}). \end{aligned}$$

Letting $n = mL + p$ with $m \in \mathbb{Z}$ and $0 \leq p < L$, we write $s_m^p(t) = s_{mL+p}(t)$. Then

$$s_m^p(t) = \sum_{k=0}^{K-1} c_{mL+p+k}\, \kappa_k\left(t, \tfrac{mL}{K} + \tfrac{p+k}{K}\right) = \sum_{k=0}^{K-1} c_{p+k}\, \kappa_k\left(t, \tfrac{p+k}{K} + \tfrac{mL}{K}\right). \qquad (13)$$

For each $0 \leq p < L$ form the Fourier series

$$\begin{aligned} G_p(t, \nu) &\equiv \tfrac{L}{K} \sum_m s_m^p(t)\, e^{-2\pi i \nu mL/K} \\ &= \tfrac{L}{K} \sum_m \sum_{k=0}^{K-1} c_{p+k}\, \kappa_k\left(t, \tfrac{p+k}{K} + \tfrac{mL}{K}\right) e^{-2\pi i \nu mL/K} \\ &= \tfrac{L}{K} \sum_{k=0}^{K-1} c_{p+k} \sum_m \kappa_k\left(t, \tfrac{p+k}{K} + \tfrac{mL}{K}\right) e^{-2\pi i \nu mL/K} \\ &= \sum_{k=0}^{K-1} c_{p+k} \sum_q \eta_k\left(t, \nu + \tfrac{qK}{L}\right) e^{-2\pi i \left(\frac{p+k}{K}\right)\left(\nu + \frac{qK}{L}\right)} \end{aligned}$$

where we have applied the Poisson Summation Formula on the last line. Assuming that $\nu \in [0, \frac{K}{L}]$ and since $\eta_k(t, \cdot)$ is supported in the interval $[0, K]$, it follows that the above sum over q is finite and therefore for each $0 \le p < L$ we have

$$G_p(t, \nu) = \sum_{k=0}^{K-1} \sum_{q=0}^{L-1} c_{p+k}\, \eta_k\left(t, \nu + \tfrac{qK}{L}\right)\, e^{-2\pi i \left(\frac{p+k}{K}\right)\left(\nu + \frac{qK}{L}\right)}.$$

Manipulating this expression, we arrive at the streamlined system

$$\widetilde{G}_p(t, \nu) = \sum_{k=0}^{K-1} \sum_{q=0}^{L-1} c_{p+k}\, e^{-2\pi i pq/L}\, \widetilde{\eta}_k\left(t, \nu + \tfrac{qK}{L}\right) \tag{14}$$

where $\widetilde{G}_p(t, \nu) = G_p(t, \nu)\, e^{2\pi i p\nu/K}$ and $\widetilde{\eta}_k(t, \nu) = \eta_k(t, \nu)\, e^{-2\pi i k\nu/K}$. In other words, we can derive a system of L equations in KL unknowns for the functions

$$\left\{\widetilde{\eta}_k\left(t, \nu + \tfrac{qK}{L}\right) : 0 \le k < K,\ 0 \le q < L, (t, \nu) \in [0, \tfrac{1}{K}] \times [0, \tfrac{K}{L}]\right\}, \tag{15}$$

in which the coefficients in the equation do not depend on (t, ν). It is clear that the matrix for this system is A, and that the set of functions in (15) completely determine κ_H.

Finally, note that since $M \subseteq U_M$, and for $(t, \nu) \in R_{K,L}$, we have $\widetilde{\eta}_k\left(t, \nu + \frac{qK}{L}\right) = 0$ unless $(k, q) = (m_j, n_j)$ for some $0 \le j \le J - 1$. Therefore (14) has no more than L nonzero terms in the double sum on the right hand side, and (14) reduces to a system of L equations in L unknowns. The matrix for this reduced system is simply a choice of L columns of the matrix A, specifically the j^{th} column of this matrix is the n_j^{th} column of the $L \times L$ matrix A_{m_j}. Call this matrix $\mathbf{A}_M$.

