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Analysis I — Problem Set 6

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6.1. Upper and Lower limits. Given sequences (a_n) and (b_n) in $\mathbb{R}$. Show that

(a) $\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \leqslant \liminf_{n\to\infty}(a_n + b_n) \leqslant \liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$;

(b) $\liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \leqslant \limsup_{n\to\infty}(a_n + b_n) \leqslant \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.

(c) If the limit of (a_n) exists, prove that $\limsup_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.

6.2. The Bolzano-Weierstrass theorem. Prove that every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

6.3. Averaging sequence. Let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ be a sequence of complex numbers converging to $a \in \mathbb{C}$. Define a sequence $(s_n)_{n\in\mathbb{N}}$ by

$$s_n := \frac{1}{n}\sum_{k=1}^{n} a_k.$$

Show that (s_n) also converges to a in $\mathbb{C}$.

6.4. Limit points. Determine the set of all limit points of the following set:

(a) $\mathbb{Q} \times \mathbb{Q}\, i$ as a subset of $(\mathbb{C},\, d_1)$,

(b) $\left\{\frac{nm+1}{m} : n \in \mathbb{Z},\, m \in \mathbb{N}\right\}$ in $(\mathbb{R},\, d_1)$,

(c) $\left\{\frac{nm+1}{m} : n \in \mathbb{Z},\, m \in \mathbb{N}\right\}$ in $(\mathbb{R},\, d_0)$,

(d) $\mathbb{Q}$ in $(\mathbb{R},\, d_0)$.

6.5. Complete metric spaces. Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a complete metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges. Is it generally true if the space X is non-complete?

6.6. (Bonus problem)Arithmetic of Cauchy Sequences. Let (a_n), (b_n), (a'_n), (b'_n) be Cauchy sequences in $\mathbb{C}$.

(a) Define (s_n), (p_n) by

$$s_n := a_n + b_n \quad \text{and} \quad p_n := a_n b_n.$$

Show that both are Cauchy sequences, too.

(b) Assume that $a_n \neq 0$ for all n. Define (q_n) by

$$q_n := \frac{1}{a_n}.$$

Show that (q_n) is a Cauchy sequence unless $a_n \longrightarrow 0$.

(c) Define the following equivalence relation on the Cauchy sequences:

$$(a_n) \sim (a'_n) \quad :\Leftrightarrow \quad |a_n - a'_n| \longrightarrow 0.$$

Show that this really is an equivalence relation and that if one sequence of an equivalence class is converging then so are all others of this class and their limits are the same.

(d) Suppose $(a_n) \sim (a'_n)$ and $(b_n) \sim (b'_n)$.
Prove that $(a_n + b_n) \sim (a'_n + b'_n)$, $(a_n b_n) \sim (a'_n b'_n)$ and $(1/a_n) \sim (1/a'_n)$, if $a_n \not\to 0$.

(e) Show that the set of all equivalence classes of Cauchy sequences form a field.