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Fall Term 2007
Thursday, 13 December
16:00 – 18:00

Analysis I — Final exam

Notes: Sign you work to certify that you adhere to the academic Code of Honor to work independently. You may use and cite all results within the script, the homeworks, and the examinations. *All answers must be justified! Show all your work!*

Each problem is worth 50 points, do any 6 of them. Points achieved beyond 300 points are bonus points.

F.1. Continuous maps.

Let f be a function mapping the metric space X to the metric space Y.

(a) Show that f is continuous if and only if the pre-image of any closed subset of Y is a closed subset of X.

(b) Assume that f is a continuous bijection of X onto Y and X is compact. Prove that f is a homeomorphism.

F.2. Uniform continuity.

Let $\{f_n\}$ be a sequence of uniformly continuous functions on $\mathbb{R}$ which converges uniformly on $\mathbb{R}$ to the function f. Prove that f is also uniformly continuous on $\mathbb{R}$.

F.3. Calculus.

Find the following limits:

(a) $\lim_{x\to 0} \dfrac{\sin x - x}{x^3}$,

(b) $\lim_{x\to 1} x^{\frac{1}{1-x}}$.

(c) Does the series $\sum_{n=1}^{\infty} \dfrac{n^3(\sqrt{2}+(-1)^n)^n}{3^n}$ converge?

F.4. Differentiability.

Can one claim that the function $F_1(x) = f(x) + g(x)$ is not differentiable at $x_0 \in \mathbb{R}$ if

(a) $f(x)$ is differentiable at x_0 and $g(x)$ is not;

(b) neither $f(x)$ nor $g(x)$ is differentiable at x_0?

Can one claim that the function $F_2(x) = f(x) \cdot g(x)$ is not differentiable at $x_0 \in \mathbb{R}$ if

(c) $f(x)$ is differentiable at x_0 and $g(x)$ is not;

(d) neither $f(x)$ nor $g(x)$ is differentiable at x_0?

F.5. Real roots.

Prove that if a polynomial P of degree n has n distinct real roots, then P' has $n-1$ distinct real roots.

F.6. Computing logarithms using roots.

(a) For $x > 0$, establish the inequality

$$\frac{x}{x+1} < \ln(x+1) < x.$$

(b) Prove that $\lim_{n\to\infty} n(\sqrt[n]{a} - 1) = \ln a$, $a > 0$.

Hint: You may use the inequality in (a) with $x = \sqrt[n]{a} - 1$ in the case $a > 1$.

F.7. Differentiation of a series.

Let $f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \ln(1 + \frac{x}{n})$ for $x \in [0, \infty)$. Show that f is differentiable on $(0, \infty)$.

F.8. Extra Bonus Problem. (5 pts)

Which concept is the following poem referring to?

A coffee cup feeling quite dazed,
Said to a donut, amazed,
An open surjective continuous injection,
You'd be plastic and I'd be glazed.