Define the $\mathbb{C}^L$–valued functions $\boldsymbol{\eta}(t, \nu)$

and $\mathbf{G}(t, \nu)$ on $R_{K,L}$ by

$$\boldsymbol{\eta}(t, \nu) = \left(\widetilde{\eta}_{m_j}\left(t, \nu + \tfrac{n_j K}{L}\right)\right)_{j=0}^{L-1}$$

and

$$\mathbf{G}(t, \nu) = \left(\widetilde{G}_p(t, \nu)\right)_{p=0}^{L-1}.$$

The system (14) can therefore be written

$$\mathbf{A}_M\, \boldsymbol{\eta}(t, \nu) = \mathbf{G}(t, \nu), \quad (t, \nu) \in R_{K,L}. \tag{16}$$

Since $\mathbf{A}_M$ is invertible, we can recover $\boldsymbol{\eta}$ pointwise from $\mathbf{G}$ which depends only on the channel output Hf. From $\boldsymbol{\eta}$ we can recover η_H and hence the kernel $\kappa_H(x, x - t)$ of H.

3.4. Boundedness and stability

Here we show that (1) holds.

LEMMA 3.2. *With f given in Section 3.2, $H \in \mathcal{H}_M$, and $\boldsymbol{\eta}$ and $\mathbf{H}$ as in* (16),

(a) $\|H(f)\|^2_{L^2(\mathbb{R})} = \iint_{R_{K,L}} \|\mathbf{H}(t,\nu)\|^2_{\mathbb{C}^L}\, dt\, d\nu.$

(b) $\|H\|^2_{\mathcal{H}} = \|\kappa_H\|^2_{L^2(\mathbb{R}^2)} = \iint_{R_{K,L}} \|\boldsymbol{\eta}(t,\nu)\|^2_{\mathbb{C}^L}\, d\nu\, dt.$

Proof. (a) Using the definition of s_n in (12) and the definition of s^p_m in (13), we have

$$\begin{aligned}
\|H(f)\|^2_{L^2(\mathbb{R})} &= \int_{-\infty}^{\infty} |Hf(t)|^2\, dt \\
&= \int_{-\infty}^{\infty} |Hf(1-t)|^2\, dt \\
&= \sum_n \int_{\frac{n}{K}}^{\frac{n+1}{K}} |Hf(1-t)|^2\, dt \\
&= \sum_n \int_0^{\frac{1}{K}} |Hf(1-t-\tfrac{n}{K})|^2\, dt \\
&= \sum_n \int_0^{\frac{1}{K}} |Hf(x_n(t))|^2\, dt \\
&= \sum_n \int_0^{\frac{1}{K}} |s_n(t)|^2\, dt \\
&= \int_0^{\frac{1}{K}} \sum_{p=0}^{L-1} \sum_m |s^p_m(t)|^2\, dt \\
&= \int_0^{\frac{1}{K}} \sum_{p=0}^{L-1} \int_0^{\frac{K}{L}} |G_p(t,\nu)|^2\, dt\, d\nu \\
&= \iint_{R_{K,L}} \sum_{p=0}^{L-1} |G_p(t,\nu)|^2\, dt\, d\nu \\
&= \iint_{R_{K,L}} \sum_{p=0}^{L-1} |\widetilde{G}_p(t,\nu)|^2\, dt\, d\nu \\
&= \iint_{R_{K,L}} \|\mathbf{H}(t,\nu)\|^2_{\mathbb{C}^L}\, dt\, d\nu.
\end{aligned}$$

(b) Similarly,

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\kappa_H(x,y)|^2\,dx\,dy &= \int_0^1\int_{-\infty}^{\infty}|\kappa_H(x,x-t)|^2\,dx\,dt \\
&= \int_0^1\int_{-\infty}^{\infty}|\kappa_H(1-t+x,x)|^2\,dx\,dt \\
&= \sum_{k=0}^{K-1}\int_0^{\frac{1}{K}}\int_{-\infty}^{\infty}|\kappa_H(1-t+x,x)|^2\,dx\,dt \\
&= \sum_{k=0}^{K-1}\int_0^{\frac{1}{K}}\int_{-\infty}^{\infty}|\kappa_k(t,x)|^2\,dx\,dt \\
&= \sum_{k=0}^{K-1}\int_0^{\frac{1}{K}}\int_{-\infty}^{\infty}|\eta_k(t,\nu)|^2\,d\nu\,dt \\
&= \sum_{k=0}^{K-1}\int_0^{\frac{1}{K}}\int_0^{K}|\eta_k(t,\nu)|^2\,d\nu\,dt \\
&= \sum_{k=0}^{K-1}\sum_{q=0}^{L-1}\int_0^{\frac{1}{K}}\int_0^{\frac{K}{L}}|\eta_k(t,\nu+q\tfrac{K}{L})|^2\,d\nu\,dt \\
&= \iint_{R_{K,L}}\sum_{k=0}^{K-1}\sum_{q=0}^{L-1}|\eta_k(t,\nu+q\tfrac{K}{L})|^2\,d\nu\,dt \\
&= \iint_{R_{K,L}}\sum_{j=0}^{L-1}|\eta_{m_j}(t,\nu+n_j\tfrac{K}{L})|^2\,d\nu\,dt \\
&= \iint_{R_{K,L}}\sum_{j=0}^{L-1}\|\boldsymbol{\eta}(t,\nu)\|_{\mathbb{C}^L}^2\,d\nu\,dt
\end{aligned}
$$

