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# Analysis I — Problem Set 3

**Issued: 17.09.07 Due: 25.09.07, noon, Analysis Mailbox in Research 1**

**3.1.** Prove that $\sqrt{2} \in \mathbb{R}$ by showing that $x \cdot x = 2$, where $x = A|B$ is the cut in $\mathbb{Q}$ with $A = \{r \in \mathbb{Q} : r \leq 0 \text{ or } r^2 < 2\}$, and $B = \mathbb{Q} \setminus A$.

*The explicit use of Dedekind cuts is not necessary to solve any of the following problems.*

**3.2.** Let $r \in \mathbb{Q}$ be rational with $r \neq 0$ and $x \in \mathbb{R}$ be irrational. Show that $x + r$ and $rx$ are also irrational. Hint: Proof by contradiction.

**3.3.** Let $A$ and $B$ be sets of real numbers, and set $A + B = \{a + b, a \in A, \quad b \in B\}$, and $A \cdot B = \{a \cdot b, \quad a \in A, b \in B\}$. Check whether the following is true for any $A$ and $B$, defined above:

(a) $\sup(A + B) = \sup A + \sup B$;

(b) $\sup(A \cdot B) = \sup A \cdot \sup B$.

**3.4.** Rational Powers

(a) Let $m,\, p \in \mathbb{Z}$ and $n,\, q \in \mathbb{N}$ such that $m/n = p/q =: r$. For $b \in \mathbb{R}$, $b > 0$, set

$$b^r := (b^m)^{1/n} \ .$$

Show that $b^r$ is well–defined, i.e. show that $(b^m)^{1/n} = (b^p)^{1/q}$.

(b) Justify why it makes sense to define $b^{1/n}$, $n \in \{2k+1 \,|\, k \in \mathbb{N}\}$, only for non–negative numbers $b$.
Hint: Consider the expression $(-8)^{\frac{1}{3}}$.

(c) Let $r,\, s \in \mathbb{Q}$. Show that $b^{r+s} = b^r b^s$ for $b > 0$.

**3.5.** Irrational Powers

(a) Fix $b \in \mathbb{R}$ with $b \geq 1$. For $x \in \mathbb{R}$ define $P(b,x) := \{b^t \,|\, t \in \mathbb{Q},\, t \leq x\}$. Prove that

$$b^r = \sup P(b,r) \quad \text{for all } r \in \mathbb{Q}.$$

Hence it makes sense to define

$$b^x = \sup P(b,x) \quad \text{for all } x \in \mathbb{R}.$$

(b) Prove that $b^{x+y} = b^x b^y$ for all $x,\, y,\, b \in \mathbb{R}$ with $b \geq 1$.