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Fall Term 2007

Analysis I — Problem Set 12

Issued: 26.11.07 Due: 04.12.07, noon

12.1. Calculate $f^{(n)}(0)$ for $f:\mathbb{R}\longrightarrow\mathbb{R},\quad x\mapsto\begin{cases} e^{-\frac{1}{x^2}}, & \text{for } x\neq 0;\\ 0, & \text{for } x=0.\end{cases}$ Recall that $f^{(n)}$ is the derivative of the function $f^{(n-1)}$, where $f^{(0)}=f$ and $f^{(1)}=f'$. (Tip: Use induction and L'Hospital.)

12.2. Assume that (f_n) is uniformly convergent on $A\subset\mathbb{R}$ to the function f_0. Assume moreover that x_0 is a cluster point of A and, $\lim_{x\to x_0} f_n(x)=f_n(x_0)$ for $n\geq N$, $N\in\mathbb{N}$.

(a) Prove that $\lim\limits_{n\to\infty}\lim\limits_{x\to x_0} f_n(x)=\lim\limits_{x\to x_0} f_0(x)$.

(b) Prove that if (f_n) is uniformly convergent on $(0,\infty)$ to the function f_0 and $\lim_{x\to\infty} f_n(x)=\alpha\in\mathbb{R}$ for $n\geq N$, $N\in\mathbb{N}$, then

$$\lim_{n\to\infty}\lim_{x\to\infty} f_n(x)=\lim_{x\to\infty} f_0(x).$$

(c) Assume that (f_n) is a sequence of functions which are differentiable on (c,d), and let $[a,b]\subset(c,d)$. If $\sum\limits_{n=1}^{\infty} f_n(x)$ converges at some $x_0\in[a,b]$ and $\sum\limits_{n=1}^{\infty} f_n'(x)$ converges uniformly on $[a,b]$, then $\sum\limits_{n=1}^{\infty} f_n(x)$ converges to a differentiable function, and

$$\left(\sum_{n=1}^{\infty} f_n(x)\right)'=\sum_{n=1}^{\infty} f_n'(x).$$

(d) Show that $f(x)=\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2+x^2}$ is differentiable on $\mathbb{R}$.

12.3. Show that $f(x)=\sum\limits_{n=1}^{\infty}\dfrac{\sin(nx^2)}{1+n^3}$ is continuously differentiable on $\mathbb{R}$ and write explicitly its derivative.

12.4. Assume that $f:\mathbb{R}\to\mathbb{R}$ is differentiable and f' is uniformly continuous on $\mathbb{R}$. Define a sequence of functions (f_n) via

$$f_n(x)=n(f(x+\frac{1}{n})-f(x)),\ x\in\mathbb{R}.$$

Then f_n converges to f' uniformly.

12.5. Prove the following inequalities:

(a) $1 - \frac{x^2}{2!} < \cos x \quad \text{for } x \neq 0,$

(b) $x - \frac{x^3}{3!} < \sin x \quad \text{for } x > 0,$

(c) $\cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \quad \text{for } x \neq 0,$

(d) $\sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!} \quad \text{for } x > 0.$