since $\eta_k(t,\nu+q\frac{K}{L})=0$ unless $(k,q)=(m_j,n_j)$. □

It is now clear that (1) holds by observing that by construction the matrix $\mathbf{A}_M$ of (16) is invertible and independent of $(t,\nu)\in R_{K,L}$. Hence, for $(t,\nu)\in R_{K,L}$, we have

$$\frac{1}{\|\mathbf{A}_M^{-1}\|^2}\,\|\boldsymbol{\eta}(t,\nu)\|_{\mathbb{C}^L}^2 \le \|\mathbf{H}(t,\nu)\|_{\mathbb{C}^L}^2 \le \|\mathbf{A}_M\|^2\,\|\boldsymbol{\eta}(t,\nu)\|_{\mathbb{C}^L}^2$$

where $\|\cdot\|$ is the Frobenius norm of a matrix, that is, the operator norm of the matrix considered as an operator on $l^2(\mathbb{Z}_L)$. Integrating this inequality over $R_{K,L}$ and applying Lemma 3.2 we obtain

$$\frac{1}{\|\mathbf{A}_M^{-1}\|}\,\|H\|_{\mathcal{H}} \le \|H(f)\|_{L^2(\mathbb{R})} \le \|\mathbf{A}_M\|\,\|H\|_{\mathcal{H}}$$

which is (1).

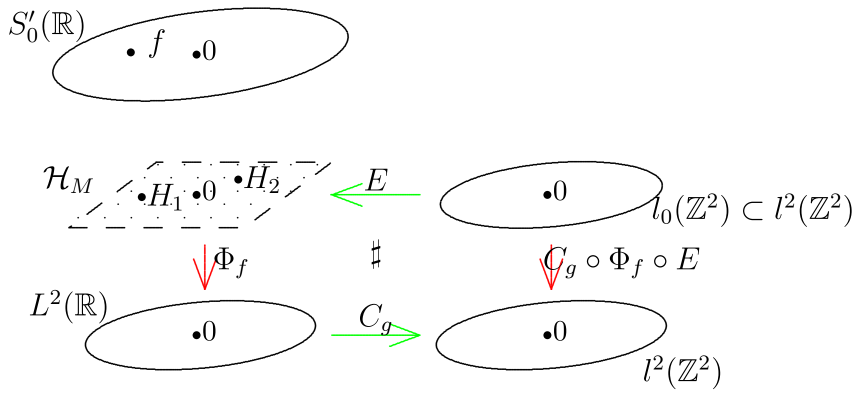

Figure 5. Strategy for the proof that $\mathcal{H}_M$ is not identifiable if $\mathrm{vol}^-(M) > 1$. We shall show that for all $f \in S_0'(\mathbb{R})$, the bounded operator $C_g \circ \Phi_f \circ E$ is not stable. The stability of the synthesis operator E and the analysis operator C_g, together with the lack of stability of $C_g \circ \Phi_f \circ E$, shows that Φ_f is not stable.

4. NECESSITY OF $VOL(M) < 1$ FOR THE IDENTIFIABILITY OF $\mathcal{H}_M$

The goal of this section is to prove the following theorem.

THEOREM 4.1. *The class $\mathcal{H}_M$ is not identifiable if* $\mathrm{vol}^-(M) > 1$.

Given a bounded and measurable subset M with $\mathrm{vol}^-(M) > 1$ we show that for *every* $f \in S_0'(\mathbb{R})$, the operator

$$\Phi_f : \mathcal{H}_M \longrightarrow L^2(\mathbb{R}), \;\; H \mapsto Hf$$

is *not* stable, that is, that inequality (1) fails to hold.

