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Analysis I — Problem Set 4

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4.1. Let $M \subset \mathbb{R}^+$ be uncountable. Show that for every $r \in \mathbb{R}$ there is a finite number of *different* elements $a_1, \dots, a_n \in M$ of M such that

$$\sum_{k=1}^{n} a_k \geq r\ .$$

(*Hint:* Among the sets $M_N := \{a \in M : a \geq \frac{1}{N}\}$, $N \in \mathbb{N}$, there has to be an infinite one.)

4.2. Let $z = (1 + \sqrt{3}i)/2$.

(a) Determine $|z|, z^{-1}, \bar{z}, z^n$ for $n \in \mathbb{N}$;

(b) Let $T \in \mathbb{C}$ be the triangle with vertices $a = 5,\ b = 6 + i,\ c = 7$. What geometric objects do $z^n a,\ z^n b,\ z^n c$ for any $n \in \mathbb{N}$ form? Draw them into the complex plane, together with T and z;

(c) Calculate $(1 + i)^n + (1 - i)^n\ \forall n \in \mathbb{Z}$. Why is the result always a real number?

4.3. Show that no order can be defined in the complex field which turns it into an ordered field.

4.4. (a) Show that the map $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R} \quad (x, y) := \sup\{\,|x_i - y_i|\ :\ i = 1, \dots, n\}$ is a metric on $\mathbb{R}^n$.

(b) Let $X = \{\,[a, b] \subset \mathbb{R} : a < b\}$ and $Y = \{\,(a, b) \subset \mathbb{R} : a < b\}$ and

$$d([a, b], [c, d]) := \inf\{\epsilon > 0\ :\ [a, b] \subset [c - \epsilon, d + \epsilon] \text{ and } [c, d] \subset [a - \epsilon, b + \epsilon]\}\ .$$

Are (X, d) and/or $(X \cup Y, \overline{d})$ metric spaces, where $\overline{d}$ is the canonical extension of d onto $X \cup Y$?

4.5. Find the limit of the sequence of complex numbers $z_n = \left(\frac{1+i}{2}\right)^n$, $n \in \mathbb{N}$. Sketch the sequence in the complex plane.

4.6. Show that for every $c \in \mathbb{R}$ there is a sequence (a_n) of rational numbers with $a_n \to c$.