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Analysis I — Problem Set 10

Issued: 12.11.07 Due: 20.11.07, noon

10.1. Compact sets. Let $C_1, \ldots, C_n$ be compact sets in the metric space (X, d). Using the definition of covering compactness, show that $\bigcap_{i=1}^{n} C_i$ is compact.

10.2. The graph of a continuous function. Consider $M \subseteq \mathbb{R}$ and a continuous function $f : M \longrightarrow \mathbb{R}$. (As usual, $\mathbb{R}$ is equipped with the euclidean metric d_2.)

(a) Prove that the *graph* $\Gamma_f := M \times f(M)$ of f is a closed subset of $M \times \mathbb{R} \subseteq \mathbb{R}^2$.

(b) Prove that if $M \subset \mathbb{R}$ is compact, then Γ_f is compact.

10.3. Complete, closed, and bounded does not imply compact. Consider the metric space $X := \mathbb{R}^{\mathbb{N}} := \{(x_n)_{n\in\mathbb{N}} : \forall n \in \mathbb{N} : \ x_n \in \mathbb{R}\}$ endowed with the metric $d_\infty((x_n), (y_n)) := \sup_n |x_n - y_n|$.

Set $\mathbf{0} := (0, 0, \ldots)$. Show that the unit ball

$$D_1^X(\mathbf{0}) := \{x \in X : d_\infty(x, \mathbf{0}) \leq 1\}$$

is closed, bounded, and complete, but not compact.

(*Hint:* It might help to consider a subset that is closed, but not compact, e.g. $\{0, 1\}^{\mathbb{N}} \subset D_1^X(\mathbf{0})$.)

10.4. Uniform continuity. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly continuous. Show that there are constants $A, B \in \mathbb{R}^+$such that $|f(x)| \leq A + B|x|$ for all $x \in \mathbb{R}$.

10.5. Connectedness. Let X be a topological space. A **separation** of X is a pair U, V of disjoint nonempty open subsets of X such that $X = U \cup V$.
X said to be **connected** if there is **no** separation of X.
X is **disconnected** if it is not connected, i.e. if there exists a separation.

(a) The unit interval $[0, 1]$ is connected.

(b) $\mathbb{R}\backslash\mathbb{Q}$ is disconnected.

10.6. Nice properties of functions with connected domain. Prove the following theorems.

(a) Consider a connected metric space (X, d_X) and an arbitrary metric space (Y, d_Y). Every *locally constant* continuous function $f : X \to Y$ is constant. (f is called locally constant if for each $x \in X$ there exists $\delta > 0$ such that f restricted to $B_\delta(x)$ is constant.)

(b) Every polynomial $p : \mathbb{R} \to \mathbb{R}$ of odd degree has a zero. (*Hint:* $p(x) = \sum_{k=0}^{d} a_k x^k$ with $a_k \in \mathbb{R}$, $a_d \neq 0$, d odd.)

(c) Every continuous *self-map* of the unit interval has a *fixed point*: if $f : [0,1] \to [0,1]$ is continuous, then there exists $x \in [0,1]$ such that $f(x) = x$.

10.7. (Bonus problem)Completeness. Consider the space $\mathbb{R}^{\mathbb{N}}$ of all sequences of real numbers, i.e. $\mathbb{R}^{\mathbb{N}} := \{(a_n)_{n\in\mathbb{N}} : a_n \in \mathbb{R}\}$.
Define the map

$d : \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{R}$, $((a_n),(b_n)) \mapsto \frac{1}{2^k}$, where k is the smallest number such that $a_k \neq b_k$.

(a) Show that d is a metric.

(b) Consider now the subspace

$$S := \left\{ (a_n) \in \mathbb{R}^{\mathbb{N}} : \lim_{N\to\infty} \sum_{n=1}^{N} a_n < \infty \right\}$$

Decide whether S is complete or not. Prove your answer!

Hint: Consider the sequence $a^{(i)} = (a_n^{(i)})_{n\in\mathbb{N}}\,(i \in \mathbb{N})$, where $a_n^{(i)} = \begin{cases} 1 \text{ if } n \leq i \\ 0 \text{ if } n > i \end{cases}$, i.e. $a^{(i)} = (\underbrace{1,\ldots,1}_{i-times}, 0, 0, \ldots)$

10.8. (Bonus problem)Uniformly continuous functions and their extension. Let (a,b) be a subspace of the real line. Show that $f : (a,b) \longrightarrow \mathbb{R}$ is uniformly continuous if and only if f can be extended to a continuous function $\hat{f} : [a,b] \longrightarrow \mathbb{R}$ such that $\hat{f}|_{(a,b)} \equiv f$.