As before, we assume without loss of generality that for some K, $L \in \mathbb{N}$, there is a $U \in \mathcal{U}_{K,L}$ such that $U \subseteq M$ and $\mathrm{vol}^-(U) > 1$. It will be sufficient to show that $\mathcal{H}_U$ is not identifiable since $\mathcal{H}_U \subseteq \mathcal{H}_M$

We shall equip $l_0(\mathbb{Z}^2)$, the space of sequences on $\mathbb{Z}^2$ with only finitely many non–zero terms, with the l^2–norm and construct a bounded and stable synthesis map $D : l_0(\mathbb{Z}^2) \longrightarrow \mathcal{H}_U$, and a bounded and stable (g, a', b')–analysis operator $C_g : L^2(\mathbb{R}) \longrightarrow l^2(\mathbb{Z}^2)$ with the property that the composition

$$C_g \circ \Phi_f \circ D : \; l_0(\mathbb{Z}^2) \longrightarrow l^2(\mathbb{Z}^2), \quad f \in S_0'(\mathbb{R})$$

is not stable. Since D and C_g are stable, we have that all operators $\Phi_f : \mathcal{H}_U \longrightarrow L^2(\mathbb{R})$, $f \in S_0'(\mathbb{R})$, are not stable. Hence, there is no f that identifies $\mathcal{H}_U$ and identification of $\mathcal{H}_M$ is impossible (see Figure 5).

We begin with a well known fact concerning the composition of Hilbert–Schmidt operators with time–frequency shifts.

LEMMA 4.2. *Let $P \in \mathcal{H}$ with spreading function $\eta_P \in S_0(\mathbb{R}\times\widehat{\mathbb{R}})$. For $p,\, r \in \mathbb{R}$ and $\omega,\, \xi \in \widehat{\mathbb{R}}$, define $\widetilde{P} = M_\omega T_{p-r} P T_r M_{\xi-\omega} \in HS$. Then $\eta_{\widetilde{P}} = e^{2\pi i r\xi} M_{(\omega,r)}\, T_{(p,\xi)}\, \eta_P$ and $\widetilde{P} \in \mathcal{H}$.*

Proof. Note that for any $f, g \in S_0(\mathbb{R})$ and $P \in \mathcal{H}$ we have

$$\begin{aligned}\langle Pf, g\rangle &= \iiint \eta_P(t,\nu) T_t M_\nu f(x)\overline{g(x)}\, dt\, d\nu\; dx \\ &= \iint \eta_P(t,\nu)\overline{\int g(x)\overline{T_t M_\nu f(x)}\, dx}\; dt\, d\nu = \langle \eta_P, V_f g\rangle\end{aligned}$$

where $V_f g(t,\nu) = \langle g, T_t M_\nu f\rangle$, $t \in \mathbb{R}$ and $\nu \in \widehat{\mathbb{R}}$. The interchange of order of integration is justified since f, g, η are in the Feichtinger algebra.

Hence, for $f, g \in S_0(\mathbb{R})$ and $s, r \in \mathbb{R}$ and $\omega, \rho \in \widehat{\mathbb{R}}$ we have

$$\langle M_\omega T_s P T_r M_\rho f, g\rangle = \langle P T_r M_\rho f, T_{-s} M_{-\omega} g\rangle = \langle \eta_P, V_{T_r M_\rho f} T_{-s} M_{-\omega} g\rangle,$$

and

$$\begin{aligned}V_{T_r M_\rho f} T_{-s} M_{-\omega} g(t,\nu) &= \langle T_{-s} M_{-\omega} g, T_t M_\nu T_r M_\rho f\rangle = \langle g, M_\omega T_s T_t M_\nu T_r M_\rho f\rangle \\ &= e^{-2\pi i\omega(s+t)}\langle g, T_{s+t} M_{\omega+\nu} T_r M_\rho f\rangle \\ &= e^{-2\pi i(\omega(s+t)+(\omega+\nu)r)}\langle g, T_{t+r+s} M_{\nu+\rho+\omega} f\rangle \\ &= e^{-2\pi i\omega(s+r)} e^{-2\pi i(\omega t+\nu r)} V_f g(t+(r+s), \nu+(\rho+\omega)).\end{aligned}$$

