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Analysis I — Problem Set 11

Issued: 19.11.07 Due: 27.11.07, noon

Each problem is worth 5 points, points achieved beyond 30 are bonus points.

11.1. Let $f, f_1, f_2, ...,$ be continuous real-valued functions on the compact metric space E, with $f = \lim_{n\to\infty} f_n$. Prove that if $f_1(p) \le f_2(p) \le f_3(p) \le ...$ for all $p \in E$ then the sequence $f_1, f_2, f_3, ...,$ converges uniformly.

11.2. Show that the closed ball in $C([0,1])$ of center 0 and radius 1 is not compact. (Hint: Consider the sequence of functions $x, x^2, x^3, ...$)

11.3. Discuss differentiability at 0 of the function:

$$f(x) = \begin{cases} x^p \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

for all $p \in \mathbb{N}_0$.

11.4. Show that f defined on $(0,2)$ by setting:

$$f(x) = \begin{cases} x^2 & \text{for rational } x \in (0,2) \\ 2x - 1 & \text{for irrational } x \in (0,2) \end{cases}$$

is differentiable only at $x = 1$ and that $f'(1) \neq 1$. Is the inverse function differentiable at $1 = y = f(1)$?

11.5. Prove that any continuous function defined on compact metric space is uniformly continuous. That is, given a compact metric space (X, d_X) and continuous $f : (X, d_X) \to (Y, d_Y)$, prove that f is uniformly continuous as well.

11.6. Let $S_i, i \in I$ be a family of connected sets in a metric space (X, d) such that $\bigcap_{i\in I} S_i \neq \emptyset$. Prove that $\bigcup_{i\in I} S_i$ is connected.

11.7. A metric space (X, d) is called *totally disconnected* if for each $x \in X$ and $\epsilon > 0$ exists a clopen set A in X with $x \in A \subseteq B_\epsilon(x)$. Show that $\mathbb{Q}$ is totally disconnected.