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Fall Term 2007

Analysis I — Problem Set 5

Issued: 01.10.07 Due: 09.10.07, 12:01 p.m.

5.1. Squeezing theorem. Let (a_n), (b_n) and (c_n) be three sequences of real numbers such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = b \in \mathbb{R}$ in $(\mathbb{R}, d_2)$. Show that $\lim_{n\to\infty} b_n = b$ in $(\mathbb{R}, d_2)$ as well.

5.2. Fibonacci sequence. Consider the sequence of real numbers which is inductively defined by $a_1 = a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$ for $n \in \mathbb{N}$

(a) Does the sequence (a_n) converge in $(\mathbb{R}, d_2)$?

(b) Show that the sequence (x_n) defined by $x_n = \frac{a_{n+1}}{a_n}$ for $n \in \mathbb{N}$ converges in $(\mathbb{R}, d_2)$ and determine its limit.

5.3. Products of sequences. Let $(a_n), (b_n) \subset \mathbb{R}$ be sequences of real numbers so that

$$\lim_{n\to\infty} a_n = +\infty, \ \lim_{n\to\infty} b_n = 0.$$

The limit of the sequence of the products $(a_n \cdot b_n)$ can realize very different values depending on the sequences a_n, b_n. Find one example for each of the following cases:

(a) $\lim_{n\to\infty} (a_n \cdot b_n) = +\infty$;

(b) $\lim_{n\to\infty} (a_n \cdot b_n) = -\infty$;

(c) $\lim_{n\to\infty} (a_n \cdot b_n) = 0$;

(d) $\lim_{n\to\infty} (a_n \cdot b_n) = c$ where $c \in \mathbb{R}$;

(e) $\lim_{n\to\infty} (a_n \cdot b_n)$ is bounded but does not converge;

(f) $\lim_{n\to\infty} (a_n \cdot b_n)$ is unbounded but does not converge to either $+\infty$ nor $-\infty$.

5.4. The distance from a point x in a metric space (X, d) to a nonempty subset $S \subseteq X$ is defined to be

$$\text{dist}(x,S) = \inf\{d(x, s) : s \in S\}.$$

Show that there exists a sequence $(s_n)_{n\in\mathbb{N}}$ in S with $\lim_{n\to\infty} s_n = x$, if and only if $\text{dist}(x,S) = 0$.

5.5. Prove that the sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ given by $x_1 = 1$ and $x_{n+1} = x_n + \frac{1}{x_n^2}$, for all $n \in \mathbb{N}$ is not bounded.

5.6. Bonus problem. Discuss the convergence of the real valued sequence given by

$$a_1 = 1, a_{n+1} = \frac{2(2a_n + 1)}{a_n + 3} \text{ for } n \in \mathbb{N}.$$

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