We have

$$\begin{aligned}\langle M_\omega T_s P T_r M_\rho f, g\rangle &= \langle \eta_P, e^{-2\pi i\omega(s+r)} M_{-(\omega,r)} T_{-(r+s,\rho+\omega)} V_f g\rangle \\ &= \langle e^{2\pi i\omega(s+r)} T_{(r+s,\omega+\rho)} M_{(\omega,r)}\eta_P, V_f g\rangle = \langle \eta_R, V_f g\rangle,\end{aligned}$$

where R is given by

$$\eta_R = e^{2\pi i\omega(s+r)} T_{(r+s,\omega+\rho)} M_{(\omega,r)}\eta_P = e^{2\pi i r(\rho+\omega)} M_{(\omega,r)} T_{(r+s,\omega+\rho)}\eta_P.$$

The choice $s = p - r$ and $\rho = \xi - \omega$ concludes the proof. □

LEMMA 4.3. *Fix $\lambda > 1$ with $1 < \lambda^4 < \frac{J}{L}$ and choose $\eta_1, \eta_2 \in \mathcal{S}(\mathbb{R})$ with values in $[0,1]$ and*

$$\eta_1(t) = \begin{cases} 1 & \text{for } t \in [\frac{\lambda-1}{2\lambda K}, \frac{\lambda+1}{2\lambda K}] \\ 0 & \text{for } t \notin [0, \frac{1}{K}] \end{cases} \quad \text{and} \quad \eta_2(\nu) = \begin{cases} 1 & \text{for } \nu \in [\frac{(\lambda-1)K}{2\lambda L}, \frac{(\lambda+1)K}{2\lambda L}] \\ 0 & \text{for } \nu \notin [0, \frac{K}{L}] \end{cases}.$$

The operator $P \in \mathcal{H}_{R_{K,L}}$ defined by $\eta_P = \eta_1 \otimes \eta_2$ has the following properties.

a) The operator family

$$\Big\{ M_{\lambda K k}\, T_{\frac{1}{K}m - \frac{\lambda L}{K}l}\, P\, T_{\frac{\lambda L}{K}l}\, M_{\frac{K}{L}n - \lambda K k} \Big\}_{k,l,m,n\in\mathbb{Z}}$$

is a Riesz basis for its closed linear span in $HS(\mathbb{R})$.

b) The operator $P \in \mathcal{H}_{R_{K,L}}$ is a time–frequency localization operator in the following sense: There exist functions $d_1, d_2 : \mathbb{R} \to \mathbb{R}_0^+$, which decay rapidly at infinity, i.e., $d_1, d_2 = O(x^{-n})$ for all $n \in \mathbb{N}$, and which have the property that for all $f \in S_0'(\mathbb{R})$ we have $|Pf(x)| \leq \|f\|_{S_0'}\, d_1(x)$, $x \in \mathbb{R}$ and $|\widehat{Pf}(\xi)| \leq \|f\|_{S_0'}\, d_2(\xi)$, $\xi \in \widehat{\mathbb{R}}$.

Proof. a) Lemma 4.2 implies that

$$\Big\{ M_{\lambda K k}\, T_{\frac{1}{K}m - \frac{\lambda L}{K}l}\, P\, T_{\frac{\lambda L}{K}l}\, M_{\frac{K}{L}n - \lambda K k} \Big\}_{k,l,m,n\in\mathbb{Z}}$$

is a Riesz basis for its closed linear span in $HS(\mathbb{R})$ if and only if

$$\Big\{ M_{\left(\lambda K k, \frac{\lambda L}{K} l\right)}\, T_{\left(\frac{1}{K}m, \frac{K}{L}n\right)}\, \eta_P \Big\}_{k,l,m,n\in\mathbb{Z}}$$

is a Riesz basis for its closed linear span in $L^2(\mathbb{R}\times\widehat{\mathbb{R}})$.

We observe that

$$\begin{aligned} &\Big\| \sum_{k,l,m,n\in\mathbb{Z}} \sigma_{k,l,m,n} M_{\left(\lambda K k, \frac{\lambda L}{K} l\right)}\, T_{\left(\frac{1}{K}m, \frac{K}{L}n\right)}\, \eta_P \Big\|^2_{L^2(\mathbb{R}^2)} \\ &\qquad = \sum_{m,n\in\mathbb{Z}} \Big\| \sum_{k,l\in\mathbb{Z}} \sigma_{k,l,m,n}\, M_{\left(\lambda K k, \frac{\lambda L}{K} l\right)}\, T_{\left(\frac{1}{K}m, \frac{K}{L}n\right)}\, \eta_P \Big\|^2_{L^2} \\ &\qquad = \sum_{m,n\in\mathbb{Z}} \Big\| \sum_{k,l\in\mathbb{Z}} \sigma_{k,l,m,n}\, M_{\left(\lambda K k, \frac{\lambda L}{K} l\right)}\, \eta_P \Big\|^2_{L^2} \end{aligned}$$

