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Fall Term 2007
Monday, 22 October
9:45 – 11:00

Analysis I — Midterm exam

Notes: Sign you work to certify that you adhere to the academic Code of Honor to work independently. You may use and cite all results within the script, the homeworks, and the examinations. *All answers must be justified! Show all your work!*

Points achieved beyond 220 points are bonus points.

M.1. Complete metric spaces.

(a) Give $X \subset \mathbb{C}$ such that (X, d_2) is a complete metric space and $Y \subset \mathbb{C}$ such that (Y, d_2) is not.

(b) Let (X, d_2), $X \subseteq \mathbb{C}$, be a complete metric space and the series $\sum_{n=1}^{\infty} x_n$ of elements of X is absolutely convergent. Prove that $\sum_{n=1}^{\infty} x_n$ converges in X.

(c) Give a counterexample to the statement in (b) if (X, d_2) is not complete.

(d) What are the subsets $X \subseteq \mathbb{C}$ such that (X, d_0) is complete? (40)

M.2. Metric spaces.

(a) Consider $\mathbb{R}^n$ equipped with the binary function

$$d_\infty(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

Show that $(\mathbb{R}^n, d_\infty)$ is a metric space.

(b) Consider the set of all bounded infinite sequences $x = (x_1, x_2, \ldots)$ of elements of $\mathbb{R}$. Prove that this is a metric space endowed with a metric

$$d_\infty(x, y) = \sup\{|x_1 - y_1|, |x_2 - y_2|, \ldots\}.$$

(40)

M.3. Series.

(a) For which $\alpha \in \mathbb{R}$ does the series $\sum_{n=1}^{\infty} \frac{1}{n(\log_2 n)^\alpha}$ converge, where $\log_2 x$ for $x > 0$ is defined by $2^{\log_2 x} = x$? (20)

(b) What is the radius of convergence of $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n(n+1)} z^n$? (20)

(c) **(Bonus problem)** Does the series $\sum_{n=1}^{\infty} \frac{(n^2)!}{n^{n^2}}$ converge? (20)

M.4. Limes superior. For a given sequence $\{a_n\}$, define $\{b_n\}$ by setting

$$b_n = \frac{1}{n}(a_1 + a_2 + \ldots + a_n), n \in \mathbb{N}.$$

Prove that $\limsup_{n\to\infty} b_n \leq \limsup_{n\to\infty} a_n$. (50)

M.5. Sequences. Suppose that a sequence $\{a_n\}$ satisfies the condition

$$|a_{n+1} - a_{n+2}| < \lambda |a_n - a_{n+1}| \text{ with a } \lambda \in (0, 1).$$

Prove that $\{a_n\}$ converges. (50)