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Analysis I — Problem Set 7

Issued: 15.10.07 Due: 30.10.07, noon

7.1. Series.

(a) Show that $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges in $\mathbb{R}$.

(b) Calculate the limit of $\sum_{k=1}^{\infty} \frac{1}{4k^2-1}$.

(c) Determine whether the series $\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$ is convergent or not.

(d) Determine the limit of $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$.

(e) Determine the limit of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$.

In the last two problems you may use the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

7.2. Radius of convergence. Find the radius of convergence of each of the following power series:

$$\text{(a)} \sum_{n=0}^{\infty} \frac{2^n}{n!} z^n, \qquad \text{(b)} \sum_{n=0}^{\infty} \frac{2^n}{n^2} z^n, \qquad \text{(c)} \sum_{n=0}^{\infty} \frac{n^3}{3^n} z^n.$$

You may use the fact that $\lim_{n\to\infty} \sqrt[n]{n} = 1$

7.3. Leibniz criterion (Alternating series test) Let (a_n) be a monotonically decreasing sequence with $a_n > 0$ and $\lim_{n\to\infty} a_n = 0$. The the series$\sum_{n=0}^{\infty}(-1)^n a_n$ converges.

7.4. Approximation of Real Numbers. Show that for every real number $x \in (0,1)$ there are integers $1 < n_1 < n_2 < \ldots$ such that

$$x = \sum_{k=1}^{\infty} \frac{1}{n_k}$$

7.5. (Bonus problem)Decimal expansions Prove that real numbers correspond bijectively to decimal expansions not terminating in an infinite string of 9's, as follows.

The decimal expansion of $x \in \mathbb{R}$ is $N.x_1x_2x_3...$ where N is the largest integer $\leq x$, x_1 is the largest integer $\leq 10(x-N)$, x_2 is the largest integer $\leq 100(x - N - \frac{x_1}{10})$, and so on.

(a) Show that each x_k, $k \in \mathbb{N}$ is a digit between 0 and 9.

(b) Show that for each $k \in \mathbb{N}$ there is an $l \geq k$ such that $x_l \neq 9$. (So far we showed that our map from $\mathbb{R}$ to decimal expansions not terminating in an infinite string of 9's is well defined.)

(c) Conversely, show that for each such expansion $N.x_1x_2x_3, ...$ not terminating in an infinite string of 9's, the sequence

$$\left(N + \sum_{k=1}^{K} x_k \cdot 10^{-k}\right)_{K \in \mathbb{N}}$$

converges to a real number x with decimal expansion $N.x_1x_2x_3....$ (This takes care of surjectivity and injectivity. Explain why!)

(d) Show that $0.9999999... = \lim_{K \to \infty} \sum_{k=1}^{K} 9 \cdot 10^{-k} = 1.0000000...$. (The mapping defined above is injective only due to our restriction to decimal expansions not terminating in an infinite string of 9's.)