where we have used the translation invariance of the L^2 norm and the fact that the support of η_P is contained in $R_{K,L}$. With $R^{\lambda}_{K,L} = [\frac{\lambda-1}{2\lambda K}, \frac{\lambda+1}{2\lambda K}] \times [\frac{(\lambda-1)K}{2\lambda L}, \frac{(\lambda+1)K}{2\lambda L}]$

$$\sum_{m,n\in\mathbb{Z}} \Big\| \sum_{k,l\in\mathbb{Z}} \sigma_{k,l,m,n}\, M_{\left(\lambda K k, \frac{\lambda L}{K} l\right)}\, \eta_P \Big\|^2_{L^2} \asymp \sum_{m,n\in\mathbb{Z}} \Big\| \sum_{k,l\in\mathbb{Z}} \sigma_{k,l,m,n}\, M_{\left(\lambda K k, \frac{\lambda L}{K} l\right)}\, \mathbf{1}_{R^{\lambda}_{K,L}} \Big\|^2_{L^2(\mathbb{R}^2)} = \frac{1}{\lambda^2 L}\, \|\{\sigma_{k,l,m,n}\}\|^2_{l^2}$$

since by definition η_P is bounded below by $\mathbf{1}_{R^{\lambda}_{K,L}}$ and bounded above by the characteristic function of finitely many translates of $R^{\lambda}_{K,L}$.

b) See [KP05], Lemma 3.4. □

Lemma 4.4 generalizes the fact that $m \times n$ matrices with $m < n$ have a non–trivial kernel and, therefore, are not stable as operators acting on $\mathbb{C}^n$. In fact, the bi–infinite matrices $M = (m_{j',j})_{j',j\in\mathbb{Z}^2}$

considered in Lemma 4.4 are not dominated by their diagonals $m_{j,j}$ — which would correspond to square matrices — but by skew diagonals $m_{j,\lambda j}$, with $\lambda > 1$, i.e., $m_{j',j}$ is small if $\|\lambda j' - j\|_\infty$ is large. The lemma is proven in [KP05]. The validity of Lemma 4.4, and therefore of Theorem 4.1 does not depend on the choice of (reasonable) topologies on domain and range. In fact, a more general version of Lemma 4.4 can be found in [Pfa05].

LEMMA 4.4. *Given* $\mathcal{M} = (m_{j',j}) : l^2(\mathbb{Z}^2) \to l^2(\mathbb{Z}^2)$. *If there exists a monotonically decreasing function* $w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *with* $w = O\left(x^{-2-\delta}\right)$, $\delta > 0$, *and constants* $\lambda > 1$ *and* $K_0 > 0$ *with* $|m_{j',j}| < w(\|\lambda j' - j\|_\infty)$ *for* $\|\lambda j' - j\|_\infty > K_0$, *then* M *is not stable, that is, for every* $\epsilon > 0$ *there is a* $\sigma \in l^2(\mathbb{Z}^2)$ *with* $\|\sigma\|_{l^2(\mathbb{Z}^2)} = 1$ *such that* $\|\mathcal{M}\sigma\|_{l^2(\mathbb{Z}^2)} < \epsilon$.

Now all pieces are in place to prove Theorem 4.1.

Proof of Theorem 4.1. Choose λ, η_1, η_2, P, d_1, and d_2 as in Lemma 4.3.

Define the synthesis operator $E : l_0(\mathbb{Z}^2) \to \mathcal{H}_U$ as follows. For $\sigma = \{\sigma_{k,p}\} \in l^2(\mathbb{Z}^2)$ write $\sigma_{k,p} = \sigma_{k,lJ+j}$ for $l \in \mathbb{Z}$ and $0 \le j < J$ and define

$$E(\sigma) = \sum_{k,l \in Z} \sum_{j=0}^{J-1} \sigma_{k,lJ+j}\ M_{\lambda K k}\, T_{\frac{1}{K} m_j + \frac{\lambda L}{K} l}\, P\, T_{-\frac{\lambda L}{K} l}\, M_{\frac{K}{L} n_j - \lambda K k}$$

Since

$$\left\{ M_{\lambda K k}\, T_{\frac{1}{K} m - \frac{\lambda L}{K} l}\, P\, T_{\frac{\lambda L}{K} l}\, M_{\frac{K}{L} n - \lambda K k} \right\}_{k,l,m,n \in \mathbb{Z}}$$

is a Riesz basis for its closed linear span in $HS(\mathbb{R})$, the subset

$$\left\{ M_{\lambda K k}\, T_{\frac{1}{K} m_j + \frac{\lambda L}{K} l}\, P\, T_{-\frac{\lambda L}{K} l}\, M_{\frac{K}{L} n_j - \lambda K k} \right\}_{k,l \in \mathbb{Z}, j \in \mathcal{J}}$$

is a Riesz basis for its closed linear span in $\mathcal{H}_U \subseteq HS(\mathbb{R})$. We conclude that E is bounded and stable.

To construct a stable (g, a', b')–analysis operator C_g, we choose as Gabor atom the Gaussian $g_0 : \mathbb{R} \to \mathbb{R}^+, \quad x \mapsto e^{-\pi x^2}$. The Gabor system $(g_0, a', b') = \{M_{ka'} T_{lb'} g_0\}$ is a frame for any $a', b' > 0$ with $a'b' < 1$, and, hence, we conclude that the analysis map given by

$$C_{g_0} : L^2(\mathbb{R}) \to l^2(\mathbb{Z}^2), \quad f \mapsto \left\{ \langle f, M_{\lambda^2 K k} T_{\frac{\lambda^2 L}{KJ} l}\, g_0 \rangle \right\}_{k,l}$$

is bounded and stable since $\lambda^2 K \frac{\lambda^2 L}{KJ} = \lambda^4 \frac{L}{J} < 1$.

For simplicity of notation, set $\alpha = K$ and $\beta = \frac{L}{KJ}$. Fix $f \in S_0'(\mathbb{R})$ and consider the composition

$$\begin{array}{ccccccc} l_0(\mathbb{Z}^2) & \stackrel{D}{\to} & \mathcal{H}_M & \stackrel{\Phi_f}{\to} & L^2 & \stackrel{C_{g_0}}{\to} & l^2(\mathbb{Z}^2) \\ \sigma & \mapsto & E\sigma & \mapsto & E\sigma\, f & \mapsto & \{ \langle E\sigma\, f,\ M_{\lambda^2 \alpha k'} T_{\lambda^2 \beta l'}\, g_0 \rangle \}_{k',l'} \end{array}.$$

Since

$$\Big(C_{g_0}\circ\Phi_f\circ E\ \{\sigma_{k,lJ+j}\}\Big)_{k',l'} = \langle \sum_{k,l}\sum_{j=0}^{J-1}\sigma_{k,lJ+j}\ M_{\lambda\alpha k}\,T_{\frac{m_j}{\alpha}+\lambda\beta lJ}\,P\,T_{-\lambda\beta lJ}\,M_{\frac{n_j}{\beta J}-\lambda\alpha k}\,f\ ,\ M_{\lambda^2\alpha k'}T_{\lambda^2\beta l'}g_0\,\rangle$$
$$= \sum_{k,l}\sum_{j=0}^{J-1}\langle\, M_{\lambda\alpha k}\,T_{\frac{m_j}{\alpha}+\lambda\beta lJ}\,P\,T_{-\lambda\beta lJ}\,M_{\frac{n_j}{\beta J}-\lambda\alpha k}\,f\ ,\ M_{\lambda^2\alpha k'}T_{\lambda^2\beta l'}g_0\,\rangle\sigma_{k,lJ+j}$$
$$= \sum_{k,l}\sum_{j=0}^{J-1} m_{k',l',k,lJ+j}\ \sigma_{k,lJ+j}\ ,$$

we see that he operator $C_{g_0}\circ\Phi_f\circ E$ is represented — with respect to the canonical basis $\{\delta(\cdot-n)\}_n$ of $l^2(\mathbb{Z}^2)$ — by the bi–infinite matrix

$$\mathcal{M}=\Big(m_{k',l',k,lJ+j}\Big)=\Big(\langle\ M_{\lambda\alpha k}\,T_{\frac{m_j}{\alpha}+\lambda\beta lJ}\,P\,T_{-\lambda\beta lJ}\,M_{\frac{n_j}{\beta J}-\lambda\alpha k}\,f\ ,\ M_{\lambda^2\alpha k'}T_{\lambda^2\beta l'}\,g_0\,\rangle\Big)\,.$$

We shall now use Lemma 4.4 to show that $\mathcal{M}$, and, therefore, $C_{g_0}\circ\Phi_f\circ E$ is not stable. Lemma 4.3, part b, together with the rapidly decaying function

$$\widetilde{d}_1=\max_{j=0,\dots,J-1}T_{\frac{m_j}{\alpha}-\lambda\beta j}d_1$$

will provide us with the necessary bounds on the matrix entries of $\mathcal{M}$. In fact, for $k,l,k',l'\in\mathbb{Z}$ and $0\le j<J$, we have

$$\begin{aligned}
|m_{k',l',k,lJ+j}| &= \Big|\langle\, M_{\lambda\alpha k}\,T_{\frac{m_j}{\alpha}+\lambda\beta lJ}\,P\,T_{-\lambda\beta lJ}\,M_{\frac{n_j}{\beta J}-\lambda\alpha k}\,f\ ,\ M_{\lambda^2\alpha k'}T_{\lambda^2\beta l'}g_0\,\rangle\Big|\\
&\le \langle\, T_{\lambda\beta(lJ+j)}\big(T_{\frac{m_j}{\alpha}-\lambda\beta j}\Big|P\,T_{-\lambda\beta lJ}\,M_{\frac{n_j}{\beta J}-\lambda\alpha k}\,f\,\Big|\,\big)\ ,\ T_{\lambda^2\beta l'}g_0\,\rangle\\
&\le \|f\|_{S_0'}\ \langle\, T_{\lambda\beta(lJ+j)}T_{\frac{m_j}{\alpha}-\lambda\beta j}d_1\ ,\ T_{\lambda^2\beta l'}g_0\,\rangle\\
&\le \|f\|_{S_0'}\ \langle\, T_{\lambda\beta(lJ+j)}\widetilde{d}_1\ ,\ T_{\lambda^2\beta l'}g_0\,\rangle\\
&\le \|f\|_{S_0'}\ (\widetilde{d}_1*g_0)\,(\lambda\beta(\lambda l'-(lJ+j)))\,,
\end{aligned}$$

and

$$\begin{aligned}
|m_{k',l',k,lJ+j}| &= \Big|\langle\, T_{\lambda\alpha k}\,M_{-\frac{m_j}{\alpha}-\lambda\beta lJ}\,\big(P\,T_{-\lambda\beta lJ}\,M_{\frac{n_j}{\beta J}-\lambda\alpha k}\,f\,\big)^{\widehat{}},\ T_{\lambda^2\alpha k'}M_{-\lambda^2\beta l'}g_0\,\rangle\Big|\\
&\le \langle\, T_{\lambda\alpha k}\Big|\big(P\,T_{-\lambda\beta lJ}\,M_{\frac{n_j}{\beta J}-\lambda\alpha k}\,f\,\big)^{\widehat{}}\,\Big|\ ,\ T_{\lambda^2\alpha k'}g_0\,\rangle\\
&\le \|f\|_{S_0'}\ (d_2*g_0)(\lambda\alpha(\lambda k'-k)).
\end{aligned}$$

where we have used the Parseval–Plancherel identity and the fact that $g_0\ge 0$, $\widehat{g_0}=g_0$, and $g_0(-x)=g_0(x)$. Since $\widetilde{d}_1$, d_2, and g_0 decay rapidly, so do $\widetilde{d}_1*g_0$ and d_2*g_0 i.e., $d_1*g_0, d_2*g_0=O(x^{-n})$ for all $n\in\mathbb{N}$. We set

$$w(x)=\|f\|_{S_0'}\ \max\big\{\widetilde{d}_1*g_0(\lambda\beta x),\ \widetilde{d}_1*g_0(-\lambda\beta x),\ d_2*g_0(\lambda\alpha x),\ d_2*g_0(-\lambda\alpha x)\big\}.$$

and obtain $|m_{k',l',k,l}| \leq w\big(\max\{|\lambda k' - k|,\ |\lambda l' - l|\}\big)$ with $w = O(x^{-n})$ for $n \in \mathbb{N}$. Lemma 4.4 implies that $\mathcal{M}$ is not stable, and, by construction, we can conclude that $C_{g_0} \circ \Phi_f \circ E$ and thus Φ_f is not stable. $\square$

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